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THE ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES OF MULTIPARAMETER STURM–LIOUVILLE SYSTEMS II

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In a previous paper we studied the asymptotic distribution of the multiparameter eigenvalues of uniformly right definite multiparameter Sturm–Liouville eigenvalue problems. In this paper we extend the analysis to deal with multiparameter Sturm–Liouville problems satisfying uniform left definiteness, and non-uniform left and right definiteness.

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1. Introduction

Consider the k-parameter Sturm–Liouville eigenvalue problem

$$u_r''(x_r)+\left(q_r(x_r)+\sum_{s=1}^{k}\lambda_s v_{rs}(x_r)\right)u_r(x_r)=0,\quad 0\leqq x_r\leqq 1,\quad r=1,\dots,k, \tag{1.1}$$

$$u_r(0)\cos\alpha_r-u_r'(0)\sin\alpha_r=0,\quad u_r(1)\cos\beta_r-u_r'(1)\sin\beta_r=0,\quad r=1,\dots,k, \tag{1.2}$$

where q_r, $r=1,\dots,k$, are real valued, continuous functions on the interval $U=[0,1]$, v_{rs}, $r,s=1,\dots,k$, are real valued, twice continuously differentiable (C^2) functions on U and α_r, $\beta_r\in[0,2\pi]$. A k-tuple $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_k)$ of real numbers is called an eigenvalue of (1.1), (1,2) if, for each r, there exists a non-trivial solution u_r of equation (1.1) satisfying the boundary conditions (1.2).

For all $\mathbf{x}=(x_1,\dots,x_k)\in U^k$ we define the determinant

$$\Delta(\mathbf{x})=\det_{1\leqq r,s\leqq k} v_{rs}(x_r).$$

The multiparameter eigenvalue problem (1.1), (1.2) is said to be *uniformly right definite* if $\Delta(\mathbf{x})>0$ for all $\mathbf{x}\in U^k$, and is said to be *right definite* if $\Delta(\mathbf{x})>0$ for almost all $\mathbf{x}\in U^k$. If the system (1.1), (1.2) is right definite then the basic result regarding the existence of eigenvalues is Klein's oscillation theorem (see [1] or [9] for the case of uniform right definiteness, and [4] for right definiteness);

Theorem 1.1. *For each multi-index* $\mathbf{i}=(i_1,\dots,i_k)$ *where* $i_1,\dots,i_k$, *are non-negative* *integers, there exists a unique eigenvalue* $\boldsymbol{\lambda}^{\mathbf{i}}$ *of* (1.1), (1.2) *such that, for each* r, *a corresponding solution of* (1.1), (1.2) *has precisely* i_r *zeros in the open interval* (0, 1). *There are no other eigenvalues.*

In [10] we studied the asymptotic behaviour of the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ of (1.1), (1.2), for large $\|\mathbf{i}\|$ (where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^k$), and we showed that if the system is uniformly right definite then the eigenvalues have the asymptotic form

$$\boldsymbol{\lambda}^{\mathbf{i}} = \pi^2\|\mathbf{i}\|^2\boldsymbol{\psi}(\mathbf{i}/\|\mathbf{i}\|) + O(\|\mathbf{i}\|^{4/3}), \tag{1.3}$$

where $\boldsymbol{\psi}: \bar{\mathbf{R}}^k_+ \cap S_1 \to \mathbf{R}^k$ is Hölder continuous and non-zero on $\bar{\mathbf{R}}^k_+ \cap S_1$ ($\mathbf{R}_\pm = \{x \in \mathbf{R}: \pm x > 0\}$, $S_1 = \{\mathbf{x} \in \mathbf{R}^k: \|\mathbf{x}\| = 1\}$). Conditions were also given which made $\boldsymbol{\psi}$ Lipschitz continuous and reduced the size of the error term in (1.3) to $O(\|\mathbf{i}\|)$, see Theorem 3.1 of [10]. In this paper we investigate the asymptotic behaviour of the eigenvalues under other definiteness conditions and discuss to what extent the behaviour described by (1.3) is preserved. Specifically, we consider uniform left definiteness and (non-uniform) right and left definiteness (each of these conditions are sufficient to ensure that an analogue of Klein's theorem is valid, see [5]).

The asymptotic behaviour of the eigenvalues has not been considered before under conditions other than uniform right definiteness. In the case where $k=2$ and the system is uniformly right definite this problem has been studied in great detail by Faierman in the papers [7], [8]. The two parameter case has also been studied recently by Browne and Sleeman in [6]. In [6] the asymptotic estimates given in [10] are improved for the case of those eigenvalues which lie in certain cones in the parameter space.

2. Notation and preliminary results

In this section we briefly recall some notation and results from [10]. We begin with a basic result on the number of zeros of solutions of a Sturm–Liouville type differential equation which was proved in [10].

Hypothesis F. *Suppose that the function* $f: U \to \mathbf{R}$ *is* C^2 *and the set* $\{x \in U: f(x) > 0\}$ *can be decomposed into the union of a finite number of disjoint, open intervals* $I_i = (a_i^1, a_i^2)$, $i = 1, \ldots, n$, *together with any of the end points* a_i^j, *which are not zeros of* f, *and there exists a constant* $K > 1$ *such that on each interval* I_i *either*:

(i)
$$\frac{|(x-a_i^1)f'(x)|}{f(x)} \leqq K, \quad \frac{|(x-a_i^1)^2 f''(x)|}{f(x)} \leqq K, \quad i = 1, \ldots, n; \tag{2.1}$$

(ii) *there is an increasing function* $\tilde{f}_i$ *such that,*

$$K^{-1}\tilde{f}_i(x) \leqq f(x) \leqq K\tilde{f}_i(x); \tag{2.2}$$

or (*i*) *holds with* a_i^1 *replaced by* a_i^2, *and* (*ii*) *holds with* $\tilde{f}$ *decreasing.*

Hypothesis UF. *Suppose that the function $f: U \times X \to \mathbf{R}$ is such that for each $\xi \in X$ the function $f(\cdot, \xi): U \to \mathbf{R}$ satisfies hypothesis F, and let $n(\xi)$, $K(\xi)$ denote the number of intervals and the constant in hypothesis F. Then f is said to satisfy hypothesis UF if $n(\xi)$ and $K(\xi)$ are uniformly bounded for $\xi \in X$, i.e. there exist constants $n > 0$, $K > 0$, such that*

$$n(\xi) \leqq n, \quad K(\xi) \leqq K, \quad \xi \in X.$$

For any function $f: U \to \mathbf{R}$, let $\|f\| = \sup\{|f(x)| : x \in U\}$ and let $[f]_+$ denote the function $x \to \max\{f(x), 0\}$, $x \in U$.

Lemma 2.1. *Consider the differential equation*

$$w''(x) + p(x)w(x) + \mu f(x, \xi)w(x) = 0, \quad x \in U, \quad \xi \in X, \tag{2.3}$$

where p is a real-valued, continuous function in U and f satisfies hypothesis UF. Then for all $\mu > 0$, and $\xi \in X$, the number of zeros $v(w)$ of any solution w of (2.3) *in the interval* $(0, 1)$ *satisfies*

$$v(w) = \pi^{-1}\mu^{1/2}\int_0^1 [f(x, \xi)]_+^{1/2}\, dx + O(1), \tag{2.4}$$

where $|O(1)| \leqq 2(n+1)(\pi^{-1}\|p\|^{1/2} + (K^2 + \|p\|)K^2 + 5)$.

