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Projectors on the Generalized Eigenspaces for Neutral Functional Differential Equations in L^p Spaces

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Abstract. We present the explicit formulas for the projectors on the generalized eigenspaces associated with some eigenvalues for linear neutral functional differential equations (NFDE) in L^p spaces by using integrated semigroup theory. The analysis is based on the main result established elsewhere by the authors and results by Magal and Ruan on non-densely defined Cauchy problem. We formulate the NFDE as a non-densely defined Cauchy problem and obtain some spectral properties from which we then derive explicit formulas for the projectors on the generalized eigenspaces associated with some eigenvalues. Such explicit formulas are important in studying bifurcations in some semi-linear problems.

1 Introduction

In this paper we consider the linear neutral functional differential equation (NFDE) in L^p spaces

$$
\begin{cases} \dfrac{d}{dt}(x(t) - L_1(x_t)) = B(x(t) - L_1(x_t)) + L_2(x_t), \\ t \geq 0, x(0) = \widehat{x} \in \mathbb{R}^n,\ x_0 = \varphi \in L^p((-r, 0), \mathbb{R}^n), \end{cases} \tag{1.1}
$$

with $x_t \in L^p((-r, 0), \mathbb{R}^n)$ satisfying $x_t(\theta) = x(t + \theta)$ for almost every $\theta \in (-r, 0)$. Here $p \in [1, +\infty)$, $r \in [0, +\infty)$, $B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, and L_j, $j = 1, 2$, are bounded linear operators from $L^p((-r, 0), \mathbb{R}^n)$ into $\mathbb{R}^n$ given by

$$
L_j(\varphi) = \int_{-r}^{0} \eta_j(\theta)\varphi(\theta)\, d\theta.
$$

Here $\eta_j \in L^q((-r, 0), M_n(\mathbb{R}))$ with $\frac{1}{p} + \frac{1}{q} = 1$, $j = 1, 2$.

The aim of this work is to derive explicit formulas for the projectors on the generalized eigenspaces associated with some eigenvalues. This problem is of particular importance in the bifurcation theory of corresponding semilinear problems. Indeed these projectors are necessary in order to study the stability of bifurcating periodic

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solutions in the context of Hopf bifurcation, and especially when computing the normal form, see for instance Hale [8], Hale and Lunel [9], and the references therein.

This kind of NFDE in the space of continuous maps $C([-r,0],\mathbb{R}^n)$ has been extensively considered (see, for example, Adimy and Ezzinbi [1]). Early work on delay differential equations in L^p spaces using semigroup methods was due to Hale [7] and Webb [14, 15]. We refer to Wu [18], Diekmann et al. [4], and Batkai and Piazzera [2] for more results and references on this topic.

Liu, Magal, and Ruan [10] proposed a general method to derive explicit formulas for the projectors on the generalized eigenspaces associated with some eigenvalues for linear functional differential equations (FDE) in the space of continuous maps:

$$\text{(1.2)} \qquad \begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \widehat{L}\,(x_t)\,, \forall t \geq 0, \\ x_0 = \varphi \in C\left([-r,0]\,,\mathbb{R}^n\right). \end{cases}$$

Their approach is based on a re-formulation of the problem in terms of an abstract non-densely defined Cauchy problem and integrated semigroup theory. One of the main difficulties of this approach is estimating the essential growth rate of some linear C_0-semigroup. This problem is solved by using a bounded perturbation result proved by Thieme [13] in the case of nondensely defined operators satisfying the Hille–Yosida property.

In the framework of L^p spaces, we will use the same approach as in [10] by reformulating problem (1.1) as an abstract non-densely defined Cauchy problem. Here the operator does not satisfy the Hille–Yosida property. The estimate of the essential spectrum uses an extension of Thieme's result proved by the authors in [5].

Note that the reformulation of the problem as an abstract non-densely defined Cauchy problem is presented in [11]. They more particularly show that the NFDE (1.1) is a particular case of a general age-structured model. The corresponding non-densely defined operator only satisfies the Hille–Yosida property for $p = 1$. Therefore in order to consider the case $p \neq 1$ we will use the theory developed in [11] for age-structured models in L^p.

The paper is organized as follows. In Section 2 we first demonstrate how to consider the NFDE problem in L^p space as an abstract non-densely defined Cauchy problem and recall some relevant results in [6, 12, 16, 17]. Then we derive some results on integrated solutions and spectral properties. Finally, in Section 3 we obtain the main result of this article, an explicit formula for the projectors on the generalized eigenspaces associated with some eigenvalues that extends the results in [10].

