Let $\Gamma$ be a $G$-symmetric graph admitting a nontrivial $G$-invariant partition $B$ of block size $v$. For blocks $B, C$ of $B$ adjacent in the quotient graph $\Gamma_B$, let $k$ be the number of vertices in $B$ adjacent to at least one vertex in $C$. In this paper we classify all possibilities for $(\Gamma, \Gamma_B, G)$ in the case where $k = v - 1 \geq 2$ and $B(\alpha) = B(\beta)$ for adjacent vertices $\alpha, \beta$ of $\Gamma$, where for a vertex of $\Gamma$, say $\gamma \in B, B(\gamma)$ denotes the set of blocks $C$ such that $\gamma$ is the only vertex in $B$ not adjacent to any vertex in $C$.

1. Introduction

A finite graph $\Gamma = (V(\Gamma), E(\Gamma))$ is said to admit a finite group $G$ as a group of automorphisms if $G$ acts on $V(\Gamma)$ in such a way that it preserves the adjacency of $\Gamma$. For such a pair $(\Gamma, G)$, if $G$ is transitive on $V(\Gamma)$ and, in its induced action, is transitive on the set $\text{Arc}(\Gamma)$ of arcs of $\Gamma$, then $\Gamma$ is said to be a $G$-symmetric graph, where an arc is an ordered pair of adjacent vertices. Roughly speaking, in most cases such a graph $\Gamma$ admits a nontrivial $G$-invariant partition, that is, a partition $B$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in B$ for $B \in B$ and $g \in G$, where $B^g := \{\alpha^g : \alpha \in B\}$. In this case $\Gamma$ is said to be an imprimitive $G$-symmetric graph. From permutation group theory [3, Corollary 1.5A], this happens precisely when $G_\alpha$ is not a maximal subgroup of $G$, where $\alpha \in V(\Gamma)$ and $G_\alpha$ is the stabiliser of $\alpha$ in $G$. For such a graph $\Gamma$ we have a natural quotient graph $\Gamma_B$ with respect to $B$, which is defined to have vertex set $B$ in which $B, C \in B$ are adjacent if and only if there exists an edge $\{\alpha, \beta\} \in E(\Gamma)$ with $\alpha \in B$ and $\beta \in C$. In the following we shall always assume that $\Gamma_B$ has at least one edge, so each block of $B$ is an independent set of $\Gamma$ (see for example [1, Proposition 22.1] and [8]). This quotient graph $\Gamma_B$ conveys a lot of information about the graph $\Gamma$, and in particular it inherits the $G$-symmetry from $\Gamma$ (under the induced action of $G$ on $B$). For $B \in B$, denote by $\Gamma_B(B)$ the neighbourhood of $B$ in $\Gamma_B$. In introducing a geometric approach to imprimitive symmetric graphs, Gardiner and Praeger [4] suggested an analysis of this quotient graph $\Gamma_B$ together with (i) the 1-design with point set $B$ and "blocks" $\Gamma(C) \cap B$ (with possible repetitions), for all $C \in \Gamma_B(B)$; and (ii) the induced bipartite subgraph $\Gamma[B, C]$ of $\Gamma$ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$, where $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$ with $\Gamma(\alpha)$.
the neighbourhood of \( \alpha \) in \( \Gamma \). Since \( \Gamma \) is \( G \)-symmetric, \( \Gamma[B, C] \) is, up to isomorphism, independent of the choice of adjacent blocks \( B, C \) of \( B \).