To show that a significant class of functions f satisfy hypothesis UF, the following lemma was proved in [**10**].

Lemma 2.2. *Suppose that X is a compact topological space and A is a complex domain containing U, and suppose that $f: A \times X \to \mathbb{C}^n$ is continuous on $A \times X$ and, for each $\xi \in X$, the function $f(\cdot, \xi) \not\equiv 0$ is analytic on A. Also suppose that $f|_{U \times X}$ (the restriction of f to $U \times X$) is real-valued. Then $f|_{U \times X}$ satisfies hypothesis UF.*

We now introduce some further notation. For any set A, we let $\bar{A}$, ∂A, $\operatorname{int} A$ denote the closure, boundary and interior of A respectively. Constants c_i, $i = 1, 2, \ldots,$ will be strictly positive and may depend on k and the functions p_r, v_{rs}, but will not depend on $\boldsymbol{\lambda}$, or on any other variables. For any two quantities a, b, possibly depending on other variables, we will use the notation $a \approx b$ if there exist constants c_1, c_2, such that

$$c_1 \leqq \frac{a}{b} \leqq c_2.$$

For $\mathbf{a} \in \mathbf{R}^k$, we write $\mathbf{a} > \mathbf{0}$ (respectively $\mathbf{a} \geqq \mathbf{0}$) if, and only if, $a_r > 0$ (respectively $a_r \geqq 0$), for all $r = 1, \ldots, k$. For any real (or vector) valued function f, we use the notation $f > 0$ (or $f > \mathbf{0}$) to mean $f(x) > 0$ (or $f(x) > \mathbf{0}$) for all x in the domain of f; the meaning of the notation $f \geqq 0$ ($f \geqq \mathbf{0}$), $f \neq 0$ ($f \neq \mathbf{0}$), etc. is defined similarly.

For each $r=1,\dots,k$, we let $T_r: D(T_r)\subset L^2(U)\to L^2(U)$ denote the self-adjoint differential operator on $L^2(U)$ associated with the differential expression on the left-hand side of (1.1), with $\boldsymbol{\lambda}=\mathbf{0}$, together with the boundary conditions (1.2). Also, for $r=1,\dots,k$, we define the functions

$$\mathbf{v}_r(x_r)=(v_{r1}(x_r),\dots,v_{rk}(x_r)),\quad \boldsymbol{\lambda}\cdot\mathbf{v}_r(x_r)=\sum_{s=1}^{k}\lambda_s v_{rs}(x_r),\quad x_r\in U,\quad \boldsymbol{\lambda}\in\mathbf{R}^k,$$

$$\phi_r(\boldsymbol{\lambda})=\int_U[\boldsymbol{\lambda}\cdot\mathbf{v}_r(x_r)]_+^{1/2}\,dx_r,\quad \boldsymbol{\phi}(\boldsymbol{\lambda})=(\phi_1(\boldsymbol{\lambda}),\dots,\phi_k(\boldsymbol{\lambda})),\quad \boldsymbol{\lambda}\in\mathbf{R}^k.$$

Clearly, the function $\boldsymbol{\phi}:\mathbf{R}^k\to\mathbf{R}^k$ is continuous and $\boldsymbol{\phi}(\boldsymbol{\lambda})\geqq\mathbf{0}$. Also, for any real $c\geqq0$,

$$\boldsymbol{\phi}(c\boldsymbol{\lambda})=c^{1/2}\boldsymbol{\phi}(\boldsymbol{\lambda}).\tag{2.5}$$

Let Q denote the set of points $\boldsymbol{\lambda}$ for which $\boldsymbol{\phi}(\boldsymbol{\lambda})>\mathbf{0}$. It follows from (2.5) that the sets Q and $\bar{Q}$ are cones (a set $A\subset\mathbf{R}^k$ is said to be a cone if, for any $\mathbf{a}\in A$, $c\mathbf{a}\in A$ for all $c>0$).

Throughout the paper we will assume that the functions $(x,\boldsymbol{\lambda})\to\boldsymbol{\lambda}\cdot\mathbf{v}_r(x)$, $(x,\boldsymbol{\lambda})\in U\times\bar{Q}$, $r=1,\dots,k$, satisfy hypothesis UF, so that the result of Lemma 2.1 holds for the differential equations (1.1).

3. Radial behaviour of $\boldsymbol{\lambda}^{\mathbf{i}}$

The estimate (1.3) shows that if the multiparameter system (1.1), (1.2) is uniformly right definite, then $\|\boldsymbol{\lambda}^{\mathbf{i}}\|\approx\|\mathbf{i}\|^2$ for large $\|\mathbf{i}\|$. In this section we consider whether this behaviour holds in general.

Suppose that, for some multi-index $\mathbf{i}$, the eigenvalue $\boldsymbol{\lambda}^{\mathbf{i}}$ exists. By definition, for each $r=1,\dots,k$, there is a solution of the differential equation (1.1) with i_r zeros in $(0,1)$. Thus, by Lemma 2.1,

$$\mathbf{i}=\pi^{-1}\boldsymbol{\phi}(\boldsymbol{\lambda}^{\mathbf{i}})+O(1),\tag{3.1}$$

where the term $O(1)$ is bounded by a constant (given in Lemma 2.1) depending on the functions p_r, v_{rs}, but not on $\boldsymbol{\lambda}$. Next, we observe that from the definition of $\boldsymbol{\phi}$,

$$\|\boldsymbol{\phi}(\boldsymbol{\lambda})\|^2\leqq c_3\|\boldsymbol{\lambda}\|,\quad \boldsymbol{\lambda}\in\mathbf{R}^k.$$

Hence, if $\|\mathbf{i}\|$ is sufficiently large,

$$\|\mathbf{i}\|^2\leqq c_4\|\boldsymbol{\lambda}^{\mathbf{i}}\|,$$

i.e. for any multiparameter problem of the form (1.1), (1.2), the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ cannot grow more slowly than $\|\mathbf{i}\|^2$.

Example. Consider the system

$$u_1''(x_1)+\lambda_1 u_1(x_1)-\lambda_2 x_1^n u_1(x_1)=0, \tag{3.2}$$

$$u_2''(x_2)+\lambda_1 u_2(x_2)=0, \tag{3.3}$$

(where n is any positive integer) with the boundary conditions

$$u_r(0)=0, \quad u_r(1)=0, \quad r=1,2. \tag{3.4}$$

For this system the determinant $\Delta(\mathbf{x})=x_1^n$, so the system is right definite, but is not uniformly right definite.

