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TRAVELLING WAVE SOLUTIONS FOR DOUBLY DEGENERATE REACTION–DIFFUSION EQUATIONS

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Abstract

This paper concerns a nonlinear doubly degenerate reaction–diffusion equation which appears in a bacterial growth model and is also of considerable mathematical interest. A travelling wave analysis for the equation is carried out. In particular, the qualitative behaviour of both sharp and smooth travelling wave solutions is analysed. This travelling wave behaviour is also verified by some numerical computations for a special case.

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1. Introduction

Travelling waves have played a very important role in the studies of many spatially diffusive models. Many pattern-forming processes in chemical and biological systems are well described by reaction–diffusion equations. The equation that we consider in this paper arises in population dynamics. It is a simplified version of a nonlinear diffusion model of pattern formation during bacterial growth. In scaled form, it reads

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((1-u)u^m \frac{\partial u}{\partial x}\right) + u(1-u), \tag{1.1}$$

where $m > 0$. The unknown u is nonnegative and represents the bacterial density, x is a space coordinate and t denotes time. This model equation appeared in the study of spatio-temporal pattern formation by bacterial colonies exemplified by the growth of bacteria of type *Bacillus subtilis* on the surface of thin agar plates [4]. Specifically, equation (1.1) is derived by simplifying the following system of reaction–diffusion

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equations [4, 10] for a bacterium $u(t, x)$ and a nutrient $n(t, x)$:

$$\frac{\partial u}{\partial t} = D_u \frac{\partial}{\partial x}\left(nu^m \frac{\partial u}{\partial x}\right) + un, \tag{1.2}$$

$$\frac{\partial n}{\partial t} = D_n \frac{\partial^2 n}{\partial x^2} - un. \tag{1.3}$$

Equation (1.1) is obtained from (1.2) and (1.3) under the assumption $D_n = 0$, which models the case of a so-called hard agar (a nutrient-poor solid agar). With this assumption, (1.1) follows upon adding (1.2) and (1.3), using the conservation of total mass in the system, namely

$$\frac{d}{dt}(u + n) = 0, \tag{1.4}$$

replacing n with $1 - u$, and a suitable rescaling.

Biologically, such nonlinear diffusion is a result of cooperative bacterial cell motion. Mathematically, (1.1) may be classified as being of nonlinear degenerate parabolic type. As a result, Satnoianu *et al.* [10], by means of numerical simulations for the special case $m = 1$, verified the existence of a travelling wave front solution of sharp type corresponding to a minimum wave speed. Such an existence result for this case was also established by Malaguti and Marcelli [5] using comparison-type techniques, by applying the upper and lower solutions method. There are also a number of papers that study reaction–diffusion equations with degenerate diffusion, in particular, in the case $D(u) = u$, where degeneracy occurs at only one point, $u = 0$ [3, 12, 13]. Garduno and Maini [3] used regular perturbation theory to calculate the form of the sharp wave front when small changes are made to the diffusion coefficient, in the form $D(u) = u + \varepsilon u^2$. Using singular perturbation theory, Sherratt [12] derived an asymptotic approximation to the smooth wave fronts with speeds close to that of the sharp front solution, and provided a comparison with earlier numerical results [13].

For an extensive review of the literature on sharp and smooth travelling wave fronts in reaction–diffusion equations with degenerate diffusion, and applications to ecology and cell biology, where density-dependent diffusion is a common feature of spatial modelling, see [12]. In ecology, such wave fronts correspond to invasions. In cell biology and bacterial colonies, they correspond to the edges of expanding populations.

The main purpose of this paper is to perform a travelling wave analysis of the general model equation (1.1) involving doubly degenerate diffusion term, and to give a more complete picture of the travelling wave behaviour of its solutions.

2. Travelling wave analysis

Both experimental data and numerical simulations [4, 7, 8, 10] indicate that the reaction–diffusion process is characterized by wave fronts moving with constant speed. Thus, we look for travelling wave solutions of (1.1) corresponding to a front and satisfying the boundary conditions

$$u(-\infty, t) = 1, \quad u(\infty, t) = 0. \tag{2.1}$$

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To find such travelling wave solutions, we let

$$u(x, t) = u(z), \quad z = x - ct, \tag{2.2}$$

where $c > 0$ is the wave speed. Substituting (2.2) into (1.1) yields

$$-c\frac{du}{dz} = \frac{d}{dz}\left((1-u)u^m\frac{du}{dz}\right) + u(1-u), \tag{2.3}$$