2 Preliminaries

Consider the neutral delay differential equation (1.1). First we transform this problem into a PDE problem. Set $u(t) = x_t$ for $t \geq 0$ and we get

$$\frac{d}{dt}[u(t,0) - L_1(u(t))] = B[u(t,0) - L_1(u(t))] + L_2(u(t)), \quad t \geq 0.$$

Let $y(t) = u(t,0) - L_1(u(t))$. We obtain that

$$\frac{dy(t)}{dt} = By(t) + L_2(u(t)),\ t \geq 0 \quad \text{and} \quad u(t,0) = L_1(u(t)) + y(t).$$

Therefore, we deduce formally that u must satisfy a PDE

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial u}{\partial \theta} = 0, \forall \theta \in (-r,0), \\ &u(t,0) = L_1(u(t)) + y(t), \\ &\frac{dy(t)}{dt} = By(t) + L_2(u(t)), \\ &y(0) = y_0 = \widehat{x} - L_1(\varphi) \in \mathbb{R}^n, \\ &u(0,\cdot) = \varphi \in L^p((-r,0),\mathbb{R}^n). \end{aligned} \tag{2.1}$$

Let $X = \mathbb{R}^n \times L^p((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$ endowed with the product norm

$$\left\| \begin{pmatrix} z_1 \\ \varphi \\ z_2 \end{pmatrix} \right\| = \|z_1\|_{\mathbb{R}^n} + \|\varphi\|_{L^p((-r,0),\mathbb{R}^n)} + \|z_2\|_{\mathbb{R}^n}$$

and $X_0 = \{0_{\mathbb{R}^n}\} \times L^p((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$. Set

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \\ y(t) \end{pmatrix}.$$

We can consider (2.1) as an abstract non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + Lv(t) + \widehat{L}v(t), t \geq 0, v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y_0 \end{pmatrix} \in \overline{D(A)}, \tag{2.2}$$

where $A\colon D(A) \subset X \to X$ is a linear operator defined by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ \varphi' \\ By \end{pmatrix}$$

with $D(A) := \{0_{\mathbb{R}^n}\} \times W^{1,p}((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$, and where $L, \widehat{L}\colon X_0 \to X$ are defined by

$$L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \widehat{L} \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} L_1(\varphi) \\ 0 \\ L_2(\varphi) \end{pmatrix}.$$

Note that $\overline{D(A)} = X_0$.

Before proceeding, we shall introduce some notations and recall some results. Let $L\colon D(L) \subset X \to X$ be a linear operator on a complex Banach space X. Denote by $\rho(L)$, $N(L)$, and $R(L)$ the resolvent set, null space, and range of L, respectively. The spectrum of L is $\sigma(L) = \mathbb{C}\backslash\rho(L)$. The *point spectrum* of L is the set

$$\sigma_P(L) := \{\lambda \in \mathbb{C} : N(\lambda I - L) \neq \{0\}\}.$$

The *essential spectrum* (in the sense of Browder [3]) of L is denoted by $\sigma_{\mathrm{ess}}(L)$ which is the set of $\lambda \in \sigma(L)$ such that at least one of the following holds: (i) $R(\lambda I - L)$ is not closed; (ii) λ is a limit point of $\sigma(L)$; (iii) $N_\lambda(L) := \bigcup_{k=1}^{\infty} N((\lambda I - L)^k)$ is infinite dimensional. Let Y be a subspace of X and $L_Y : D(L_Y) \subset Y \to Y$ denote the part of L on Y, which is defined by

$$L_Y x = Lx, \quad \forall x \in D(L_Y) := \{x \in D(L) \cap Y : Lx \in Y\}.$$

Definition 2.1 Let $L\colon D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}_{t\geq 0}$ on a Banach space X. We define $\omega_0(L) \in [-\infty, +\infty)$, the *growth bound* of L, by

$$\omega_0(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\mathcal{L}(X)})}{t}.$$

The *essential growth bound* $\omega_{0,\mathrm{ess}}(L) \in [-\infty, +\infty)$ of L is defined by

$$\omega_{0,\mathrm{ess}}(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\mathrm{ess}})}{t},$$

where $\|T_L(t)\|_{\mathrm{ess}}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{\mathrm{ess}} = \kappa(T_L(t)B_X(0,1)).$$

Here $\mathrm{B}_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$, and for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$$

is the Kuratovsky measure of non-compactness.

In the following theorem, the existence of the projector was first proved by Webb [16, 17] and the fact that there is a finite number of points of the non-essential spectrum was proved by Engel and Nagel [6].

Theorem 2.2 *Let $L\colon D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}_{t\geq 0}$ on a Banach space X. Then*

$$\omega_0(L) = \max(\omega_{0,\mathrm{ess}}(L), \max_{\lambda\in\sigma(L)\backslash\sigma_{\mathrm{ess}}(L)} \mathrm{Re}(\lambda)).$$

Assume in addition that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Let $\gamma \in (\omega_{0,\mathrm{ess}}(L), \omega_0(L)]$ be fixed. Then the subset $\{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) \geq \gamma\} \subset \sigma_P(L)$ is nonempty, finite, and contains only poles of the resolvent of L. Moreover, there exists a finite rank bounded linear operator of projection $\Pi\colon X \to X$ satisfying the following properties:

(i) $\Pi(\lambda I - L)^{-1} = (\lambda I - L)^{-1}\Pi, \forall \lambda \in \rho(L)$;
(ii) $\sigma(L_{\Pi(X)}) = \{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) \geq \gamma\}$;
(iii) $\sigma(L_{(I-\Pi)(X)}) = \sigma(L) \setminus \sigma(L_{\Pi(X)})$.