We now study the behaviour of the eigenvalues $\boldsymbol{\lambda}^{(1,m)}$ as $m\to\infty$. For a solution of (3.3) to have exactly m interior zeros it is necessary that $\lambda_1\approx m^2$. For a solution of (3.2) to have exactly 1 interior zero it is necessary that $\phi_1(\boldsymbol{\lambda})\approx 1$. Now,

$$\phi_1(\boldsymbol{\lambda})=\int_0^{(\lambda_1/\lambda_2)^{1/n}}(\lambda_1-\lambda_2 x_1^n)^{1/2}\,dx_1=\int_0^1(\lambda_1-\lambda_1 z^n)^{1/2}(\lambda_1/\lambda_2)^{1/n}\,dz$$

$$=\frac{\lambda_1^{1/2+1/n}}{\lambda_2^{1/n}}\int_0^1(1-z^n)^{1/2}\,dz$$

(substituting $x_1=(\lambda_1/\lambda_2)^{1/n}z$). Therefore we must have $\lambda_2\approx\lambda_1^{n/2+1}\approx m^{n+2}$, and so $\|\boldsymbol{\lambda}^{(1,m)}\|\approx m^{n+2}$.

This example shows that if we weaken uniform right definiteness to right definiteness then the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ can grow faster than $\|\mathbf{i}\|^{\alpha}$, for any positive α. We will now describe a simple condition which ensures that $\|\boldsymbol{\lambda}^{\mathbf{i}}\|\approx\|\mathbf{i}\|^2$.

For each $r=1,\dots,k$, let

$$N_r=\{\boldsymbol{\lambda}:\boldsymbol{\lambda}\cdot\mathbf{v}_r\leqq 0\}, \quad P_r=\{\boldsymbol{\lambda}:\boldsymbol{\lambda}\cdot\mathbf{v}_r\geqq 0\}.$$

The sets N_r are cones and, for each r, $\mathbf{0}\in N_r$ and $\mathbf{0}\in\partial N_r$ unless $N_r=\mathbf{R}^k$, which only happens if $v_{rs}\equiv 0$ for all s.

Lemma 3.1. *Suppose that the cones N_r satisfy*

$$\bar{Q}\cap\bigcap_{r=1}^{k}\partial N_r=\{0\}. \tag{3.5}$$

Then there exist constants c_5, $c_6>0$, such that for all $\boldsymbol{\lambda}\in\bar{Q}$,

$$c_5\|\boldsymbol{\lambda}\|\leqq\|\boldsymbol{\phi}(\boldsymbol{\lambda})\|^2\leqq c_6\|\boldsymbol{\lambda}\|. \tag{3.6}$$

Proof. We know that the second inequality holds in general, so we must prove the first inequality. Let $m_r(\lambda)=\max\{\lambda\cdot\mathbf{v}_r(x_r): x_r\in U\}$. If $\lambda\in Q$, then for each r, $m_r(\lambda)>0$. Now suppose that there exists a sequence of unit vectors $\lambda^n\in Q$, $n=1,2,\dots$, such that $m_r(\lambda^n)\to 0$, for all $r=1,\dots,k$. By taking a subsequence, if necessary, we may suppose that $\lambda^n\to\lambda^\infty\in\bar{Q}$, $\lambda^\infty\neq\mathbf{0}$. Also, for each r we have $m_r(\lambda^n)>0$, $n=1,2,\dots$, but $m_r(\lambda^\infty)=0$ (by continuity), so $\lambda^\infty\in\partial N_r$, which, since $\lambda^\infty\neq\mathbf{0}$, contradicts (3.5). Thus there exists $c_7>0$ such that for any unit vector $\lambda\in\bar{Q}$ there is an r for which $m_r(\lambda)\geqq c_7$. Hence, since the functions v_{rs} are C^2, it follows that for any $\lambda\in\bar{Q}$, there is an r, and an interval $U_\lambda\subset U$ of length $|U_\lambda|\geqq c_8$, such that

$$\lambda\cdot\mathbf{v}_r(x_r)\geqq c_9\|\lambda\|,\quad x_r\in U_\lambda,$$

and so the result follows from the definition of ϕ.

We observe that in the above example $\lambda=(0,1)\in\bar{Q}\cap\partial N_1\cap\partial N_2$, which is the direction of rapid growth of the eigenvalues $\lambda^{(1,m)}$.

Corollary 3.2. *If* (3.5) *holds, then there exists a constant* $c_{10}>0$ *such that if the eigenvalue* $\lambda^{\mathbf{i}}$ *exists and* $\lambda^{\mathbf{i}}\in\bar{Q}$, *then*

$$\|\lambda^{\mathbf{i}}\|\leqq c_{10}\|\mathbf{i}\|^2. \tag{3.7}$$

We necessarily have $\lambda^{\mathbf{i}}\in\bar{Q}$ if, for instance, the operators, T_r are negative definite (which is assumed in the left definite case, see Section 4). Thus, in this case (3.5) implies (3.7). Similarly, (3.5) implies (3.7) if the operators T_r can be made negative definite by an eigenvalue transformation (as is the case under uniform right definiteness, for example).

The condition (3.5) in Lemma 3.1 does not depend on any particular definiteness conditions, but various such conditions ensure that it holds. For instance, it follows easily from Lemma 2.4 of [10] that uniform right definiteness implies (3.5). Also, it will be shown below that left definiteness (uniform and non-uniform) implies (3.5) (see Section 4 below for the definition of left definiteness and Sections 4 and 6 for the proof of this result). The above example shows that right definiteness does not imply (3.5); however, it will be shown in Section 5 that if the system is right definite and, for each r, the functions $\{v_{rs}: s=1,\dots,k\}$ are linearly independent then (3.5) holds.

4. The uniformly left definite case

Let

$$V(\mathbf{x})=\begin{pmatrix} v_{11}(x_1) & \dots & v_{1k}(x_1)\\ \vdots & & \vdots\\ v_{k1}(x_k) & \dots & v_{kk}(x_k)\end{pmatrix},\ \mathbf{x}\in U^k.$$

The determinant $\Delta(\mathbf{x})=\det V(\mathbf{x})$, and we define $\Delta_{rs}(\mathbf{x})$ to be the cofactor of $v_{rs}(x_r)$ in the expansion of the determinant $\Delta(\mathbf{x})$.

Definition. We say that the multiparameter system (1.1), (1.2) is *uniformly left definite* if the following two conditions hold:
LD_t: each operator T_r, $r=1,\dots,k$, is strictly negative definite.
LD_δ^s: $\Delta_{rs}(\mathbf{x})>0$, for all $\mathbf{x}\in U^k$, $r,s=1,\dots,k$.
We say that the multiparameter system (1.1), (1.2) is *left definite* if LD_t and the following conditions holds:
LD_δ: $\Delta_{rs}(\mathbf{x})>0$, for almost all $\mathbf{x}\in U^k$, $r,s=1,\dots,k$.

Clearly a uniformly left definite system is left definite. Note that what we are calling uniform left definiteness has often been called simply left definiteness in the literature. Note also that conditions LD_δ^s is different to the condition used in the definition of (uniform) left definiteness in, for example, [11]; however, it is shown in [2] that the definitions are equivalent (after an invertible linear transformation of the eigenvalues, if necessary).

