CLASS–NUMBER PROBLEMS FOR CUBIC NUMBER FIELDS

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1. Introduction

Let $M$ be any number field. We let $D_M$, $d_M$, $h_M$, $\zeta_M$, $A_M$ and $\text{Reg}_M$ be the discriminant, the absolute value of the discriminant, the class-number, the Dedekind zeta–function, the ring of algebraic integers and the regulator of $M$, respectively. We set $c = \frac{3 + 2\sqrt{2}}{2}$. If $q$ is any odd prime we let $(\cdot/q)$ denote the Legendre's symbol. We let $D_P$ and $d_P$ be the discriminant and the absolute value of the discriminant of a polynomial $P$.

**Lemma A** (See [Sta, Lemma 3] and [Hof, Lemma 2]). Let $M$ be any number field. Then, $\zeta_M$ has at most one real zero in

$$L \left[ 1 - \frac{1}{c \log d_M}, 1 \right];$$

if such a zero exists, it is simple and is called a Siegel zero.

**Lemma B** (See [Lou 2]). Let $M$ be a number field of degree $n = r_1 + 2r_2$ where $M$ has $r_1$ real conjugate fields and $2r_2$ complex conjugate fields. Let $s_0 \in \left[ (1/2), 1 \right]$ be such that $\zeta_M(s_0) \leq 0$. Then,

$$\text{Res}_{s=1}(\zeta_M) \geq (1 - s_0) d_M^{(s_0 - 1)/2} \left( 1 - \frac{2r_1}{d_M^{s_0/2n}} - \frac{2\pi r_2}{d_M^{s_0/n}} \right).$$

2. Lower bounds for class–numbers of cubic number fields

Let $K$ be a cubic number field. If $K/Q$ is normal then $K$ is a cyclic cubic number field. Let $f_K$ be its conductor. Then $d_K = f_K^2$ and $\zeta_K(s) = \zeta(s)L(s, \chi)$ $L(s, \bar{\chi})$ where $\chi$ is a primitive cubic Dirichlet character modulo $f_K$. Hence, we get
\(\zeta_K(s) \leq 0, s \in [0,1]\). With \(s_0 = 1 - \left(2 / \log d_K\right)\) Lemma B provides the following lower bound which improves the one given in [Let]

\[\text{Res}_{s=1}(\zeta_K) \geq \frac{1}{3 \log(f_K)}, f_K \geq 10^6.\]

Lettl used his lower bound to determine all the simplest cubic number fields with small class-numbers. Here, the simplest cubic number fields are real cubic cyclic numbers fields defined as being the splitting fields of the polynomials \(P(X) = X^3 - aX^2 - (a + 3)X - 1 (a \geq 0)\) whose discriminants \(d_p = (a^2 + 3a + 9)^2\) are square of a prime \(p = a^2 + 3a + 9\), which implies \(A_K = \mathbb{Z}[\varepsilon]\) where \(\varepsilon > 1\) is the only real root of \(P(X)\) greater than one (we have \(\varepsilon \in \mathbb{N}, a + 1, a + 2\)). He found that there are 7 simplest cubic fields with class-number one, and none with class-number two or three.

In the same spirit, Lazarus determined all the simplest quartic fields with class-number 1 or 2. Here, the simplest quartic number fields are real quartic cyclic number fields defined as being the splitting fields of the polynomials \(P(X) = X^4 - 2aX^3 - 6X^2 + 2aX + 1 (a \geq 0)\) whose discriminants \(d_p = 256d_a^3\) are such that \(d_a = a^2 + 4\) is odd-square-free. He found that there are 6 simplest quartic fields with class-number one, and 3 with class-number two.

From now on, we assume that \(K\) is not normal. Thus, the normal closure \(N\) of \(K\) is a sextic number field with Galois group the symmetric group \(S_3\).