The purpose of this paper is to classify a family of imprimitive symmetric graphs and the corresponding quotients and groups. This makes partial contribution to our project of the study of \( G \)-symmetric graphs \( \Gamma \) with \( k = v - 1 \geq 2 \), where \( v := |B| \) is the block size of \( B \) and \( k := |\Gamma(C) \cap B| \) is the size of each part of the bipartition of \( \Gamma[B, C] \). It seems that this case is rather rich in both theory and examples: In [7, Section 6] a natural construction of a subclass of such graphs was discovered, and this was further developed in [9, 10]. In [5] such graphs \( \Gamma \) with \( \Gamma_B \) a complete graph and \( G \) a 3-transitive subgroup of \( PGL(2, q) \) were determined and characterised, for any prime power \( q \). In [11] an intertwined relationship between \( G \)-symmetric graphs with \( k = v - 1 \geq 2 \) and certain kinds of \( G \)-point- and \( G \)-block-transitive 1-designs was revealed. For such a graph \( \Gamma \) and a vertex \( \alpha \) of \( \Gamma \), we denote by \( B(\alpha) \) the unique block of \( B \) containing \( \alpha \). Since \( k = v - 1 \geq 2 \), we may define

\[
(1) \quad B(\alpha) := \{ C \in B : \Gamma(C) \cap B(\alpha) = B(\alpha) \setminus \{\alpha\} \}
\]

and set

\[
(2) \quad m := |B(\alpha)|.
\]

Thus \( B(\alpha) \) is the set of blocks of \( B \) which are adjacent to \( B(\alpha) \) in \( \Gamma_B \) but contain no vertex adjacent to \( \alpha \) in \( \Gamma \). Since \( G \) is transitive on \( V(\Gamma) \), the integer \( m \) does not depend on the choice of \( \alpha \). It seems that the size of \( B(\alpha) \cap B(\beta) \), for adjacent vertices \( \alpha, \beta \) of \( \Gamma \), influences a lot the structure of \( \Gamma \). For example, we shall see in Lemma 2.1(c) that, if it is greater than \( m/2 \), then \( \Gamma \) is forced to be an almost cover [9] of \( \Gamma_B \), that is, \( \Gamma[B, C] \) is a matching of \( v - 1 \) edges. In this paper, we investigate the extreme case where \( B(\alpha) = B(\beta) \) for adjacent vertices \( \alpha, \beta \) of \( \Gamma \), and (without loss of generality) \( \Gamma_B \) is connected. In this case, we shall prove that the group \( G \) is rather restrictive and all of \( \Gamma \), \( \Gamma_B \) and \( \Gamma[B, C] \) can be determined explicitly, namely \( \Gamma \cong (v + 1) \cdot K_m^v \), \( \Gamma_B \cong K_m^{v+1} \). \( \Gamma \) is an almost cover of \( \Gamma_B \), and \( G \) is an extension of a group by any 3-transitive group of degree \( v + 1 \) (see Theorem 3.1 and Remark 3.2). Here we denote by \( K_m^n \) the complete \( n \)-partite graph with \( m \) vertices in each part of its \( n \)-partition, and by \( n \cdot \Sigma \) the union of \( n \) vertex-disjoint copies of a given graph \( \Sigma \).

2. **Preliminary**

For terminology and notation on graphs and permutation groups, the reader is referred to [1] and [3], respectively. Let \( \Gamma \) be a \( G \)-symmetric graph admitting a nontrivial \( G \)-invariant partition \( B \) such that \( k = v - 1 \geq 2 \). For two vertices \( \alpha, \beta \) of \( \Gamma \), if \( B(\alpha) \in B(\beta) \) and \( B(\beta) \in B(\alpha) \) hold simultaneously, then we say that \( \alpha, \beta \) are mates, and that \( \alpha \) is
the mate of $\beta$ in $B(\alpha)$ (so $\beta$ is the mate of $\alpha$ in $B(\beta)$ as well). Define $\Gamma'$ to be the graph with vertex set $V(\Gamma')$ in which $\alpha, \beta$ are adjacent if and only if they are mate. Then $\Gamma'$ is $G$-symmetric ([7, Proposition 3]). One can see that the set $\{B(\alpha) : \alpha \in B\}$ is a $G_B$-invariant partition of $\Gamma_B(B)$, and hence $G_B$ induces an action on it, where $G_B$ is the setwise stabiliser of $B$ in $G$. Clearly, for $(\alpha, \beta) \in \text{Arc}(\Gamma')$, the value of $|B(\alpha) \cap B(\beta)|$ is between 0 and $m$, and is independent of the choice of such $(\alpha, \beta)$ since $\Gamma$ is $G$-symmetric. Part (c) of the following lemma gives an upper bound for this integer in terms of $m$ and the valency of $\Gamma[B, C]$.