In Theorem 2.2 the projector Π is the projection on the direct sum of the generalized eigenspaces of L associated with all points $\lambda \in \sigma(L)$ with $\mathrm{Re}(\lambda) \geq \gamma$. As a consequence of Theorem 2.2 we have the following corollary.

Corollary 2.3 *Let $L\colon D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}_{t\geq 0}$ on a Banach space X. Assume that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Then*

$$\{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}(L)\} \subset \sigma_P(L)$$

and each $\widehat{\lambda} \in \{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}(L)\}$ is a pole of the resolvent of L. That is, $\widehat{\lambda}$ is isolated in $\sigma(L)$, and there exists an integer $k_0 \geq 1$ (the order of the pole) such that the Laurent expansion of the resolvent takes the following form

$$(\lambda I - L)^{-1} = \sum_{n=-k_0}^{\infty} (\lambda - \widehat{\lambda})^n B_n^{\widehat{\lambda}},$$

where $\{B_n^{\widehat{\lambda}}\}$, $n \geq -k_0$ are bounded linear operators on X and the above series converges in the norm of operators whenever $|\lambda - \widehat{\lambda}|$ is small enough.

The following result is due to Magal and Ruan [12, Lemma 2.1, Proposition 3.5].

Theorem 2.4 *Let $(X, \|\cdot\|)$ be a Banach space and $L\colon D(L) \subset X \to X$ be a linear operator. Assume that $\rho(L) \neq \varnothing$ and L_0, the part of L in $\overline{D(L)}$, is the infinitesimal generator of a linear C_0-semigroup $\{T_{L_0}(t)\}_{t\geq 0}$ on the Banach space $\overline{D(L)}$. Then $\sigma(L) = \sigma(L_0)$.*

Let $\Pi_0\colon \overline{D(L)} \to \overline{D(L)}$ be a bounded linear operator of projection. Assume that

$$\Pi_0(\lambda I - L_0)^{-1} = (\lambda I - L_0)^{-1}\Pi_0,\ \forall \lambda > \omega,\ \omega \in \mathbb{R},$$
$$\Pi_0(\overline{D(L)}) \subset D(L_0),\ \ \textit{and}\ \ L_0|_{\Pi_0(\overline{D(L)})}\ \textit{is bounded.}$$

Then there exists a unique bounded linear operator of projection Π on X satisfying the following properties:

(i) $\Pi|_{\overline{D(L)}} = \Pi_0$.
(ii) $\Pi(X) \subset \overline{D(L)}$.
(iii) $\Pi(\lambda I - L)^{-1} = (\lambda I - L)^{-1}\Pi, \forall \lambda > \omega$.

Moreover, for each $x \in X$ we have the following approximation formula:

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda(\lambda I - L)^{-1} x.$$

Now we return to the Cauchy problem (2.2). We first have the following property.

Lemma 2.5 *The resolvent sets of A and $A+L$ satisfy $\rho(A) = \rho(A+L) = \rho(B)$. We have the following explicit formulas for the resolvents of A and $A+L$:*

$$(2.3) \qquad (\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix}$$

$$\iff \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda\theta}\alpha + \int_\theta^0 e^{\lambda(\theta - l)}\varphi(l)dl, \ \forall \theta \in (-r, 0) \\ \widehat{y} = (\lambda I - B)^{-1} y, \end{cases}$$

$$(2.4) \quad (\lambda I - (A+L))^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix}$$

$$\iff \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda\theta}[(\lambda I - B)^{-1} y + \alpha] + \int_\theta^0 e^{\lambda(\theta - l)}\varphi(l)dl, \ \forall \theta \in (-r, 0) \\ \widehat{y} = (\lambda I - B)^{-1} y. \end{cases}$$

Proof We only prove the result for $A+L$. The proof for A is similar. Let us first prove that $\rho(A+L) \subset \rho(B)$. We only need to show that $\sigma(B) \subset \sigma(A+L)$. Let $\lambda \in \sigma(B)$. Then there exists $\widehat{y} \in \mathbb{C}^n \setminus \{0\}$ such that $B\widehat{y} = \lambda\widehat{y}$. If we consider $\widehat{\varphi}(\theta) = e^{\lambda\theta}\widehat{y}$, we have

$$(A+L) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} -\widehat{\varphi}(0) + \widehat{y} \\ \widehat{\varphi}' \\ B\widehat{y} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \lambda\widehat{\varphi} \\ \lambda\widehat{y} \end{pmatrix}.$$

Thus $\lambda \in \sigma(A+L)$. This implies that $\sigma(B) \subset \sigma(A+L)$. On the other hand, if $\lambda \in \rho(B)$, for $\begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} \in X$ we must have $\begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} \in D(A)$ such that

$$(\lambda I - (A+L)) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} \iff \begin{cases} \widehat{\varphi}(0) - \widehat{y} = \alpha \\ \lambda\widehat{\varphi} - \widehat{\varphi}' = \varphi \\ \lambda\widehat{y} - B\widehat{y} = y \end{cases}$$

$$\iff \begin{cases} \widehat{\varphi}(\widehat{\theta}) = e^{\lambda\widehat{\theta}}[(\lambda I - B)^{-1} y + \alpha] + \int_{\widehat{\theta}}^0 e^{\lambda(\widehat{\theta} - l)}\varphi(l)dl, \ \forall \widehat{\theta} \in (-r, 0) \\ \widehat{y} = (\lambda I - B)^{-1} y. \end{cases}$$

Therefore, we obtain that $\lambda \in \rho(A+L)$ and the formula in (2.4) holds. ■

Since B is a matrix on $\mathbb{R}^n$, we have $\omega_0(B) := \max_{\lambda \in \sigma(B)} \mathrm{Re}(\lambda)$ and the following lemma.