**Lemma C.** Let \(K/Q\) be a non-normal cubic extension with normal closure \(N = KL\) where \(L = Q(\sqrt{d_K})\) is quadratic. It holds \(\zeta_N^2 = \zeta_K^2\zeta_L\). Hence, \(d_N = d_Kd_L\), and \(d_N\) divides \(d_K^3\). Finally, \(\zeta_K\) does not have any real zero in the range \([1 - (1/(3\epsilon \log d_K^3)), 1]\).

**Proof.** The first point is proved page 227 of [Cas-Fro] using Artin \(L\)-series formalism. The second point follows from the first one using the functional equations satisfied by these zeta-functions (see [Lou 1]). According to the first point, any real zero in \([0,1]\) of \(\zeta_K\) is a multiple zero of \(\zeta_N\). Hence, the fourth point follows from Lemma A. It remains to prove the third point. According to Stickelberger’s theorem, we have \(D_K \equiv 0, 1 \pmod{4}\). Hence, \(D_L\) divides \(D_K\), and we may define \(f \geq 1\) by means of \(D_K = f^2 D_L\).

According to Lemmata A, B and C, we get:
THEOREM 1. Let $\mathbf{K}$ be a non-normal cubic number field. It holds

$$h_\mathbf{K}\text{Reg}_\mathbf{K} \geq \frac{1}{55}\frac{\sqrt{d_\mathbf{K}}}{\log d_\mathbf{K}}, \quad d_\mathbf{K} \geq 4 \cdot 10^5.$$

3. Computation of the class-number of a non-abelian cubic number field $\mathbf{K}$ with negative discriminant

We make use of the results of [Bar-Lox-Wil] and [Bar-Wil-Ban]. Set $\Phi(s) = (\zeta_\mathbf{K} / \zeta)(s) = \sum_{j \geq 1} \alpha(j)e^{-\zeta}$ Then,

$$h_\mathbf{K}\text{Reg}_\mathbf{K} = \frac{\Phi(1)}{C} = \sum_{j \geq 1} \frac{\alpha(j)}{jC} e^{-jC} + \sum_{j \geq 1} \alpha(j)E(jC)$$

where

$$E(y) = \int_y^\infty \frac{e^{-x}}{x} dx = -\log x - \sum_{j \geq 1} \frac{(-1)^j}{j(j!)} x^j,$$

where $\gamma = 0.577215664901...$ is the Euler’s constant, and where $C = 2\pi / \sqrt{d_\mathbf{K}}$.

Now, $j \mapsto \alpha(j)$ is a multiplicative function such that $\alpha(p^n) = F(p^n) - F(p^{n-1})$ where $F(k)$ is the number of distinct ideals of $\mathbf{K}$ with norm $k \geq 1$. Moreover, if $p$ does not divide $d_\mathbf{K}$, then

$$(p) = \begin{cases} (p) & \text{which implies } (D_\mathbf{K}/p) = +1 \text{ (Type I)}, \\ \mathcal{P}_1\mathcal{P}_2 & \text{if and only if } (D_\mathbf{K}/p) = -1 \text{ (Type II)}, \\ \mathcal{P}_1^2\mathcal{P}_2 & \text{which implies } (D_\mathbf{K}/p) = +1 \text{ (Type III)}. \end{cases}$$

If $p$ divides $d_\mathbf{K}$, then with $f$ as in the proof of Lemma C we have

$$(p) = \begin{cases} \mathcal{P}_1^2\mathcal{P}_2 & \text{if } p \text{ does not divide } f \text{ (Type IV)}, \\ \mathcal{P}_1^3 & \text{if } p \text{ divides } f \text{ (Type V)}. \end{cases}$$

If the ring of algebraic integers of $\mathbf{K}$ is generated by an algebraic integer $x_\mathbf{K}$ and if $P_\mathbf{K}(X)$ is the minimum polynomial over $\mathbb{Q}$ of $x_\mathbf{K}$, then we are in the Type I or Type III cases according as $P_\mathbf{K}(X)$ does not have or has at least one root modulo $p$, and we are in the Type IV or Type V cases according as $P_\mathbf{K}(X)$ has a double or a triple root modulo $p$.