**Lemma 2.1.** Let $(\Gamma, G)$ be as above, and let $s$ be the valency of $\Gamma[B, C]$ (for adjacent blocks $B, C$ of $B$). Then the following (a)-(c) hold.

(a) The valency of $\Gamma$ is equal to $ms(v - 1)$, and the valency of $\Gamma_B$ is equal to $mv$ ([7, Theorem 5(a)]).

(b) $G_B$ is doubly transitive on $\{B(\alpha) : \alpha \in B\}$ ([7, Theorem 5(b)]).

(c) For $(\alpha, \beta) \in \text{Arc}(\Gamma)$, we have $|B(\alpha) \cap B(\beta)| \leq m/s$. In particular, if $|B(\alpha) \cap B(\beta)| > m/2$, then $\Gamma[B, C] \cong (v - 1) \cdot K_2$.

**Proof:** We need to prove (c) only. Let $n = |B(\alpha) \cap B(\beta)|$ for $(\alpha, \beta) \in \text{Arc}(\Gamma)$. Let $B = B(\alpha), C = \Gamma_B(B) \setminus B(\alpha)$, and set $\Gamma(\alpha) \cap C = \{\beta_1, \ldots, \beta_s\}$. Then $B(\alpha) \cap B(\beta_i)$, for $i = 1, \ldots, s$, are pairwise disjoint with each containing $n$ blocks of $B(\alpha)$. So we have $sn \leq m$, as required. In particular, if $n > m/2$, then we must have $s = 1$ and thus $\Gamma[B, C] \cong (v - 1) \cdot K_2$.

The following example shows that the case where $B(\alpha) = B(\beta)$ for adjacent vertices $\alpha, \beta$ of $\Gamma$ can occur. For a finite set $I$, we denote by $I^{(2)}$ the set of ordered pairs of distinct elements of $I$.

**Example 2.2.** Let $X$ be a finite group acting 3-transitively on a finite set $I$ of degree $v + 1 \geq 4$, and $Y$ a finite group acting on a finite set $J$ of degree $m \geq 1$. We require that $Y$ is 2-transitive on $J$ whenever $m \geq 2$. Then $G := X \times Y$ is transitive on $V := I^{(2)} \times J$ in its action defined by $(i, h, j)(x, y) := (i^x, h^x, j^y)$ for $(i, h, j) \in V$ and $(x, y) \in G$. Define $\Gamma$ to be the graph with vertex set $V$ in which $(i, h, j), (i', h', j')$ are adjacent if and only if $i \neq i'$ and $h = h'$. Then $\Gamma \cong (v + 1) \cdot K_m^v$, and the assumptions on $X, Y$ imply that $\Gamma$ is $G$-symmetric. Clearly, $\Gamma$ admits $B := \{[i, j] : i \in I, j \in J\}$ as a $G$-invariant partition, where $[i, j] := \{(i, h, j) : h \in I \setminus \{i\}\}$. We have $\Gamma_B \cong K_m^{v+1}$ with $[i, j], [i', j']$ adjacent if and only if $i \neq i'$. Also, we have $\Gamma[B, C] \cong (v - 1) \cdot K_2$ for adjacent blocks $B, C$ of $B$ (hence $k = v - 1 \geq 2$). Moreover, for adjacent vertices $\alpha = (i, h, j), \alpha' = (i', h, j')$ of $\Gamma$, we have $B(\alpha) = B(\alpha') = \{[h, \ell] : \ell \in J\}$, and hence $|B(\alpha)| = m$. 