with boundary conditions

$$u(-\infty) = 1, \quad u(\infty) = 0. \tag{2.4}$$

The standard method for analysing (2.3) with (2.4) is to write it as a phase plane system. Therefore, we define

$$v = (1-u)u^{m-1}\frac{du}{dz} \tag{2.5}$$

to obtain

$$\frac{du}{dz} = \frac{v}{(1-u)u^{m-1}}, \tag{2.6}$$

$$\frac{dv}{dz} = -\frac{(c+v)v + u^m(1-u)^2}{u^m(1-u)}, \tag{2.7}$$

which is a singular ordinary differential equation (ODE) system. In order to resolve this singularity of (2.6)–(2.7), one can use the following reparameterization. Let τ be such that $d\tau/dz = 1/(u^m(1-u)) > 0$ for all z. Then, in terms of τ, the system (2.6)–(2.7) becomes

$$\frac{du}{d\tau} = uv, \tag{2.8}$$

$$\frac{dv}{d\tau} = -(c+v)v - u^m(1-u)^2, \tag{2.9}$$

which is not singular. Moreover, given that $1/(u^m(1-u)) > 0$ for $0 < u < 1$, the dynamics given by systems (2.6)–(2.7) and (2.8)–(2.9) are the same in the half plane $\{(u, v) \mid 0 < u < 1, -\infty < v < +\infty\}$. Thus, we analyse the dynamics of (2.8)–(2.9).

The ODE system (2.8)–(2.9) has the following three stationary points:

$$p_1 = (1, 0), \quad p_0 = (0, 0), \quad p_c = (0, -c).$$

Hence, searching for travelling wave solutions of (1.1) is equivalent to looking for heteroclinic trajectories of (2.8)–(2.9), which connect the above stationary points. This is also equivalent to looking for solutions in graph form $v = v(u) < 0$, $0 < u < 1$, satisfying the ODE

$$\frac{dv}{du} = -\frac{(c+v)v + u^m(1-u)^2}{uv}. \tag{2.10}$$

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Thus, to analyse the local behaviour of solution trajectories near the stationary points, we use (2.8)–(2.9) as well as (2.10). A linear analysis of (2.8)–(2.9) about p_1 shows that p_1 is a non-hyperbolic point, and the eigenvalues of the Jacobian matrix associated with (2.8)–(2.9) at p_1 are $\lambda_1 = 0$ and $\lambda_2 = -c$, with corresponding eigenvectors $\underline{e_1} = (1, 0)$ and $\underline{e_2} = (1, -c)$. Clearly $\underline{e_2}$ points towards p_1, so that any travelling wave must originate from p_1 along the $\underline{e_1}$ direction. Because of the nature of this point, we can apply the centre manifold theorem [1] and use a Taylor expansion of (2.10) around p_1 to obtain an approximation of the trajectory locally around p_1. Hence, we find that

$$v_1(u) = -\frac{(1-u)^2}{c} + \cdots . \tag{2.11}$$

Then, in this case, from (2.8) we obtain

$$\frac{du}{d\tau} \approx -\frac{(1-u)^2}{c},$$

so that $u \to 1$ as $\tau \to -\infty$. Then (2.6) gives, for $(u, v) \approx (1, 0)$,

$$\frac{du}{dz} \approx -\frac{1-u}{c},$$

so that $u \to 1$ as $z \to -\infty$.

Linearizing (2.8)–(2.9) at p_c shows that p_c is a saddle point. The eigenvalues of the Jacobian matrix associated with (2.8)–(2.9) at p_c are $\lambda_3 = -c$ and $\lambda_4 = c$, with corresponding eigenvectors $\underline{e_3} = (1, 0)$ and $\underline{e_4} = (0, 1)$. In this case, a travelling wave trajectory (type I solution) approaches p_c along the eigenvector $\underline{e_3}$, being the tangent to the right-stable manifold, giving, more precisely,

$$v_c(u) = -c + \frac{u^m}{(m+1)c} + \cdots . \tag{2.12}$$

Hence, in this case we obtain

$$\frac{du}{d\tau} \approx -cu + \frac{u^{m+1}}{(m+1)c},$$

so that $u \to 0$ as $\tau \to \infty$. Then (2.6) gives, for $(u, v) \approx (0, -c)$,

$$\frac{du}{dz} \approx -cu^{1-m},$$

so that u reaches zero at a finite value of z, i.e. $z_{\min} < \infty$. More precisely, we have the asymptotic behaviour