Lemma 2.6 *For each $\omega_A > \omega_0(B)$, $(\omega_A, +\infty) \subset \rho(A)$ and there exists $M_A \geq 1$ such that*

$$(2.5) \qquad \|(\lambda I - A)^{-n}\|_{\mathcal{L}(X_0)} \leq \frac{M_A}{(\lambda - \omega_A)^n}, \ \forall n \geq 1, \forall \lambda > \omega_A.$$

Moreover, $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0$, $\forall x \in X$.

Proof Let $\omega_A > \omega_0(B)$. From Lemma 2.5 we obtain that $(\omega_A, +\infty) \subset \rho(B) = \rho(A)$. We can define the equivalent norm on $\mathbb{R}^n$

$$|y| := \sup_{t \geq 0} e^{-\omega_A t} \|e^{Bt} y\|, \quad y \in \mathbb{R}^n.$$

Then we have $|e^{Bt} y| \leq e^{\omega_A t} |y|$, $\forall t \geq 0$ and $\|y\| \leq |y| \leq M_A \|y\|$, where

$$M_A := \sup_{t \geq 0} \|e^{(B-\omega_A I)t}\|_{M_n(\mathbb{R})}.$$

Moreover, for each $\lambda > \omega_A$, we have

$$|(\lambda I - B)^{-1} y| = \Big| \int_0^{+\infty} e^{-\lambda s} e^{Bs} y \, ds \Big| \leq \frac{|y|}{\lambda - \omega_A}.$$

We define $|\cdot|$ the equivalent norm on X by

$$\left| \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} \right| = |\alpha| + \|\varphi\|_{\omega_A} + |y|,$$

where

$$\|\varphi\|_{\omega_A} := |e^{-\omega_A \cdot} \varphi(\,\cdot\,)|_{L^p} = \Big(\int_{-r}^0 |e^{-\omega_A \theta} \varphi(\theta)|^p d\theta \Big)^{1/p}.$$

Using (2.3) and the above results, we obtain that for $\begin{pmatrix} 0 \\ \varphi \\ y \end{pmatrix} \in X_0$,

$$\begin{aligned} \left| (\lambda I - A)^{-1} \begin{pmatrix} 0 \\ \varphi \\ y \end{pmatrix} \right| &\leq \Big| e^{-\omega_A \cdot} \int_{\cdot}^0 e^{\lambda(\cdot - l)} \varphi(l) \, dl \Big|_{L^p} + |(\lambda I - B)^{-1} y| \\ &\leq |e^{(\lambda - \omega_A)\cdot}|_{L^1} |e^{-\omega_A \cdot} \varphi(\,\cdot\,)|_{L^p} + \frac{1}{\lambda - \omega_A} |y| \leq \frac{1}{\lambda - \omega_A} \big[\|\varphi\|_{\omega_A} + |y| \big]. \end{aligned}$$

Therefore, (2.5) holds. The last part of the proof is trivial. ■

As an immediate consequence of the above lemma and by applying the results proved in Magal and Ruan [11, Lemma 2.2, Proposition 2.5], we obtain the following lemma.

Lemma 2.7 *A_0 the part of A in X_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0}$ of bounded linear operators on X_0, which is defined by*

$$T_{A_0}(t) \begin{pmatrix} 0 \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{A_0}(t) \varphi \\ e^{Bt} y \end{pmatrix}, \tag{2.6}$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(\theta)=\begin{cases}\varphi(t+\theta) & \text{if } t+\theta\le 0,\\ 0 & \text{if } t+\theta>0.\end{cases}$$

Moreover, A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X, which is defined by

$$S_A(t)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}=\begin{pmatrix}0\\ \alpha 1_{[-t,0]}(\,\cdot\,)+\int_0^t \widehat{T}_{A_0}(l)\varphi dl\\ \int_0^t e^{Bl}y dl\end{pmatrix}.$$

Proof See [11, Section 6] for more details. ∎

We set $X_1=\mathbb{R}^n\times\{0_{L^p}\}\times\{0_{\mathbb{R}^n}\}$. Then we have $X=X_1\oplus X_0$.

By using the same argument as in the proof of Theorem 6.6 in [11] we obtain the following result.