4. Explicit class-number problem for non-normal cubic number fields

**First example.** In the same way we got Theorem 1, we get
THEOREM 2. (a) (See [Fro-Tay, Chapter 5]) Let \( l \geq 1 \) be an integer. Set 
\[ P_l(X) = X^3 + lX - 1. \] Then \( P_l(X) \) is irreducible in \( \mathbb{Q}[X] \), has negative discriminant \( D_l = -d_l = -4(l^3 + 27) \), and has exactly one real root \( x_l \). Set \( \varepsilon_l = 1/x_l \). Then, \( l < \varepsilon_l < l + 1 \). Set \( K_l = \mathbb{Q}(x_l) = \mathbb{Q}(\varepsilon_l) \). Then \( K_l \) is a real cubic number field with negative discriminant and \( \varepsilon_l \) is the fundamental unit greater than one of the cubic order \( \mathbb{Z}[x_l] = \mathbb{Z}[\varepsilon_l] \).

(b) Assume that the ring of algebraic integers \( \mathbf{A}_l \) of \( K_l \) is equal to \( \mathbb{Z}[\varepsilon_l] \). Then,
\[ h_1 \geq \frac{\sqrt{d_l}}{20 \log^2(d_l)} \text{ when } d_l \geq 2 \cdot 10^5 \text{ and } h_1 > 3 \text{ when } l > 400 \]

(here \( h_1 \) is the class-number of \( K_l \)). Hence, there are 5 such \( K_l \) with class-number one, namely \( K_1, K_2, K_3, K_6 \) and \( K_{11} \); there are 2 such \( K_l \) with class-number two, namely \( K_4 \) and \( K_7 \); there are 3 such \( K_l \) with class-number three, namely \( K_8, K_{15} \) and \( K_{17} \).

LEMMA D. (a) The ring of algebraic integers of \( K_l \) is equal to \( \mathbb{Z}[\varepsilon_l] \) if and only if \( 3^3 \) does not divide \( l \), and \( p^2 \) does not divide \( d_l = 4l^3 + 27 \) for any prime \( p \geq 5 \).

(b) Under this assumption, we have:
(i) For any prime \( p \neq 3 \) that divides \( d_l \) we have \( F(p^n) = n + 1 \) and \( \alpha(p^n) = 1, n \geq 1 \).
(ii) If \( p = 3 \) divides \( d_l \), then \( F(3^n) = 1 \) and \( \alpha(3^n) = 0, n \geq 1 \).
(iii) If \( p \) does not divide \( d_l \) and \((-d_l/p) = -1, \) then \( \alpha(p^n) = (1 + (-1)^n)/2. \)
(iv) If \( p \) does not divide \( d_l \) and \((-d_l/p) = +1, \) then \( \alpha(p^n) = \frac{1}{2} \) if \( P_l(X) \) has at least one root modulo \( p \).

\[ \alpha(p^n) = \begin{cases} 
 1 & \text{if } n \equiv 0 \pmod{3}, \\
 -1 & \text{if } n \equiv 1 \pmod{3}, \\
 0 & \text{if } n \equiv 2 \pmod{3}, 
\end{cases} \]

\( \alpha(p^n) = \frac{n + 1}{(n + 1/3)} \)

Proof. Point (b) follows from Section 3 (see [Bar-Lox-Wil, Table 2]).