Suppose that (1.1), (1.2) is left definite. We define the cones $Q_\pm\subset Q$ by

$$Q_\pm=\{\boldsymbol{\lambda}\in\mathbf{R}^k:\exists\mathbf{x}\in U^k \text{ such that } \boldsymbol{\lambda}\cdot\mathbf{v}_r(x_r)>0, r=1,\dots,k, \quad\text{and}\quad \pm\Delta(\mathbf{x})>0\}.$$

The cones $Q_\pm$ are the analogues, in the present setting, of the cones $C_\pm$ defined in section 4.1 of [3] in an abstract setting. For any $\boldsymbol{\lambda}\in Q$, there are open intervals $U_r\subset U$ such that $\boldsymbol{\lambda}\cdot\mathbf{v}_r(x_r)>0$, $x_r\in U_r$, $r=1,\dots,k$, so, from LD_δ, there is a point $\mathbf{x}\in U^k$ such that

$$\Delta(\mathbf{x})\lambda_l=\sum_{r,s=1}^{k}\Delta_{rl}(\mathbf{x})v_{rs}(x_r)\lambda_s=\sum_{r=1}^{k}\Delta_{rl}(\mathbf{x})\boldsymbol{\lambda}\cdot\mathbf{v}_r(x_r)>0,\quad l=1,\dots,k.$$

It follows that $\Delta(\mathbf{x})\neq0$, so $\boldsymbol{\lambda}\in Q_-\cup Q_+$ and the components λ_l, $l=1,\dots,k$, of $\boldsymbol{\lambda}$ are non-zero and have the same sign as $\Delta(\mathbf{x})$. Thus, $Q=Q_-\cup Q_+$, and $Q_\pm\subset\mathbf{R}^k_\pm$.

In the left definite case we have the following Klein type oscillation theorem, see Corollary 5.6 of [5].

Theorem 4.1. *Suppose that the multiparameter system* (1.1), (1.2) *is left definite. Then for each multi-index* $\mathbf{i}$, *and each non-empty cone* $Q_\pm$ *there exists a unique eigenvalue* $\boldsymbol{\lambda}^{\mathbf{i}\pm}\in Q_\pm$ *of* (1.1), (1.2) *such that, for each* r, *a corresponding solution of* (1.1), (1.2) *has precisely* i_r *zeros in the open interval* (0, 1). *There are no other eigenvalues.*

We remark that Corollary 5.6 of [5] uses a slightly weaker condition than LD_t, but condition LD_t is required in the proofs of the main theorems below to ensure that the eigenvalues lie in the set $\bar{Q}$.

Note also that $Q_\pm\neq\emptyset\Leftrightarrow\exists\mathbf{x}\in U^k$ such that $\pm\Delta(\mathbf{x})>0$. To avoid the trivial case of no eigenvalues we assume that $\Delta\not\equiv0$, and hence, by reordering the system if necessary, we may suppose that $\Delta(\mathbf{x})>0$ for some $\mathbf{x}$, and so $Q_+\neq\emptyset$. We will discuss the distribution of the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}+}$ in Q_+; a similar discussion applies to the $\boldsymbol{\lambda}^{\mathbf{i}-}\in Q_-$ if $Q_-\neq\emptyset$.

For the remainder of this section we suppose that the system (1.1), (1.2) is uniformly left definite. The distribution of the eigenvalues in the (non-uniform) left definite case will be discussed in Section 6.

We now define the matrices $V_r(\mathbf{x})$, $r=1,\dots,k$, to be the matrices obtained by replacing the rth row of $V(\mathbf{x})$ with the vector $\omega=(1,\dots,1)$. Condition LD^s_δ implies that for each r and each $\mathbf{x}\in U^k$, $\det V_r(\mathbf{x})>0$, so the matrix $V_r(\mathbf{x})$ is non-singular. Also, let $\omega^\perp$ denote the set $\{\lambda\in\mathbf{R}^k:\lambda\cdot\omega=0\}$.

Lemma 4.2. *For any $v\in\omega^\perp$, there exist integers $r_\pm$ such that*

$$\pm v\cdot \mathbf{v}_{r\pm}\geqq c_{11}\|v\|.$$

Proof. Choose any unit vector $v\in\omega^\perp$. Now suppose that for some r, and all $r'\neq r$, there exist points $x^0_{r'}$ such that $v\cdot\mathbf{v}_{r'}(x^0_{r'})=0$. Then letting $\mathbf{x}^0=(x^0_1,\dots,x^0_k)$, with $\mathbf{x}^0_r=\frac{1}{2}$ say, we have $V_r(\mathbf{x}^0)v=\mathbf{0}$ (where $V_r(\mathbf{x}^0)v$ denotes the standard matrix product, with v regarded as a column matrix), which contradicts the non-singularity of the matrix $V_r(\mathbf{x}^0)$. Hence, for each r there is an $r'\neq r$ such that $v\cdot\mathbf{v}_{r'}\neq 0$; therefore, letting $R(v)$ denote the set of integers r for which $v\cdot\mathbf{v}_r\neq 0$, the number of elements of $R(v)$ is at least 2. Now suppose that $v\cdot\mathbf{v}_r>0$, $r\in R(v)$. By the definition of $R(v)$, there exists a point $\mathbf{x}\in U^k$ such that

$$v\cdot\mathbf{v}_r(x_r)=p_r,\quad r=1,\dots,k, \tag{4.1}$$

where $p_r>0$, $r\in R(v)$, and $p_r=0$, $r\notin R(v)$. From (4.1) and LD^s_δ we obtain

$$\Delta(\mathbf{x})v_l=\sum_{r=1}^k\Delta_{rl}(\mathbf{x})\sum_{s=1}^k v_{rs}(x_r)v_s=\sum_{r=1}^k\Delta_{rl}(\mathbf{x})p_r>0,\quad l=1,\dots,k,$$

so the numbers v_r, $r=1,\dots,k$, are all non-zero and have the same sign. However, this contradicts the hypothesis that $v\cdot\omega=0$, and so our supposition that $v\cdot\mathbf{v}_r>0$, $r\in R(v)$, must be false. Similarly, we cannot have $v\cdot\mathbf{v}_r<0$, $r\in R(v)$. Therefore, letting $R_\pm(v)$ denote the set of integers r for which $\pm v\cdot\mathbf{v}_r>0$, each set $R_\pm(v)$ must be non-empty for any unit vector $v\in\omega^\perp$.

Now suppose that there exists a sequence of unit vectors $v^n\in\omega^\perp$, $n=1,2,\dots$, such that for all $r\in R_+(v^n)$, $\min\{v^n\cdot\mathbf{v}_r(x_r):x_r\in U\}<n^{-1}$. By taking a subsequence, if necessary, we may assume that $v^n\to v^\infty\in\omega^\perp$, and the sets $R_\pm(v^n)$, $n=1,2,\dots$, are constant, and equal to $R_\pm$, say. By continuity, for each $r\in R_+$ the function $v^\infty\cdot\mathbf{v}_r$ has a zero, so $r\notin R_+(v^\infty)$; for each $r\in R_-$ the function $v^\infty\cdot\mathbf{v}_r\leqq 0$, so $r\notin R_+(v^\infty)$; for each $r\notin R_+\cup R_-$ the function $v^\infty\cdot\mathbf{v}_r$ has a zero, so $r\notin R_+(v^\infty)$. However, this contradicts the fact proved above that $R_+(v^\infty)$ must be non-empty. Therefore there exists a constant $c_{12}>0$ such that for all unit vectors $v\in\omega^\perp$, we have $v\cdot\mathbf{v}_r\geqq c_{12}$, for some $r\in R_+(v)$. We can obtain a similar result for the set $R_-(v)$, so the lemma follows.