$$u(z) \approx (mc(z_{\min} - z))^{1/m}.$$

A local analysis of (2.8)–(2.9) about p_0 shows that p_0 is a non-hyperbolic point. The eigenvalues of the Jacobian matrix associated with (2.8)–(2.9) at p_0 are $\lambda_5 = 0$ and $\lambda_6 = -c$, with corresponding eigenvectors $\underline{e_5} = (1, 0)$ and $\underline{e_6} = (0, 1)$. The centre manifold theorem and Taylor expansion of (2.10) around p_0 give the approximation of the trajectory locally around p_0, in the form

$$v_0(u) = -\frac{u^m}{c} + \cdots . \tag{2.13}$$

In this case, any travelling wave trajectory (type II solution) must enter p_0 along $\underline{e}_5$, being the tangent to the centre manifold. Hence, (2.13) gives

$$\frac{du}{d\tau} \approx -\frac{u^{m+1}}{c} \quad \text{and} \quad \frac{du}{dz} \approx -\frac{u}{c},$$

so that $u \to 0$ as $\tau \to \infty$, and as $z \to \infty$.

For completeness, we use the monotonicity properties of the trajectories with respect to the speed parameter c to establish the uniqueness of the trajectory connecting p_1 and p_c (type I solution). From (2.11), we find that for any $c > 0$ and $u < 1$,

$$\frac{\partial}{\partial c} v_1(u, c) \simeq \frac{(1-u)^2}{c^2} > 0.$$

Hence, $v_1(u, c)$ is strongly monotonically increasing in c for every $0 < u < 1$; see Figure 1. Similarly, from (2.12) we have, for small u,

$$\frac{\partial}{\partial c} v_c(u, c) \simeq -1 - \frac{u^m}{(m+1)c^2} < 0.$$

Hence, $v_c(u, c)$ is strongly monotonically decreasing in c for every $0 < u < 1$. Therefore, we deduce that if there is a solution trajectory connecting p_1 to p_c, then this is a finite sharp travelling wave solution with a minimum speed $c_{\min}$. All other solutions are the trajectories (type II solutions) connecting p_1 to p_0, and corresponding to smooth travelling wave solutions with speeds larger than the minimum speed. In addition, by using arguments like that in [2], one can establish the existence of such types of solution trajectories.

To conclude, we present an argument for the following result. This travelling wave behaviour will also be shown numerically for special cases.

Equation (1.1) admits:

(i) a travelling wave solution $u(x, t) = u(x - ct) = u(z)$ for a minimum speed $c_{\min} > 0$, satisfying $u(-\infty) = 1$, $u(z) = 0$ for all $z \geq z_{\min}$, where $z_{\min} < \infty$, $du/dz(-\infty) = 0$, $du/dz(z_{\min}) = -c_{\min}$, and $du/dz(z) = 0$ for all $z \geq z_{\min}$;
(ii) a travelling wave solution $u(x, t) = u(x - ct) = u(z)$ for speeds larger than the minimum speed, satisfying $u(-\infty) = 1$ and $u(\infty) = 0$.

In order to determine speeds c and find trajectories corresponding to travelling wave solutions, we solve numerically the phase trajectory equation (2.10) using an adaptive step Runge–Kutta scheme of fourth order [9] for initial conditions that have been estimated from (2.11). As a result, we compute the unique trajectory from p_1 to p_c for the minimum speed $c_{\min}$, and trajectories from p_1 to p_0 as the speed increases

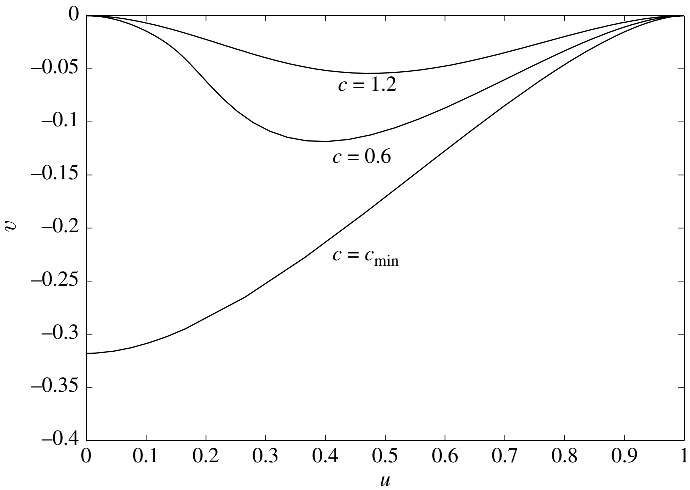

FIGURE 1. The solution trajectories for the case $m = 2$ and different wave speeds.