Now, we proceed with the proof of point (a). We note that the integral bases of all cubic fields are determined in [Alb]. Though, we give a simple proof for this special case. Let \( t \) be an integer and set
\[ y_{p,l}(t) = \frac{x_l^2 + tx_l + t^2 + l}{p}. \]

Then \( y_{p,l}(t) \) is a root of the following polynomial
\[ Y^3 - \frac{P'_l(t)}{p} Y^2 + \frac{3tP_l(t)}{p^2} Y - \frac{P_l(t)^3}{p^3} \]
(being the characteristic polynomial of the linear map $z \mapsto y_{p,l}(t)z$ of the three $\mathbb{Q}$-dimensional vector space $K_i$, then $y_{p,l}(t)$ is indeed a root of this polynomial). Hence, $y_{p,l}(t)$ is an algebraic integer provided that $P_i(t) \equiv 0 \pmod{p^3}$ and $P_i(t) \equiv 0 \pmod{p}$.

Now, assume that $\mathbb{Z}[x_i]$ is the ring of algebraic integers of $K_i$ and let $p$ be any prime which divides $d_i = 4t^3 + 27$. Hence $p$ is odd.

First, assume that $p \neq 3$. Then $p$ does not divide $l$ and we take $t = t_{l,p}$ such that $2lt_{l,p} \equiv 3 \pmod{p}$ and where we hence write $2lt_{l,p} = 3 + ap$. Then,

$$4t^2P_i(t_{p,i}) = 3(3 + ap)^2 + 4t^3 \equiv d_i \equiv 0 \pmod{p},$$

$$8t^3P_i(t_{p,i}) = (3 + ap)^2 + 4t^3(3 + ap) - 8t^3 \equiv d_i + apd_i \equiv d_i \pmod{p^2}.$$  

Hence, if $p^2$ would divide $d_i$ then $y_{p,i}(t_{p,i})$ would be an algebraic integer of $K_i$ and $p$ would divide the index of $\mathbb{Z}[x_i]$ in $A_i$. Contradiction.

Second, assume that $p = 3$. If $3^2$ would divide $l$, then $y_{3,i}(1)$ would be an algebraic integer. Hence, $3$ would divide the index of $\mathbb{Z}[x_i]$ in $A_i$. Contradiction.

Conversely, first assume that if $p \neq 3$ divides $d_i$ then $p^2$ does not divides $d_i$. Since $d_i = (A_i : \mathbb{Z}[x_i])^2d_{K_i}$, then $p$ does not divides this index $(A_i : \mathbb{Z}[x_i])$ which is thus a 3-power. Note that if 3 divides this index, then 3 divides $d_i$, i.e., 3 divides $l$.

Second, it is known that if $x$ is an algebraic integer whose minimum polynomial is $p$-Eisenstein for a prime $p$, then $p$ does not divides the index of $\mathbb{Z}[x]$ in the ring of algebraic integers of the number field $\mathbb{Q}(x)$ (see [Nar, Lemma 2.2, page 60]). Hence, if we can find an integer $a$ such that $P_i(a) \equiv P_i'(a) \equiv 0 \pmod{3}$ and $P_i(a) \not\equiv 0 \pmod{3^3}$, then the minimum polynomial $Q_i(Y) = P_i(Y + a) = Y^3 + 3aY^2 + P_i(a)Y + P_i(a)$ of $y_i(a) = x_i - a$ is then 3-Eisenstein. Hence, 3 does not divide the index $(A_i : \mathbb{Z}[x_i])$, which implies the desired result. Now, assume that 3 divides $l$ but that $3^2$ does not divide $l$. Then, $Q_i(Y)$ being 3-Eisenstein we get the desired result.