Lemma 4.3. *If the system* (1.1), (1.2) *is uniformly left definite then condition* (3.5) *holds.*

Proof. Suppose that $\lambda\neq\mathbf{0}$ and $\lambda\in\bar{Q}\cap\bigcap_{r=1}^k\partial N_r$. Since the set $Q=Q_-\cup Q_+$ is a cone and $\bar{Q}_\pm\subset\bar{\mathbf{R}}^k_\pm$, there exists a vector $v\in\omega^\perp$ such that $\lambda+v\in Q$. However, since $\lambda\in\bigcap_{r=1}^k$

∂N_r, we have $\boldsymbol{\lambda}\cdot\mathbf{v}_r \leqq 0$ for all r, and it follows from Lemma 4.2 that $\boldsymbol{\nu}\cdot\mathbf{v}_{r'} \leqq 0$, for some r', hence $(\boldsymbol{\lambda}+\boldsymbol{\nu})\cdot\mathbf{v}_{r'} \leqq 0$, which contradicts $\boldsymbol{\lambda}+\boldsymbol{\nu}\in Q$ and so proves the lemma.

The discussion on Section 3 now yields the following corollary.

Corollary 4.4. *If the system* (1.1), (1.2) *is uniformly left definite then* $\|\boldsymbol{\lambda}^{\mathbf{i}+}\| \approx \|\mathbf{i}\|^2$, *for all* $\mathbf{i}$. *A similar result holds for the eigenvalues* $\boldsymbol{\lambda}^{\mathbf{i}-}$, *if they exist.*

Lemma 4.5. *Suppose that* $\boldsymbol{\lambda}\in\bar{Q}_+$ *and* $\|\boldsymbol{\lambda}\|=1$. *Then for any* $\boldsymbol{\mu}\in\mathbf{R}^k$ *with* $\|\boldsymbol{\mu}\|\leqq 1$ and $\boldsymbol{\lambda}+\boldsymbol{\mu}\in\bar{Q}_+$, *we have*:

(i)
$$\|\boldsymbol{\phi}(\boldsymbol{\lambda}+\boldsymbol{\mu})-\boldsymbol{\phi}(\boldsymbol{\lambda})\| \geqq c_{13}\|\boldsymbol{\mu}\|^3; \tag{4.2}$$

(ii) *if, for some* ε, $0<\varepsilon<1$, *we have* $\boldsymbol{\phi}(\boldsymbol{\lambda})\geqq(\varepsilon,\dots,\varepsilon)$ *and* $\boldsymbol{\phi}(\boldsymbol{\lambda}+\boldsymbol{\mu})\geqq(\varepsilon,\dots,\varepsilon)$, *then*

$$\|\boldsymbol{\phi}(\boldsymbol{\lambda}+\boldsymbol{\mu})-\boldsymbol{\phi}(\boldsymbol{\lambda})\| \geqq \varepsilon c_{14}\|\boldsymbol{\mu}\|. \tag{4.3}$$

Proof. Since $\boldsymbol{\lambda}\in\bar{Q}_+\subset\bar{\mathbf{R}}^k_+$ and $\boldsymbol{\omega}=(1,\dots,1)$, $\boldsymbol{\mu}$ can be written in the form $\boldsymbol{\mu}=\alpha\boldsymbol{\lambda}+\boldsymbol{\nu}$, where $\boldsymbol{\nu}\in\boldsymbol{\omega}^{\perp}$ and α, $\boldsymbol{\nu}$ are uniquely defined and satisfy

$$c_{15}\|\boldsymbol{\mu}\| \leqq |\alpha|+\|\boldsymbol{\nu}\| \leqq c_{16}\|\boldsymbol{\mu}\|. \tag{4.4}$$

We may suppose that $\alpha\geqq 0$ (the case $\alpha<0$ can be dealt with similarly). Now suppose that $\alpha>c_{17}\|\boldsymbol{\nu}\|^{1/2}$, where c_{17} is sufficiently large (the specific criterion will be given below). Then, if $\phi_r(\boldsymbol{\lambda})\geqq c_{18}$ (this is true for some r by Lemmas 3.1 and 4.3), we have

$$\begin{aligned}\phi_r(\boldsymbol{\lambda}+\boldsymbol{\mu})-\phi_r(\boldsymbol{\lambda}) &= \phi_r((1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu})-\phi_r(\boldsymbol{\lambda})\\ &\geqq \phi_r((1+\alpha)\boldsymbol{\lambda})-\phi_r(\boldsymbol{\lambda})-c_{19}\|\boldsymbol{\nu}\|^{1/2}\\ &= ((1+\alpha)^{1/2}-1)\phi_r(\boldsymbol{\lambda})-c_{19}\|\boldsymbol{\nu}\|^{1/2}\\ &\geqq c_{20}\alpha \geqq c_{21}\|\boldsymbol{\mu}\|,\end{aligned}$$

if c_{17} is sufficiently large (the first inequality above is obtained from the integral defining $\phi_r((1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu})$ by applying the general inequality $[a+b]_+^{1/2}\geqq[a]_+^{1/2}-2|b|^{1/2}, a,b\in\mathbf{R}$; the second inequality uses the fact that $\alpha\leqq c_{16}$).

Now suppose that $\alpha\leqq c_{17}\|\boldsymbol{\nu}\|^{1/2}$, (and hence $\|\boldsymbol{\mu}\|\leqq c_{22}\|\boldsymbol{\nu}\|^{1/2}$, by (4.4), since $\|\boldsymbol{\nu}\|\leqq c_{23}\|\boldsymbol{\nu}\|^{1/2}$) and suppose that

$$\boldsymbol{\nu}\cdot\mathbf{v}_r \geqq c_{24}\|\boldsymbol{\nu}\|$$

(this is true for some r by Lemma 4.2). This inequality is (2.25) in the proof of Lemma

2.6 of [10] (in the notation used there, we have $(1+\alpha)\boldsymbol{\lambda}=\boldsymbol{\lambda}^1$, $(1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu}=\boldsymbol{\lambda}^2$ and $\boldsymbol{\nu}=\boldsymbol{\lambda}^2-\boldsymbol{\lambda}^1$), and leads to inequality (2.31) in [10], which in the present case is

$$\phi_r((1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu})\geqq\phi_r((1+\alpha)\boldsymbol{\lambda})+c_{25}\|\boldsymbol{\nu}\|^{3/2}.$$

Similarly, if the hypotheses of part (ii) of the lemma hold, inequality (2.31) in [10] leads to

$$\phi_r((1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu})\geqq\phi_r((1+\alpha)\boldsymbol{\lambda})+c_{26}\varepsilon\|\boldsymbol{\nu}\|.$$

Thus, in the general case we have

$$\begin{aligned}\phi_r(\boldsymbol{\lambda}+\boldsymbol{\mu})-\phi_r(\boldsymbol{\lambda})&=\phi_r((1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu})-\phi_r(\boldsymbol{\lambda})\\&\geqq((1+\alpha)^{1/2}-1)\phi_r(\boldsymbol{\lambda})+c_{25}\|\boldsymbol{\nu}\|^{3/2}\\&\geqq c_{27}\|\boldsymbol{\mu}\|^3,\end{aligned}$$

and, if the hypotheses of part (ii) of the lemma hold, we have

$$\begin{aligned}\phi_r(\boldsymbol{\lambda}+\boldsymbol{\mu})-\phi_t(\boldsymbol{\lambda})&=\phi_r((1+\alpha)\boldsymbol{\lambda}+\boldsymbol{\nu})-\phi_r(\boldsymbol{\lambda})\\&\geqq((1+\alpha)^{1/2}-1)\phi_r(\boldsymbol{\lambda})+c_{26}\varepsilon\|\boldsymbol{\nu}\|\\&\geqq c_{28}\varepsilon(\alpha+\|\boldsymbol{\nu}\|)\geqq c_{29}\varepsilon\|\boldsymbol{\mu}\|.\end{aligned}$$

These results prove the lemma.