beyond $c_{\min}$. The minimum speed can be computed as follows. Taking $\delta_1 = 0.0001$ and $u_{\text{small}} = 0.005$, we start with $u_{\text{start}} = 1 - \delta_1$, terminate at $u_{\text{end}} < u_{\text{small}}$, and equate $v(u_{\text{small}}, c) = v_c(u_{\text{small}}, c)$ by using the following iteration scheme [6]:

$$c = c - \delta(v(u_{\text{small}}, c) - u_c(u_{\text{small}}, c)),$$

where δ is the relaxation factor and $u_c(u, c)$ is given by (2.12), for $c = c_0$. The stopping criterion is as follows: choose a suitable δ, say $\delta = 0.3$, and iterate until $|c - c_{\min}| < \epsilon$, where ϵ is some tolerance value.

For the special case $m = 1$, the minimum speed was found to be $c_{\min} \approx 0.561\,465$, while for the special case $m = 2$ for which the plots are shown, $c_{\min} \approx 0.318\,015$. It is clear that the minimum speed of the sharp type wave front, $c_{\min}$, decreases as m increases. In Figure 1, we plot solution trajectories for different speeds, $c = c_{\min}$ and $c > c_{\min}$. Figure 2 shows the solution trajectory connecting p_1 to p_c with local forms given by equations (2.11) and (2.12). Figure 3 shows the solution trajectory connecting p_1 to p_0 with local forms given by equations (2.11) and (2.13).

To obtain approximations of the wave profiles corresponding to the solution trajectories shown in Figures 1–3, we solve the ODE system (2.6)–(2.7) for increasing z using the fourth-order Runge–Kutta method with step size control. The initial values of u, v at some value of z, say $z = 0$, have been estimated from (2.11). Figure 4 plots the numerical solution of (2.6)–(2.7) for the minimum speed, showing a wave of sharp type. Figure 5 shows the solution of (2.6)–(2.7) for a speed larger than the minimum speed, showing a smooth wave. The sharp and smooth travelling waves established by this analysis are known solutions of (1.1). The transition from the sharp type wave to the smoother waves can be shown by solving the partial differential

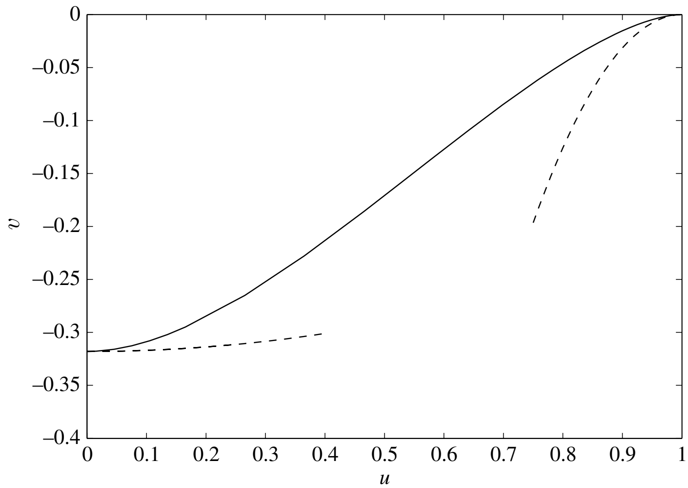

FIGURE 2. The solution trajectory for the minimum speed, as in Figure 1, with the local forms (dashed lines) around p_1 and p_c given by equations (2.11) and (2.12), respectively.

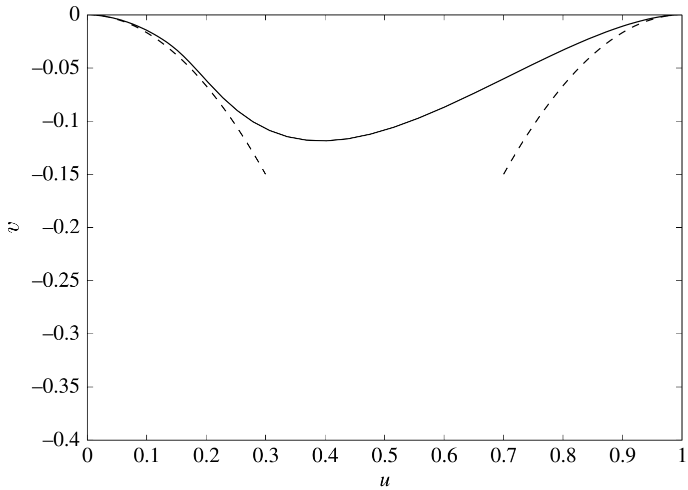

FIGURE 3. The solution trajectory for the large speed $c = 0.6$, as in Figure 1, with the local forms (dashed lines) around p_1 and p_0 given by equations (2.11) and (2.13), respectively.