Lemma 2.8 *For each $\tau>0$, each $h_1\in L^p((0,\tau),X_1)$, and each $h_2\in L^1((0,\tau),X_0)$, there exists a unique integrated solution of the Cauchy problem*

$$\frac{dv(t)}{dt}=Av(t)+h(t),\quad t\in[0,\tau], h=h_1+h_2,\ \text{and } v(0)=v_0:=\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\\ y_0\end{pmatrix},$$

which is given by $v(t)=T_{A_0}(t)v_0+d/dt(S_A*h)(t)$, $\forall t\in[0,\tau]$, and we also have the following estimate for each $t\in[0,\tau]$,

$$\|v(t)\|\le M_A e^{\omega_A t}\|v_0\|+\Big(\int_0^t\|h_1(s)\|^p ds\Big)^{1/p}+M_A\int_0^t e^{\omega_A(t-s)}\|h_2(s)\|\,ds.$$

Furthermore,

$$v(t)=\begin{pmatrix}0\\ u(t)\\ y(t)\end{pmatrix}$$

with

$$\begin{pmatrix}u(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}\widehat{T}_{A_0}(t)\varphi\\ e^{Bt}y_0\end{pmatrix}+\begin{pmatrix}h_1(t+.)1_{[-t,0]}(\,\cdot\,)+\int_0^t\widehat{T}_{A_0}(t-s)h_{21}(s)\,ds\\ \int_0^t e^{B(t-s)}h_{22}(s)\,ds\end{pmatrix}.$$

Here $h_2(t)=(0,h_{21}(t),h_{22}(t))$.

By using the same argument as in the proof of Theorem 6.6 in [11], we also note that if $h\in C^1([0,\tau],X)$, then

$$\begin{aligned}\Big\|\frac{d}{dt}(S_A*h)(t)\Big\|&=\Big\|\frac{d}{dt}(S_A*Ph)(t)+\frac{d}{dt}(S_A*(I-P)h)(t)\Big\|\\&\le\Big(\int_0^t\|Ph(s)\|^p\,ds\Big)^{1/p}+M_A\int_0^t e^{\omega_A(t-s)}\,\|(I-P)h(s)\|\,ds,\end{aligned}$$

for all $t \in [0,\tau]$, where $P\colon X \to X$ is defined by

$$Px = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}, \ \forall x = \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} \in X.$$

Let

$$\Gamma(t,h) = \Big(\int_0^t \|Ph(s)\|^p ds\Big)^{1/p} + M_A \int_0^t e^{\omega_A(t-s)} \|(I-P)h(s)\| ds, \ \forall t \in [0,\tau].$$

We obtain that

$$\begin{aligned}\Gamma(t,h) &= \Big(\int_0^t \|Ph(s)\|^p \, ds\Big)^{1/p} + M_A \int_0^t e^{\omega_A(t-s)} \|(I-P)h(s)\| \, ds \\ &\le t^{1/p} \sup_{s\in[0,t]} \|Ph(s)\| + M_A \Big(\int_0^t e^{q\omega_A(t-s)} \, ds\Big)^{1/q} \Big(\int_0^t \|(I-P)h(s)\|^p \, ds\Big)^{1/p} \\ &\le t^{1/p} \|P\| \sup_{s\in[0,t]} \|h(s)\| + M_A \Big(\int_0^t e^{q\omega_A(t-s)} ds\Big)^{1/q} t^{1/p} \|I-P\| \sup_{s\in[0,t]} \|h(s)\| \\ &\le \delta(t) \sup_{s\in[0,t]} \|h(s)\|, \ \forall t \in [0,\tau],\end{aligned}$$

where $1/p + 1/q = 1$ and $\delta(t) = t^{1/p}\|P\| + t^{1/p} M_A (\int_0^t e^{q\omega_A(t-s)} \, ds)^{1/q} \|I-P\|$ satisfying $\lim_{t\to 0^+} \delta(t) = 0$. Hence, we get $\|L+\widehat{L}\|_{\mathcal{L}(X_0,X)} \delta(t) < 1$ for t small enough. Therefore, by using the perturbation result proved in [11, Theorem 3.1] we know that $A+L+\widehat{L}$ statisfies the same properties as A. In particular, $(A+L+\widehat{L})_0$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L+\widehat{L})_0}(t)\}_{t\ge 0}$ of bounded linear operators on X_0.

From the definition of $A+L+\widehat{L}$ in (2.2) and the fact that

$$\begin{aligned} D(A) &:= \{0_{\mathbb{R}^n}\} \times W^{1,p}((-r,0),\mathbb{R}^n) \times \mathbb{R}^n, \\ \overline{D(A)} &= \{0_{\mathbb{R}^n}\} \times L^p((-r,0),\mathbb{R}^n) \times \mathbb{R}^n, \end{aligned}$$

we know that

$$\begin{aligned} &D((A+L+\widehat{L})_0) \\ &= \left\{ \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times W^{1,p}((-r,0),\mathbb{R}^n) \times \mathbb{R}^n \;\middle|\; -\varphi(0) + y + L_1(\varphi) = 0 \right\}. \end{aligned}$$

Lemma 2.9 *The point spectrum of* $(A+L+\widehat{L})_0$ *is the set*

$$\sigma_P((A+L+\widehat{L})_0) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},$$