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3. Main result and the proof

Unexpectedly, the graphs $\Gamma$ in Example 2.2 are the only $G$-symmetric graphs with $\Gamma_B$ connected such that $k = v - 1 \geq 2$ and $B(\alpha) = B(\beta)$ for adjacent vertices $\alpha, \beta$ of $\Gamma$, and $\Gamma_B, \Gamma[B, C]$ are as shown therein. More precisely, we have the following theorem, which is the main result of this paper.

**Theorem 3.1.** Suppose that $\Gamma$ is a $G$-symmetric graph admitting a nontrivial $G$-invariant partition $B$ such that $k = v - 1 \geq 2$. Suppose further that $\Gamma_B$ is connected and that $B(\alpha) = B(\beta)$ for adjacent vertices $\alpha, \beta$ of $\Gamma$. Let $m = |B(\alpha)|$. Then $\Gamma \cong (v + 1) \cdot K_m$, $\Gamma_B \cong K_m^{v+1}$, $\Gamma[B, C] \cong (v - 1) \cdot K_2$ for adjacent blocks $B, C$ of $\Gamma$, and the induced action of $G$ on the natural $(v + 1)$-partition $B$ of $\Gamma_B$ is 3-transitive. Moreover, the vertices of $\Gamma$ can be labelled by ordered triples of integers such that the following (a)-(c) hold (where we set $I := \{0, 1, \ldots, v\}$ and $J := \{1, 2, \ldots, m\}$):

(a) $V(\Gamma) = I^2 \times J$, and two vertices $(i, h, j), (i', h', j') \in V(\Gamma)$ are adjacent in $\Gamma$ if and only if $i \neq i'$ and $h = h'$.

(b) $B = \left\{ [i, j] : i \in I, j \in J \right\}$, where $[i, j] := \{(i, h, j) : h \in I \setminus \{i\}\}$, and $[i, j], [i', j']$ are adjacent blocks if and only if $i \neq i'$.

(c) $B = \{i : i \in I\}$, where $i = \{[i, j] : j \in J\}$.

Conversely, the graph $\Gamma$ defined in (a) together with the group $G = X \times Y$ satisfies all conditions of the theorem, where $X$ is a group acting 3-transitively on $I$, $Y$ is a group acting on $J$ which is 2-transitive if $m \geq 2$, and the action of $G$ on $V(\Gamma)$ is as defined in Example 2.2.

**Proof:** By our assumption we have $|B(\alpha) \cap B(\beta)| = m > m/2$ for $(\alpha, \beta) \in \text{Arc}(\Gamma)$. Thus Lemma 2.1(c) implies

(i) $\Gamma[D, E] \cong (v - 1) \cdot K_2$ for adjacent blocks $D, E$ of $B$.

Let $B$ be a block of $B$ and let $\alpha_1, \alpha_2, \ldots, \alpha_v$ be vertices of $B$. For each $\alpha_i \in B$, we label (in an arbitrary way) the $m$ blocks in $B(\alpha_i)$ by $[i, j], j \in J$. Also, we label the unique mate $\beta_{ij}$ of $\alpha_i$ in the block $[i, j]$ by $(i, 0, j), j \in J$. For each block $[i, j]$ and for each $h \in I \setminus \{i\}$ distinct from $i$, (i) implies that $[i, j]$ contains a unique vertex adjacent to $\alpha_h$. We label such a vertex in $[i, j]$ by $(i, h, j)$. In view of (i) one can see that each vertex in $[i, j]$ receives a unique label, and that the labels of distinct vertices in $[i, j]$ have distinct second coordinates. Therefore, for each $i \in I \setminus \{0\}$ and $j \in J$, we may identify the block $[i, j]$ with the set $\{(i, h, j) : h \in I \setminus \{i\}\}$.

In particular, this implies that

(ii) $B((i, h, j)) = B(\alpha_h) = \{[h, 1], [h, 2], \ldots, [h, m]\}$.

In particular, (ii) implies that

(iii) $[i, j], [i', j']$ are adjacent blocks, for distinct $i, i' \in I \setminus \{0\}$ and any $j, j' \in J$.