equation problem for asymptotic initial conditions. If the initial conditions have decay of the form $u(x, 0) = \exp(-ax)$, then the speed of the waves follows the dispersion relation

$$c = \begin{cases} 1/a, & a < 1/c_{\min}, \\ c_{\min}, & a > 1/c_{\min}. \end{cases} \tag{2.14}$$

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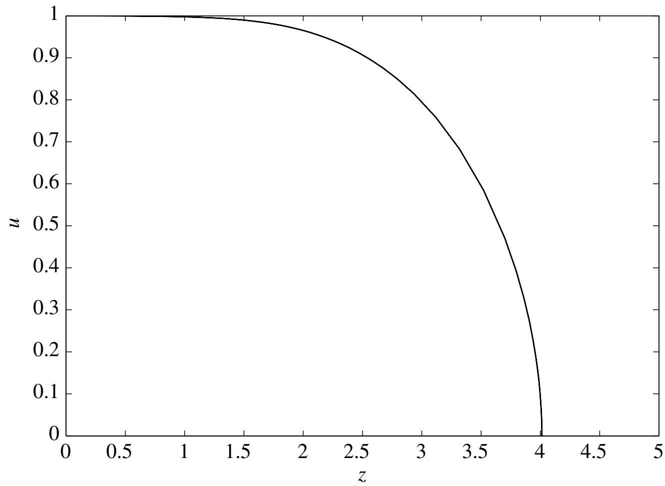

FIGURE 4. The sharp travelling wave solution corresponding to the solution trajectory in Figure 2 in the case $m = 2$, with the minimum speed $c_{\min} \approx 0.318\,015$.

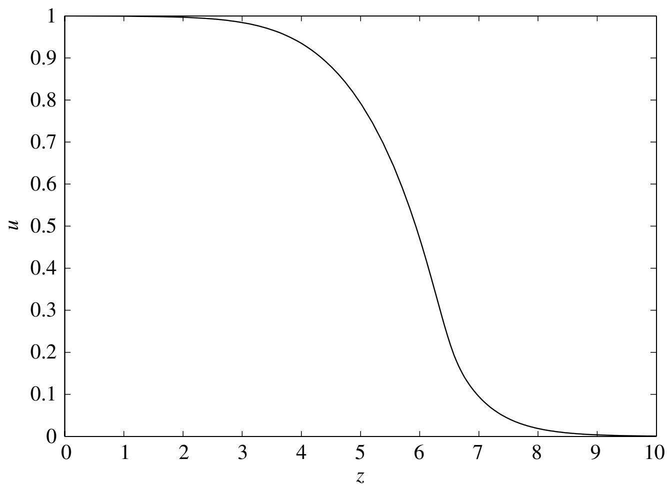

FIGURE 5. The smooth travelling wave solution corresponding to the solution trajectory in Figure 3 in the case $m = 2$, with speed $c = 0.6 > c_{\min}$.

This dispersion relation can be derived by applying leading edge analysis to the evolving wave, similar to that performed in [10]. This form of wave speed dependence on initial conditions is familiar from parabolic partial differential equations [11].

3. Conclusion

We have considered a doubly degenerate reaction–diffusion equation which appears in a bacterial growth model that incorporates cell movement in the nonlinear diffusion

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term and cell proliferation in the reaction term. This model equation describes the spatio-temporal pattern formation by a bacterium and a nutrient when assuming conservation of total mass. We have performed a travelling wave analysis for the equation. In particular, we have analysed the qualitative behaviour of both sharp and smooth travelling wave solutions. To verify this travelling wave behaviour, we have presented some numerical computations for a special case by solving the travelling wave equation as an initial-value problem and using a suitable iteration scheme, giving approximations of the minimum speed for which the wave is of sharp type, and smooth wave profiles for speeds larger than this minimum speed. These results concerning the travelling wave behaviour provide more understanding of the structure of the model equation. In addition, our computations lead to accurate values for the minimum speed of the sharp type wave front, in comparison with previous computations [4, 10].

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