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where

$$\Delta(\lambda) = (\lambda I - B)[I - L_1(e^{\lambda\cdot} I)] - L_2(e^{\lambda\cdot} I) = (\lambda I - B)\Big[I - \int_{-r}^{0} e^{\lambda\theta}\eta_1(\theta)\,d\theta\Big] - \int_{-r}^{0} e^{\lambda\theta}\eta_2(\theta)\,d\theta. \tag{2.7}$$

Proof Let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma_P((A + L + \widehat{L})_0)$ if and only if there exist $\varphi \in W^{1,p}((-r, 0), \mathbb{C}^n) \setminus \{0\}$ and $y \in \mathbb{C}^n$ such that

$$\varphi'(\theta) = \lambda\varphi(\theta), \forall\theta \in (-r, 0),$$
$$By + L_2(\varphi) = \lambda y \quad \text{and} \quad \varphi(0) = y + L_1(\varphi).$$

Hence we obtain that

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0), \quad \lambda y - By - L_2(e^{\lambda\cdot}\varphi(0)) = 0, \quad \text{and} \quad y = \varphi(0) - L_1(e^{\lambda\cdot}\varphi(0)).$$

Therefore,

$$\varphi \neq 0 \Longleftrightarrow \varphi(0) \neq 0 \text{ and } (\lambda I - B)\left[\varphi(0) - L_1(e^{\lambda\cdot}\varphi(0))\right] - L_2\left(e^{\lambda\cdot}\varphi(0)\right) = 0.$$

The proof is complete. ■

From the discussion in this section and the results we recalled above, we obtain the following proposition.

Proposition 2.10 *$(A + L + \widehat{L})_0$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L+\widehat{L})_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0. Moreover,*

$$\omega_{0,\mathrm{ess}}((A + L + \widehat{L})_0) = \omega_{0,\mathrm{ess}}(A_0) = -\infty,$$
$$\omega_0((A + L + \widehat{L})_0) = \max_{\lambda\in\sigma_P((A+L+\widehat{L})_0)} \mathrm{Re}(\lambda),$$
$$\sigma(A + L + \widehat{L}) = \sigma((A + L + \widehat{L})_0) = \sigma_P((A + L + \widehat{L})_0) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},$$

and each $\lambda_0 \in \sigma(A+L+\widehat{L})$ is a pole of $(\lambda I - (A+L+\widehat{L}))^{-1}$. For each $\gamma \in \mathbb{R}$, the subset $\{\lambda \in \sigma((A + L + \widehat{L})_0) : \mathrm{Re}(\lambda) \geq \gamma\}$ is either empty or finite.

Proof We only need to prove that $\omega_{0,\mathrm{ess}}((A+L+\widehat{L})_0) = \omega_{0,\mathrm{ess}}(A_0) = -\infty$. From (2.6) it is easy to know that for $t > r$, $T_{A_0}(t)$ is compact. Hence $\omega_{0,\mathrm{ess}}(A_0) = -\infty$. Since for each $t > 0$ $(L + \widehat{L})T_{A_0}(t)$ is compact, the result follows by applying Theorem 1.2 in [5]. ■

3 Projectors on the Eigenspaces

Let $\lambda_0 \in \sigma(A + L + \widehat{L})$. From Proposition 2.10 we already know that λ_0 is a pole of $(\lambda I - (A + L + \widehat{L}))^{-1}$ of finite order $k_0 \geq 1$. This means that λ_0 is isolated in $\sigma(A + L + \widehat{L})$ and the Laurent expansion of the resolvent around λ_0 takes the following form:

$$(\lambda I - (A + L + \widehat{L}))^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda - \lambda_0)^n B_n^{\lambda_0}.$$

The bounded linear operator $B_{-1}^{\lambda_0}$ is the projector on the generalized eigenspace of $(A + L + \widehat{L})$ associated with λ_0. The goal of this section is to provide a method to compute $B_{-1}^{\lambda_0}$. We remark that

$$(\lambda - \lambda_0)^{k_0} (\lambda I - (A + L + \widehat{L}))^{-1} = \sum_{m=0}^{+\infty} (\lambda - \lambda_0)^m B_{m-k_0}^{\lambda_0}.$$

So we have the following approximation formula

$$B_{-1}^{\lambda_0} = \lim_{\lambda \to \lambda_0} \frac{1}{(k_0 - 1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \big((\lambda - \lambda_0)^{k_0} (\lambda I - (A + L + \widehat{L}))^{-1} \big).$$

In order to give an explicit formula for $B_{-1}^{\lambda_0}$, we need the following results.

Lemma 3.1 *For each $\lambda \in \rho(A + L + \widehat{L})$, we have the following explicit formula for the resolvent of $A + L + \widehat{L}$,*

$$(\lambda I - (A + L + \widehat{L}))^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix}$$
$$\iff \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda\theta}\Phi_\lambda + \int_\theta^0 e^{\lambda(\theta - l)} \varphi(l)\, dl, \\ \widehat{y} = \Phi_\lambda - L_1(e^{\lambda \cdot}\Phi_\lambda) - L_1(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l)\, dl) - \alpha, \end{cases}$$

where Φ_λ is defined by

$$\text{(3.1)} \quad \Phi_\lambda = \Delta(\lambda)^{-1} \Big[(\lambda I - B) \Big(L_1 \Big(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l)\, dl \Big) + \alpha \Big) + L_2 \Big(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + y \Big]$$

with $\Delta(\lambda)$ defined in (2.7).