The computational method of the third section and Lemma D provide the following table of class-numbers $h_i$ of these number fields $K_i$ for the first values of $l \geq 1$ such that $\mathbb{Z}[\varepsilon_i]$ is the ring of algebraic integers of $K_i$. According to the more extensive class-number computation of $h_i$ for $l \leq 400$, we easily get Theorem 2(b).
Second example. Let $K = \mathbb{Q}(\sqrt[3]{d})$, $d \geq 2$, be a real pure cubic number field. Assume that $d$ is cube-free and define $a$ and $b$ by means of $(a, b) = 1$ and $d = ab^2$. Then, $D_K = -3(ab)^2$ or $D_K = -27(ab)^2$ according as $d \equiv \pm 1 \pmod{9}$ or not. Now, $L = \mathbb{Q}(\sqrt[3]{-3})$ and $\zeta_L$ does not have any real zero in $[0, 1[$. Hence, we get $\zeta_N(s) \leq 0$, $s \in [0, 1]$. According to Lemma C, we have $\text{Res}_{s=1}(\zeta_N) = (\text{Res}_{s=1}(\zeta_K))^2 \text{Res}_{s=1}(\zeta_L)$ and we may apply Lemma B with $s_0 = 1 - (2/\log d_N)$, which provides an optimal lower bound on $\text{Res}_{s=1}(\zeta_N)$. Since $\text{Res}_{s=1}(\zeta_L) = \pi/3\sqrt{3}$, we get the following lower bound that improves the one given in Theorem 1:

**Theorem 3** (See [Bar-Lou]). Let $K$ be a real pure cubic number field. Then,

$$h_K \text{Reg}_K \geq \frac{1}{9} \sqrt{\frac{d_K}{\log d_K}}, \quad d_K \geq 3 \cdot 10^4.$$ 

**Theorem 4.** When $m \geq 1$ is such that $m^3 \pm 1$ is cube-free, we set $K_{\pm m} = \mathbb{Q}(\sqrt[3]{m^3 \pm 1})$ which is a pure cubic real number field. There are 2 such $K_{\pm m}$ with class-number one, namely $K_{+1}$ and $K_{-2}$; there does not exist any such $K_{\pm m}$ with class-number two; and there are 3 such $K_{\pm m}$ with class-number three, namely $K_{-2}$, $K_{-3}$ and $K_{+3}$.

Proof. Set $d_{\pm m} = m^3 \pm 1$ and $\omega_{\pm m} = \sqrt[3]{d_{\pm m}}$. Then, $\varepsilon_{\pm m} = \pm \sqrt[3]{\omega_{\pm m} - m} = \omega_{\pm m} + m\omega_{\pm m} + m^2$ is a unit of $A_{K_{\pm m}}$, and we have $1 < \varepsilon_{\pm m} \leq 4\omega_{\pm m}^2 = (8d_{\pm m})^{2/3}$. Hence,
and according to Theorem 3 we get
\[ \text{Reg}_{K_{\pm m}} \leq \log \varepsilon_{\pm m} \leq \frac{2}{3} \log(8d_{\pm m}) \leq \frac{2}{3} \log(3d_{K_{\pm m}}), \]

and according to Theorem 3 we get
\[ h_{K_{\pm m}} \geq \frac{1}{6} \sqrt{\frac{d_{K_{\pm m}}}{\log^3(3d_{K_{\pm m}})}}, \quad d_{K_{\pm m}} \geq 3 \cdot 10^4. \]

which implies \( h_{K_{\pm m}} > 3 \) for \( d_{K_{\pm m}} > 1.1 \cdot 10^6 \). Note that \( m > 72 \) implies \( d_{K_{\pm m}} \geq 3d_m > 1.1 \cdot 10^6 \). Now, the computation of \( a \) and \( b \) for \( d_{\pm m} = m^3 \pm 1 \) and \( 1 \leq m \leq 72 \) yields \( d_{K_{\pm m}} \leq 1.1 \cdot 10^6 \) if and only if \( 1 \leq m \leq 6 \) when \( d_{\pm m} = m^3 + 1 \), and \( 2 \leq m \leq 7 \) when \( d_{\pm m} = m^3 - 1 \) (note that \( d_{\pm m} \) must be cube-free). The class-numbers of these number fields may be found in [Hos-Wad], and they provide the desired result.

\[ \square \]