We now let $\bar{\mathbf{R}}^k_\varepsilon$ denote the closed set in $\mathbf{R}^k$ consisting of those vectors $\boldsymbol{\lambda}\neq\mathbf{0}$ such that $\boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|\geqq(\varepsilon,\dots,\varepsilon)$, together with $\mathbf{0}$. Also, we let $\boldsymbol{\phi}_+$ denote the restriction of $\boldsymbol{\phi}$ to the set $\bar{Q}_+$.

Theorem 4.6. *The mapping $\boldsymbol{\phi}_+:\bar{Q}_+\to\bar{\mathbf{R}}^k_+$ is a homeomorphism. Let $\boldsymbol{\phi}_+^{-1}:\bar{\mathbf{R}}^k_+\to\bar{Q}_+$ denote the inverse of this homeomorphism. The restriction of $\boldsymbol{\phi}_+^{-1}$ to $\bar{\mathbf{R}}^k_+\cap S_1$ is Hölder continuous with exponent $\frac{1}{3}$. The eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}+}$ of the multiparameter problem satisfy*

$$\boldsymbol{\lambda}^{\mathbf{i}+}=\pi^2\boldsymbol{\phi}_+^{-1}(\mathbf{i})+O(\|\mathbf{i}\|^{5/3})=\pi^2\|\mathbf{i}\|^2\boldsymbol{\phi}_+^{-1}(\mathbf{i}/\|\mathbf{i}\|)+O(\|\mathbf{i}\|^{5/3}),$$

for all multi-indices $\mathbf{i}\neq\mathbf{0}$. For any ε with $0<\varepsilon\leqq1$, the restriction of $\boldsymbol{\phi}_+^{-1}$ to $\bar{\mathbf{R}}^k_\varepsilon\cap S_1$ is Lipschitz continuous with Lipschitz constant $c_{30}\varepsilon^{-1}$, and for all $\mathbf{i}\in\bar{\mathbf{R}}^k_\varepsilon$,

$$\boldsymbol{\lambda}^{\mathbf{i}+}=\pi^2\boldsymbol{\phi}_+^{-1}(\mathbf{i})+\varepsilon^{-1}O(\|\mathbf{i}\|)=\pi^2\|\mathbf{i}\|^2\boldsymbol{\phi}_+^{-1}(\mathbf{i}/\|\mathbf{i}\|)+\varepsilon^{-1}O(\|\mathbf{i}\|).$$

Proof. The proof is almost identical to the proof of Theorem 3.1 of [**10**], using the results of Lemma 4.5.

We observe that the error estimate $O(\|\mathbf{i}\|^{5/3})$ is larger than the corresponding estimate $O(\|\mathbf{i}\|^{4/3})$ obtained in Theorem 3.1 of [**10**]. This is due to the fact that we cannot assert that for all $\boldsymbol{\lambda}\neq\mathbf{0}$ there is an r such that $\boldsymbol{\lambda}\cdot\mathbf{v}_r\neq 0$, as we can under uniform right definiteness (c.f. Lemma 2.4 in [**10**] and Lemma 4.2 above). Probably both these estimates are too pessimistic for many multiparameter systems (in particular, see the third part of Theorem 3.1 of [**10**]).

5. The right definite case

In this section we suppose that the system (1.1), (1.2) is right definite. Lemma 3.1 gave a condition which ensures that the radial behaviour of the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ is as described in (1.3) for large $\|\mathbf{i}\|$ (assuming, in addition, that condition LD_t holds; see the discussion in Section 3). We will now discuss a condition which ensures that the 'angular' behaviour of the $\boldsymbol{\lambda}^{\mathbf{i}}$ is as described in (1.3), at least in subcones of $\bar{Q}$ which lie 'strictly' inside Q.

We begin with some constructions which will enable us to deal with the loss of uniformity compared with the uniform right definite case. For any $\mathbf{u}=(u_1,\ldots,u_k)\in\bigoplus_{r=1}^k L^2(U)$, let $W(\mathbf{u})$ denote the $k\times k$ matrix with (r,s) element equal to $(v_{rs}u_r,u_r)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(U)$. It can easily be seen that right definiteness implies that if $\mathbf{u}$ has $u_r\neq 0$, $r=1,\ldots,k$, then $\det W(\mathbf{u})\neq 0$, i.e. the matrix $W(\mathbf{u})$ is non-singular.

Lemma 5.1. *For any $\boldsymbol{\lambda}\in\mathbf{R}^k$ there exists r such that $\boldsymbol{\lambda}\in N_r\cup P_r$.*

Proof. Suppose that there exists $\boldsymbol{\lambda}\neq\mathbf{0}$ such that $\boldsymbol{\lambda}\notin N_r\cup P_r$ for all $r=1,\ldots,k$ (i.e. the functions $\boldsymbol{\lambda}\cdot\mathbf{v}_r$ attain both positive and negative values). Then since the rth element of the matrix product $W(\mathbf{u})\boldsymbol{\lambda}$ is given by

$$\int_0^1 \boldsymbol{\lambda}\cdot\mathbf{v}_r(x_r)|u_r(x_r)|^2\,dx_r,$$

we can choose non-zero functions $u_r\in L^2(U)$, $r=1,\ldots,k$, such that $W(\mathbf{u})\boldsymbol{\lambda}=\mathbf{0}$, which contradicts right definiteness, and so proves the lemma.

Now, for each $r=1,\ldots,k$, let

$$Z_r=\{\boldsymbol{\lambda}\in\mathbf{R}^k:\boldsymbol{\lambda}\cdot\mathbf{v}_r\equiv 0 \text{ on a subset of } U \text{ with positive measure}\}.$$

The set Z_r need not be a subset of $N_r\cup P_r$, but if Z_r intersects $N_r\cup P_r$ then $Z_r\cap(N_r\cup P_r)\subset\partial N_r\cup\partial P_r$ (to see this suppose that $\boldsymbol{\lambda}\in Z_r\cap(N_r\cup P_r)$). Right definiteness implies that at least one of the functions v_{rs}, $s=1,\ldots,k$, is non-zero somewhere on the interval on which $\lambda\cdot\mathbf{v}_r\equiv 0$, so there exist points $\lambda^{\pm}$ arbitrarily close to λ for which $\lambda^+\notin P_r$, $\lambda^-\notin N_r$).