Proof Let $\lambda \in \rho(A + L + \widehat{L})$ and $\gamma > 0$ large enough such that $\operatorname{Re}(\lambda) > \omega_0(B) - \gamma$. So we obtain that $\lambda \in \rho(B - \gamma I)$. Consider the linear operators $A_\gamma \colon D(A) \subset X \to X$ and $L_\gamma \colon X_0 \to X$ defined respectively by

$$A_\gamma \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} -\varphi(0) + y \\ \varphi' \\ (B - \gamma I) y \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \in D(A)$$

and

$$L_\gamma \left(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \right) = \begin{pmatrix} L_1(\varphi) \\ 0 \\ L_2(\varphi) + \gamma y \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \in X_0.$$

From Lemma 2.5, we know that $\lambda \in \rho(A_\gamma)$ and

$$(3.2) \quad (\lambda I - A_\gamma)^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix}$$

$$\iff \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda\theta}[((\lambda + \gamma) I - B)^{-1} y + \alpha] + \int_\theta^0 e^{\lambda(\theta - l)} \varphi(l)\, dl, \\ \widehat{y} = [(\lambda + \gamma) I - B]^{-1} y. \end{cases}$$

Moreover, the operator $[\lambda I - (A_\gamma + L_\gamma)]$ is invertible if and only if $I - L_\gamma(\lambda I - A_\gamma)^{-1}$ is invertible, and

$$(3.3) \qquad (\lambda I - (A_\gamma + L_\gamma))^{-1} = (\lambda I - A_\gamma)^{-1}[I - L_\gamma(\lambda I - A_\gamma)^{-1}]^{-1}.$$

We also know that

$$[I - L_\gamma(\lambda I - A_\gamma)^{-1}] \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\varphi} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix}$$

is equivalent to

$$(3.4) \qquad \widetilde{\varphi} = \varphi,$$

$$(3.5) \qquad \widetilde{\alpha} - L_1(e^{\lambda \cdot}\widetilde{\alpha}) - L_1(e^{\lambda \cdot} y_\lambda) = L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + \alpha,$$

and, noting that $\widetilde{y} - \gamma[(\lambda + \gamma)I - B]^{-1}\widetilde{y} = (\lambda I - B) y_\lambda$,

$$(3.6) \qquad -L_2(e^{\lambda \cdot}\widetilde{\alpha}) - L_2(e^{\lambda \cdot} y_\lambda) + (\lambda I - B) y_\lambda = L_2\Big(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + y,$$

where $y_\lambda = (\lambda + \gamma)I - B)^{-1}\widetilde{y}$. By computing $(\lambda I - B)\times$ (3.5) + (3.6), we get

$$\begin{aligned} &(\lambda I - B)[\widetilde{\alpha} - L_1(e^{\lambda \cdot}\widetilde{\alpha})] - L_2(e^{\lambda \cdot}\widetilde{\alpha}) + (\lambda I - B)[y_\lambda - L_1(e^{\lambda \cdot} y_\lambda)] - L_2(e^{\lambda \cdot} y_\lambda) \\ &\quad = (\lambda I - B)\Big(L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + \alpha \Big) + L_2\Big(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + y, \end{aligned}$$

i.e.,

$$\Delta(\lambda)(\widetilde{\alpha}+y_\lambda) = (\lambda I - B)\Big(L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)dl\Big) + \alpha\Big) + L_2\Big(\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big) + y.$$

We know from (3.5) that

$$\widetilde{\alpha} - L_1(e^{\lambda\cdot}(\widetilde{\alpha}+y_\lambda)) = L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big) + \alpha.$$

Therefore we deduce that $I - L_\gamma(\lambda I - A_\gamma)^{-1}$ is invertible if and only if $\Delta(\lambda)$ is invertible. Moreover,

$$[I - L_\gamma(\lambda I - A_\gamma)^{-1}]^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = \begin{pmatrix}\widetilde{\alpha}\\ \widetilde{\varphi}\\ \widetilde{y}\end{pmatrix}$$

is equivalent to

$$\widetilde{\varphi} = \varphi, \tag{3.7}$$

$$\widetilde{\alpha} = L_1(e^{\lambda\cdot}\Phi_\lambda) + L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big) + \alpha \tag{3.8}$$

and

$$\widetilde{y} = \big((\lambda+\gamma)I - B\big)\,[\Phi_\lambda - \widetilde{\alpha}], \tag{3.9}$$

where Φ_λ is defined by (3.1). Note that $A + L + \widehat{L} = A_\gamma + L_\gamma$. By using (3.2), (3.3), (3.7), (3.8), and (3.9), we obtain that

$$(\lambda I - (A + L + \widehat{L}))^{-1}\begin{pmatrix}(\alpha\\ \varphi\\ y\end{pmatrix} = (\lambda I - (A_\gamma + L_\gamma))^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \widehat{\varphi}\\ \widehat{y}\end{pmatrix}$$