5. Another class-number problem

Let \( K \) be a real quadratic number field of discriminant \( D > 0 \). The ring of algebraic integers \( A_K = \mathbb{Z} \left[ \frac{D + \sqrt{D}}{2} \right] \) of \( K \) writes \( A_K = \mathbb{Z}[\varepsilon] \) where \( \varepsilon = \frac{u + v\sqrt{D}}{2} > 1 \) is a unit of \( A_K \) if and only if \( v = 1 \), hence if and only if \( D = m^2 \pm 4 \), \( m \geq 1 \). In that case, \( \varepsilon_D \leq \varepsilon = \frac{m + \sqrt{D}}{2} \leq \sqrt{D + 4} \) where \( \varepsilon_D \) is the fundamental unit of \( K \). According to the Brauer-Siegel theorem, there are only finitely many real quadratic number fields with discriminants \( D = m^2 \pm 4 \) of given class-number. Up to now, no one knows how to make the Brauer-Siegel effective in the real quadratic case, without assuming a suitable generalized Riemann hypothesis (see [Mol-Wil]). In contrast to the real quadratic case, let \( K \) be a cubic number field with negative discriminant whose ring of algebraic integers \( A_K \) is generated by a unit \( \varepsilon \), i.e. such that \( A_K = \mathbb{Z}[\varepsilon] \). This clearly amounts to saying that \( A_K = \mathbb{Z}[\varepsilon_K] \), where \( \varepsilon_K > 1 \) is the fundamental unit of \( K \). According to Theorem 1 and the following Proposition 5, we get the effective Corollary 6 that would enable one to explicitly determine all the cubic number fields of negative discriminants whose have small class-numbers and whose rings of algebraic integers are generated by units.

**Proposition 5.** A polynomial \( P(X) \) is said of type \((T)\) if it is a monic irreducible cubic polynomial with integral coefficients (say of the form \( P(X) = X^3 - aX^2 + bX - 1 \)) with exactly one real root \( \varepsilon_p \) (i.e. \( D_p < 0 \)) which satisfies \( \varepsilon_p > 1 \) (i.e. we have
Then, there exists an effective constant \( c_1 > 0 \) such that for any polynomial of type \((T)\) we have \( d_p = |D_p| \geq c_1 \varepsilon_p^{3/2} \).

**Corollary 6.** Let \( h \) be a positive integer. Then, there are only finitely many cubic number fields \( K \) of negative discriminants \( D_K \) which have class-number \( h \) and whose rings of algebraic integers write \( \mathcal{O}_K = \mathbb{Z}[\varepsilon_K] \), where \( \varepsilon_K > 1 \) is the fundamental unit of \( K \). Moreover, there exists \( c_2 > 0 \) effective such that \( |D_K| \leq c_2 h^2 \log^4 h \) for these number fields.

**Proof of Proposition 5.** Set \( \varepsilon = \varepsilon_p, x_1 = \varepsilon, \) and let \( x_2 = \alpha + i\beta \) and \( x_3 = \alpha - i\beta \) be the two complex conjugate roots of this polynomial. Then,

\[
1 = x_1 x_2 x_3 = \varepsilon (\alpha^2 + \beta^2),
\]

\[
b = x_1 x_2 + x_1 x_3 + x_2 x_3 = 2\varepsilon\alpha + (\alpha^2 + \beta^2) = 2\varepsilon\alpha + (1/\varepsilon),
\]

\[
a = x_1 + x_2 + x_3 = \varepsilon + 2\alpha.
\]

Hence, we have \( |\alpha| \leq 1/\sqrt{\varepsilon} < 1 \) and \( \varepsilon < \alpha + 2 \), which implies \( a \geq 0 \). Moreover, we have \( |b| \leq \sqrt{2\varepsilon} + 1 \leq \sqrt{2\alpha + 2} + 1 \), which implies that there are only finitely many polynomials of type \((T)\) with \( 0 \leq a \leq 17 \). Hence, we may assume \( a \geq 18 \).

We have \(-D_p = 4(a^3 + b^3) - a^2 b^2 - 18ab + 27\).