Moreover, if two vertices $(i, h, j), (i', h', j')$ are adjacent, then by (ii) and our assumption we must have $B(\alpha_h) = B((i, h, j)) = B((i', h', j')) = B(\alpha_{h'})$, which is true only when
\[ h = h' \]. This, together with (i) and (iii), implies the following assertion.

(iv) For distinct \( i, i' \in I \setminus \{0\} \) and any \( j, j' \in J \), two labelled vertices \((i, h, j), (i', h', j')\) of \( \Gamma \) are adjacent if and only if \( h = h' \). In other words, for adjacent blocks \( D = [i, j], E = [i', j'] \) of \( B \), the bipartite subgraph \( \Gamma[D, E] \) of \( \Gamma \) is the matching of \( v - 1 \) edges joining \((i, h, j)\) and \((i', h', j')\), for \( h \in I \setminus \{i, i'\} \).

Therefore, \((i, i', j)\) and \((i', i, j')\) are mates and hence, for the graph \( \Gamma' \) defined at the beginning of the previous section, we have

\[ \Gamma'(i, h, j) = \{(h, i, j') : j' \in J\} \]

Now let us examine a particular labelled vertex, say \((i, h, j)\). From Lemma 2.1(a) and (i) above, the valency of \( \Gamma \) is \( m(v - 1) \), and hence the neighbourhood \( \Gamma((i, h, j)) \) of \((i, h, j)\) contains \( m(v - 1) \) vertices. From (iv) we have \( \{(i', h, j') : i' \in I \setminus \{0, h, i\}, j' \in J\} \subseteq \Gamma((i, h, j)) \) and this contributes \( m(v - 2) \) neighbours of \((i, h, j)\). Note that \( \alpha_h \) is also a neighbour of \((i, h, j)\). Apart from these, there are \( m - 1 \) remaining neighbours of \((i, h, j)\), which we denote by \( \delta_2, \ldots, \delta_m \), respectively. By (i) these vertices \( \delta_2, \ldots, \delta_m \) belong to distinct blocks, say \( B_2, \ldots, B_m \) of \( B \). For each \( \delta_t \), we have \( B(\delta_t) = B((i, h, j)) = B(\alpha_h) = \{[h, 1], [h, 2], \ldots, [h, m]\} \) by (ii) and our assumption. In particular, this implies that all the blocks \([h, \ell]\), for \( \ell \in J \), are adjacent to the block \( B_t \). On the other hand, from (v) we have \( \Gamma'(h, h', \ell) = \{(h', t) : t \in J\} \) for each \( (h, h', \ell) \in [h, \ell] \setminus \{\beta_{ht}\} \). In other words, the \( m \) mates of each vertex in \([h, \ell] \setminus \{\beta_{ht}\} \) are in \( \bigcup_{h' \in I \setminus \{0, h\}, \ell \in J} [h', \ell] \). So the only possibility is that \( \beta_{ht} \) is the mate of \( \delta_t \) in \([h, \ell] \), for each \( \ell \in J \). Consequently, we have

\[ \beta_{h1} = \cdots = \beta_{hm} = \{B, B_2, \ldots, B_m\}, \]

and hence none of \( B, B_2, \ldots, B_m \) coincides with \([i, j]\) for any \( i \in I \setminus \{0\} \) and \( j \in J \).