Lemma 5.2. *Suppose that*

$$Z_r\cap(N_r\cup P_r)\subset\{\mathbf{0}\}\cup\bigcup_{t=1}^{k}(\operatorname{int}N_t\cup\operatorname{int}P_t),\quad r=1,\dots,k. \tag{5.1}$$

Then, for any ε *with* $0<\varepsilon\leqq 1$, *there is a number* $\gamma(\varepsilon)>0$ *such that for any* $\lambda\in\mathbf{R}^k$ *there is an integer* r, *for which* $\lambda\in N_r\cup P_r$ *and*

$$|\lambda\cdot\mathbf{v}_r(x_r)|\geqq\gamma(\varepsilon)\|\lambda\|,\quad x_r\notin U_r(\lambda,\varepsilon),$$

where the set $U_r(\lambda,\varepsilon)\subset U$ *has measure* $|U_r(\lambda,\varepsilon)|<\varepsilon$.

Remark. If $Z_r=\{\mathbf{0}\}$, $r=1,\dots,k$ (e.g. if the functions v_{rs}, $s=1,\dots,k$, are analytic and linearly independent for each r), then condition (5.1) is satisfied.

Proof. Suppose that the assertion of the lemma is false and there exists an $\varepsilon>0$ and a sequence of unit vectors λ^n, $n=1,2,\dots$, such that, for those r for which $\lambda^n\in N_r\cup P_r$, the sets

$$U_r^n=\{x_r\in U:|\lambda^n\cdot\mathbf{v}_r(x_r)|\leqq 1/n\}\subset U$$

have measure $|U_r^n|\geqq\varepsilon$. By taking a subsequence, if necessary, we may assume that $\lambda^n\to\lambda^\infty\neq\mathbf{0}$, and that there is a set of integers R such that $\lambda^n\in N_r\cup P_r\Leftrightarrow r\in R$, for all n. Letting $U_r^\infty=\bigcap_{N=1}^\infty\bigcup_{n=N}U_r^n$, $r\in R$, we have $|U_r^\infty|\geqq\varepsilon$ (see Q.2 in Exercises 13.2, p. 340 of [12]), and by continuity, $\lambda^\infty\cdot\mathbf{v}_r(x_r)=0$, $x_r\in U_r^\infty$. Thus: for $r\in R$ we have $\lambda^\infty\in Z_r$, and so $\lambda^\infty\notin\operatorname{int}N_r\cup\operatorname{int}P_r$ (since $Z_r\cap(N_r\cup P_r)\subset\partial N_r\cup\partial P_r$); for $r\notin R$ we have $\lambda^\infty\notin\operatorname{int}N_r\cup\operatorname{int}P_r$ (since $\lambda^n\notin N_r\cup P_r$, $n=1,2,\dots$). This contradicts condition (5.1), and so complete the proof of the lemma.

Lemma 5.3. *Suppose that* (5.1) *holds and that* $0<\varepsilon<1$. *There exists a number* $\delta(\varepsilon)>0$ *such that if* $\lambda^i\in\bar{Q}_+$, $\|\lambda^i\|\leqq 1$, $\phi(\lambda^i)\geqq(\varepsilon,\dots,\varepsilon)$, $i=1,2$, *and* $\|\lambda^2-\lambda^1\|$ *is sufficiently small, then*

$$\|\phi(\lambda^2)-\phi(\lambda^1)\|\geqq\delta(\varepsilon)\|\lambda^2-\lambda^1\|. \tag{5.2}$$

Proof. Let $\mu=\lambda^2-\lambda^1$. By definition, for any r,

$$\phi_r(\lambda^2)-\phi_r(\lambda^1)=\int_U\delta_r(x_r)\,dx_r, \tag{5.3}$$

where $\delta_r(x_r)=[\lambda^1\cdot\mathbf{v}_r(x_r)+\mu\cdot\mathbf{v}_r(x_r)]_+^{1/2}-[\lambda^1\cdot\mathbf{v}_r(x_r)]_+^{1/2}$. Now, the maximum of the func-

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tions $\boldsymbol{\lambda}^i \cdot \mathbf{v}_r$ in U is greater than ε^2, and since the functions v_{rs} are C^2, it follows that $\boldsymbol{\lambda}^i \cdot \mathbf{v}_r \geqq \varepsilon^2/2$ on an interval U_r of length at least $2c_{31}\varepsilon^2$. Also, by Lemma 5.2, there is an r such that

$$\boldsymbol{\mu}\cdot\mathbf{v}_r(x_r) \geqq \gamma(c_{31}\varepsilon^2)\|\boldsymbol{\mu}\|, \quad x_r \notin U_r', \tag{5.4}$$

where $U_r' \subset U$ is an interval with $|U_r'| < c_{31}\varepsilon^2$ or $(\boldsymbol{\mu}\cdot\mathbf{v}_r(x) \leqq -\gamma(c_{31}\varepsilon^2)\|\boldsymbol{\mu}\|$, $x \in U_r'$, in which case we interchange $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ to obtain (5.4)). Thus, the set $U_r'' = U_r \cap (U \backslash U_r')$ has measure $|U_r''| \geqq c_{31}\varepsilon^2$, and

$$\boldsymbol{\mu}\cdot\mathbf{v}_r(x_r) \geqq \gamma(c_{31}\varepsilon^2)\|\boldsymbol{\mu}\|, \quad \boldsymbol{\lambda}^1\cdot\mathbf{v}_r(x_r) \geqq \varepsilon^2/2, \quad x_r \in U_r''.$$

It follows from these inequalities that

$$\delta_r(x_r) \geqq \frac{\gamma(c_{31}\varepsilon^2)\|\boldsymbol{\mu}\|}{(\varepsilon^2/2)^{1/2} + (\gamma(c_{31}\varepsilon^2)\|\boldsymbol{\mu}\|)^{1/2}} \geqq \gamma(c_{31}\varepsilon^2)\|\boldsymbol{\mu}\|\varepsilon^{-1}, \quad x_r \in U_r'',$$

if $\|\boldsymbol{\mu}\|$ is sufficiently small. The inequality (5.2) now follows from this estimate, (5.3) and the estimate for $|U_r''|$.

Note that $\delta(\varepsilon)$ may tend to zero as $\varepsilon \to 0$.

Theorem 5.4. *If* (3.5) *and* (5.1) *hold then the mapping* $\boldsymbol{\phi}: Q \to \mathbf{R}_+^k$ *is a homeomorphism, with inverse* $\boldsymbol{\phi}^{-1}: \mathbf{R}_+^k \to Q$. *Also, for any* ε *with* $0 < \varepsilon \leqq 1$, *the restriction of* $\boldsymbol{\phi}^{-1}$ *to* $\bar{\mathbf{R}}_\varepsilon^k \cap S_1$ *is Lipschitz continuous with Lipschitz constant* $\delta(\varepsilon)^{-1}$. *If condition* LD_t *also holds then, for all* $\mathbf{i} \in \mathbf{R}_\varepsilon^k$,

$$\boldsymbol{\lambda}^{\mathbf{i}} = \pi^2\boldsymbol{\phi}^{-1}(\mathbf{i}) + \delta(\varepsilon)^{-1}O(\|\mathbf{i}\|) = \pi^2\|\mathbf{i}\|^2\boldsymbol{\phi}^{-1}(\mathbf{i}/\|\mathbf{i}\|) + \delta(\varepsilon)^{-1}O(\|\mathbf{i}\|).$$

Proof. The proof is almost identical to the proof of Theorem 3.1 of **[10]** using Lemmas 3.1 and 5.3.