First, we assume \( a \geq 0 \). Since \( 0 < 2\varepsilon\alpha + (1/\varepsilon) < 2\sqrt{\varepsilon} + 2\alpha \) (since \( 0 \leq \alpha \leq 1/\sqrt{\varepsilon} \) and \( \varepsilon > 1 \)), we get \( 1 \leq b < 2\sqrt{a} \). Now \( b \mapsto f(b) = -D_p = 4b^3 - a^2 b^2 - 18ab + 4a^3 + 27 \) is decreasing in the range \([1, 2\sqrt{a}]\) (since \( a \geq 9 \)). So, we write \( 4a = m^2 + r \), with \( m \geq 0 \) and \( 0 \leq r \leq 2m \), which provides \( 16d_p \geq 16f(m) \) if \( r \geq 1 \), and \( 16d_p \geq 16f(m - 1) \) if \( r = 0 \). Noticing that \( 16f(b) = (4a)^2(4a - b^2) - 72b(4a - b^2) - 8b^4 + 432 \), we thus get

\[
16d_p \geq \begin{cases} r(m^2 + r^2 - 72m - 8m^3 + 432 \geq m^4 - 8m^3 + 2m^2 - 72m + 433 & \text{if } r \geq 1, \\
m^5 - m^4 - 8m^3 - 120m^2 + 192m + 368 & \text{if } r = 0. 
\end{cases}
\]

Since \( 4a \leq m^2 \) and \( \varepsilon < a + 2 \), we get the desired result.

Second, we assume \( a \leq 0 \). Then, \( b = 2\varepsilon\alpha + (1/\varepsilon) \leq 1/\varepsilon < 1 \), i.e. \( b \leq 0 \). We set \( B = -b \). Now \( g(B) = -D_p = -4B^3 - a^2 B^2 + 18kB + 4a^3 + 27 \) is decreasing on \([1, +\infty[\) (since \( a \geq 9 \), and \( g(\sqrt{4a + 1}) < 0 \) (since \( a \geq 16 \)). Hence, we get \( B < \sqrt{4a + 1} \). So, we write \( 4a + 1 = m^2 + r \), with \( m \geq 0 \) and \( 0 \leq r \leq 2m \). Since \( g(0) \geq g(1) \) (since \( a \geq 18 \)), we have \( 16d_p \geq 16g(m) \) if \( r \geq 1 \), and \( 16d_p \geq 16g(m - 1) \) if \( r = 0 \). We thus get
\[ \begin{align*}
16d_p & \geq \\
&= \begin{cases}
(r-1)(m^2 + r-1)^2 + 72m(r-1) + 8m^3 + 432 \geq 8m^3 + 432 & \text{if } r \geq 1, \\
2m^5 - 2m^4 + 4m^3 + 124m^2 - 262m + 566 & \text{if } r = 0.
\end{cases}
\]

As in the first case, we get the desired result.

Let us note that when \( P(X) = X^3 - M^2X^2 - 2MX - 1 \) we have \( d_p = 4M^3 + 27 \) and \( M^2 < \varepsilon_p < M^2 + 1 \) (\( M \geq 2 \)) which implies \( d_p \approx 4\varepsilon_p^{3/2} \).

\[ \square \]

6. Conclusion

Let \( h \) and \( n \) be two given positive integers. Are there only finitely many number fields \( K \) of degree \( n \) with class-number \( h \) such that their rings of algebraic integers are generated by units? More precisely, let \( n \geq 1 \) be a given positive integer and let \( K = \mathbb{Q}(x) \) be a number field of degree \( n \) where \( x \) is an algebraic unit which is a root of any irreducible monic polynomial of the form \( P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + 1 \). If we assume that the ring of algebraic integers of \( K \) is equal to \( \mathbb{Z}[x] \) and that \( K \) has a unit group of rank 1, is that true that the class-number of \( K \) tends to infinity with \( d_K = d_p \)?

REFERENCES


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