We know from (iii) that the blocks \([i', j']\), for \( i' \in I \setminus \{0, h\} \) and \( j' \in J \), are all adjacent to \([h, \ell] \). Besides these \( m(v - 1) \) blocks, \( B, B_2, \ldots, B_m \) are the only blocks of \( B \) adjacent to \([h, \ell] \) in \( \Gamma_B \) since \( \Gamma_B \) has valency \( mv \) (Lemma 2.1(a)). Therefore, if we apply the procedure above to another vertex \((i', h, j')\), we would get the same blocks \( B_2, \ldots, B_m \). In other words, these blocks are independent of the choice of the vertex \((i, h, j)\) (depending only on \( h \)), and hence they are adjacent to the block \([i, j]\) for any \( i \in I \setminus \{0\} \) and \( j \in J \). Moreover, since the mate \( \delta_t \) of \( \beta_{ht} \) in \( B_t \) is unique, the vertices \( \delta_2, \ldots, \delta_m \) are also independent of the choice of \((i, h, j)\) and thus they are common neighbours of all such vertices \((i, h, j)\). Thus, since the valency of \( \Gamma_B \) is \( mv \), \( B, B_2, \ldots, B_m \) are the only unlabelled blocks of \( B \). From this and by a similar argument to that above, we see that for each \( h \in I \setminus \{0\} \), all the vertices \((i, h, j), i \in I \setminus \{0, h\}, j \in J \), have a common neighbour in each \( B_t \), which we now label by \((0, h, t)\). Since for distinct \( h, h' \) the vertices \((i, h, j), (i', h', j')\) have different neighbours in \( B_t \), the vertices of \( B_t \) receive pairwise distinct labels. Now let us label \( B, B_2, \ldots, B_m \) with \([0, 1], [0, 2], \ldots, [0, m]\), respectively, and label each \( \alpha_h \) with \((0, h, 1)\).
Then all the vertices of $\Gamma$ and all the blocks of $B$ have been labelled. From the labelling above, the validity of (a) and (b) follows immediately.

Since the valency of $\Gamma$ is $m(v - 1)$, the argument above also shows that for each $h \in I$ the connected component of $\Gamma$ containing the vertex $\alpha_h$ is the complete $v$-partite graph $K_m^v$ with $v$-partition $\{(i, h, j) : j \in J\} : i \in I$, where we set $\alpha_0 = \beta_{11}$. Hence we have $\Gamma \cong (v + 1) \cdot K_m^v$. Also, $\Gamma_B$ is the complete $(v + 1)$-partite graph $K_{m+1}^{v+1}$ with $(v + 1)$-partition $B := \{i : i \in I\}$, where $i := B(\alpha_i) = \{[i, j] : j \in J\}$ for $i \in I$. Clearly, $(\Gamma_B)_B \cong K_{v+1}^{v+1}$ and $B$ is a $G$-invariant partition of $B$. From Lemma 2.1(b), $G_B$ is doubly transitive on $\{B(\gamma) : \gamma \in B\}$. The setwise stabiliser in $G$ of the block 0 contains $G_B$ as a subgroup, and so is doubly transitive on the neighbourhood $B \setminus \{0\}$ of 0 in $(\Gamma_B)_B$. Therefore, $G$ is 3-transitive on $B$.

Finally, for $G = X \times Y$ with $X$ triply transitive on $I$ and $Y$ doubly transitive on $J$ whenever $m \geq 2$, Example 2.2 shows that the graph $\Gamma$ defined in (a) satisfies all the conditions in the theorem.

**Remark 3.2.** In Theorem 3.1, $G$ may or may not be faithful on $B$. (This can be seen from Example 2.2, where the action of $G$ on $B$ is permutationally isomorphic to the action of $X$ on $I$ which is not necessarily faithful.) Let $K$ be the kernel of the action of $G$ on $B$, and set $H := G/K$. Then $H$ is 3-transitive and faithful on $B$ of degree $v + 1$, and $G$ is an extension of $K$ by $H$. From the classification of finite highly transitive permutation groups (see for example [2, 6]), $H$ is one of the following: $S_{v+1} (v \geq 3)$, $A_{v+1} (v \geq 4)$, $M_{v+1} (v = 10, 11, 21, 22, 23)$, $M_{11} (v = 11)$, $AGL(d, 2) (v = 2^d - 1)$, $Z_2^d . A_7 (v = 15)$, and $PSL(2, v) \leq H \leq PGL(2, v) (v$ a prime power). Example 2.2 shows that $m = |B(\alpha)|$ defined in (2) can be any positive integer and $H$ can be any group listed above.

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