We will now give a simple criterion for condition (3.5) to be satisfied when the system is right definite.

Lemma 5.5. *If, for each* r, *the functions* $\{v_{rs}: s = 1, \ldots, k\}$ *are linearly independent then* (3.5) *holds.*

Proof. Suppose that $\boldsymbol{\lambda} \neq \mathbf{0}$, $\boldsymbol{\lambda} \in \bar{Q} \cap \bigcap_{r=1}^k \partial N_r$, and let $\boldsymbol{\lambda}^n \in Q$, $n = 1, 2, \ldots$, be a sequence of vectors such that $\boldsymbol{\lambda}^n \to \boldsymbol{\lambda}$. It follows from Lemma 5.1 that, after taking a subsequence if necessary, there is an r such that $\boldsymbol{\lambda}^n \in P_r$ for all n. But, by assumption, $\boldsymbol{\lambda} \in \partial N_r$, so by continuity we must have $\boldsymbol{\lambda}\cdot\mathbf{v}_r \equiv 0$. However, this contradicts the linear independence of the functions $\{v_{rs}: s = 1, \ldots, k\}$ and so proves the lemma.

Remark. The conditions (3.5) and (5.1) are independent of each other in the sense that either can hold without the other. To see this let

$$f_0(x)=\begin{cases}1-2x, & 0\leqq x\leqq \frac{1}{2},\\ 0, & \frac{1}{2}\leqq x\leqq 1,\end{cases} \quad f_1(x)=1-x, \quad 0\leqq x\leqq 1,$$

and consider the 2×2 arrays of coefficient functions v_{rs} given by

$$\text{(i)} \begin{pmatrix}1 & -f_1\\ 1 & f_0\end{pmatrix}, \quad \text{(ii)} \begin{pmatrix}1 & f_1\\ -1 & f_1\end{pmatrix}.$$

It can be verified that both these arrays are right definite, but for (i), (3.5) holds and (5.1) does not, while for (ii), (3.5) does not hold and (5.1) does.

6. The left definite case

In this section we suppose that the multiparameter system (1.1), (1.2) is left definite as defined in Section 4. As noted in Section 4, the oscillation theorem, Theorem 4.1, is valid under this hypothesis. Again we will assume that $\Delta(\mathbf{x})>0$ for some $\mathbf{x}$, and so $Q_+\neq\emptyset$, and we will discuss the distribution of the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}+}$ in Q_+. A similar discussion applies to the $\boldsymbol{\lambda}^{\mathbf{i}-}\in Q_-$ if $Q_-\neq\emptyset$.

For any $\mathbf{u}=(u_1,\ldots,u_k)\in\bigoplus_{r=1}^k L^2(U)$ we let $W_r(\mathbf{u})$, $r=1,\ldots,k$, denote the matrix obtained by replacing the rth row of $W(\mathbf{u})$ (defined in Section 5) with the vector $\boldsymbol{\omega}=(1,\ldots,1)$. Left definiteness implies that for each r and each $\mathbf{u}$ with $u_r\neq 0$, $r=1,\ldots,k$, $\det W_r(\mathbf{u})>0$, so the matrix $W_r(\mathbf{u})$ is non-singular.

Lemma 6.1. *For any $\mathbf{v}\in\boldsymbol{\omega}^{\perp}$ there exist integers $r_{\pm}$ such that*

$$\pm\mathbf{v}\cdot\mathbf{v}_{r\pm}(x_r)>0, \quad \text{a.e.} \quad x_{r\pm}\in U.$$

Proof. The proof of this lemma is similar to the proof of Lemma 4.2, but using the matrices $W_r(\mathbf{u})$ instead of the matrices $V_r(\mathbf{x})$, as in the proof of Lemma 5.1.

Lemma 6.2. *If the system* (1.1), (1.2) *is left definite then condition* (3.5) *holds.*

Proof. The proof of this lemma is similar to the proof of Lemma 4.3, but using Lemma 6.1, rather than Lemma 4.2.

Corollary 6.3. *If the system* (1.1), (1.2) *is left definite then* $\|\boldsymbol{\lambda}^{\mathbf{i}+}\|\approx\|\mathbf{i}\|^2$, *for all* $\mathbf{i}$.

Thus, of all the definiteness conditions considered in this paper, only (non-uniform) right definiteness allows the eigenvalues to grow faster than $\|\mathbf{i}\|^2$.

Lemma 6.4. *Suppose that*

$$Z_r \cap (N_r \cup P_r) \cap \omega^\perp \subset \{\mathbf{0}\} \cup \bigcup_{t=1}^{k} (\operatorname{int} N_t \cup \operatorname{int} P_t), \quad r = 1, \ldots, k. \tag{6.1}$$

Then, for any ε *with* $0 < \varepsilon \leqq 1$, *there is a number* $\gamma(\varepsilon) > 0$ *such that for any* $\mathbf{v} \in \omega^\perp$, *there are integers* $r_\pm$, *for which*

$$\pm \mathbf{v} \cdot \mathbf{v}_r(x_{r\pm}) \geqq \gamma(\varepsilon)\|\mathbf{v}\|, \quad x_{r\pm} \notin U_{r\pm}(\mathbf{v}, \varepsilon),$$

where the sets $U_{r\pm}(\mathbf{v}, \varepsilon) \subset U$ *have measures* $|U_{r\pm}(\mathbf{v}, \varepsilon)| < \varepsilon$.

Proof. The proof of this lemma is similar to the proof of Lemma 5.2.

Remark. When $k = 2$, it follows from Lemma 6.1 that $Z_r \cap \omega^\perp = \{\mathbf{0}\}$, so in this case condition (6.1) holds automatically.

We can now use Lemma 6.4 to obtain the following theorem on the distribution of the multiparameter eigenvalues in the left definite case, using similar methods to those used to obtain Theorems 4.6 and 5.4.

Theorem 6.5. *If* (6.1) *holds then the mapping* $\phi_+ : Q_+ \to \mathbf{R}^k_+$ *is a homeomorphism, with inverse* $\phi_+^{-1} : \mathbf{R}^k_+ \to Q_+$. *Also, for any* ε *with* $0 < \varepsilon \leqq 1$, *there is a number* $\delta(\varepsilon) > 0$ *such that the restriction of* ϕ_+^{-1} *to* $\bar{R}^k_\varepsilon \cap S_1$ *is Lipschitz continuous with Lipschitz constant* $\delta(\varepsilon)^{-1}$ *and, for all* $\mathbf{i} \in \bar{\mathbf{R}}^k_\varepsilon$,

$$\lambda^{\mathbf{i}+} = \pi^2 \phi_+^{-1}(\mathbf{i}) + \delta(\varepsilon)^{-1} O(\|\mathbf{i}\|) = \pi^2 \|\mathbf{i}\|^2 \phi_+^{-1}(\mathbf{i}/\|\mathbf{i}\|) + \delta(\varepsilon)^{-1} O(\|\mathbf{i}\|).$$

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