$$\iff \begin{cases}\widehat{\varphi}(\theta) = e^{\lambda\theta}\Phi_\lambda + \displaystyle\int_\theta^0 e^{\lambda(\theta - l)}\varphi(l)\,dl,\\[2ex] \widehat{y} = \Phi_\lambda - L_1(e^{\lambda\cdot}\Phi_\lambda) - L_1\Big(\displaystyle\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)dl\Big) - \alpha,\end{cases}$$

where Φ_λ is defined by (3.1). ∎

The map $\lambda \to \Delta(\lambda)$ from $\mathbb{C}$ into $M_n(\mathbb{C})$ is differentiable, and

$$\Delta^{(1)}(\lambda) := \frac{d\Delta(\lambda)}{d\lambda} = I - \int_{-r}^0 (e^{\lambda\theta} + \lambda\theta e^{\lambda\theta})\eta_1(\theta)d\theta + B\int_{-r}^0 \theta e^{\lambda\theta}\eta_1(\theta)d\theta - \int_{-r}^0 \theta e^{\lambda\theta}\eta_2(\theta)d\theta.$$

So the map $\lambda \to \Delta(\lambda)$ is analytic and

$$\Delta^{(n)}(\lambda) := \frac{d^n \Delta(\lambda)}{d\lambda^n} = -\int_{-r}^{0} (n\theta^{n-1}e^{\lambda\theta} + \lambda\theta^n e^{\lambda\theta})\eta_1(\theta)d\theta + B\int_{-r}^{0} \theta^n e^{\lambda\theta}\eta_1(\theta)d\theta - \int_{-r}^{0} \theta^n e^{\lambda\theta}\eta_2(\theta)d\theta, n \geq 2.$$

We know that the inverse function $\psi\colon L \to L^{-1}$ of a linear operator $L \in \mathrm{Isom}(X)$ is differentiable, and $D\psi(L)\widehat{L} = -L^{-1} \circ \widehat{L} \circ L^{-1}$. Applying this result, we deduce that $\lambda \to \Delta(\lambda)^{-1}$ from $\rho(A + L + \widehat{L})$ into $M_n(\mathbb{C})$ is differentiable, and $\frac{d}{d\lambda}\Delta(\lambda)^{-1} = -\Delta(\lambda)^{-1}\big(\frac{d}{d\lambda}\Delta(\lambda)\big)\,\Delta(\lambda)^{-1}$. Therefore, we obtain that the map $\lambda \to \Delta(\lambda)^{-1}$ is analytic and has Laurent expansion around λ_0

$$\Delta(\lambda)^{-1} = \sum_{n=-\widehat{k}_0}^{+\infty} (\lambda - \lambda_0)^n \Delta_n. \tag{3.10}$$

From the following lemma we know that $\widehat{k}_0 = k_0$.

Lemma 3.2 *Let $\lambda_0 \in \sigma(A + L + \widehat{L})$. Then the following statements are equivalent:*

(i) *λ_0 is a pole of order k_0 of $(\lambda I - (A + L + \widehat{L}))^{-1}$.*
(ii) *λ_0 is a pole of order k_0 of $\Delta(\lambda)^{-1}$.*
(iii) *$\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{k_0}\Delta(\lambda)^{-1} \neq 0$ and $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{k_0+1}\Delta(\lambda)^{-1} = 0$.*

Proof The proof follows from the explicit formula of the resolvent of $A + L + \widehat{L}$ obtained in Lemma 3.1. ∎

Lemma 3.3 *The matrices $\Delta_{-1}, \dots, \Delta_{-k_0}$ in (3.10) satisfy*

$$\Delta_{k_0}(\lambda_0)\begin{pmatrix}\Delta_{-1}\\ \Delta_{-2}\\ \vdots\\ \Delta_{-k_0+1}\\ \Delta_{-k_0}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

and $\begin{pmatrix}\Delta_{-k_0} & \Delta_{-k_0+1} & \cdots & \Delta_{-2} & \Delta_{-1}\end{pmatrix}\Delta_{k_0}(\lambda_0) = \begin{pmatrix}0 & \cdots & 0\end{pmatrix}$, *where*

$$\Delta_{k_0}(\lambda_0) = \begin{pmatrix}\Delta(\lambda_0) & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2! & \cdots & & \Delta^{(k_0-1)}(\lambda_0)/(k_0-1)!\\ 0 & \Delta(\lambda_0) & \Delta^{(1)}(\lambda_0) & \cdots & & \Delta^{(k_0-2)}(\lambda_0)/(k_0-2)!\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ & & & & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2!\\ & & & & \Delta(\lambda_0) & \Delta^{(1)}(\lambda_0)\\ 0 & 0 & 0 & \cdots & 0 & \Delta(\lambda_0)\end{pmatrix}.$$

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Proof We have

$$(\lambda-\lambda_0)^{k_0} I=\Delta(\lambda)\Big(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\Big)=\Big(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\Big)\Delta(\lambda).$$

Hence,

$$\begin{aligned}(\lambda-\lambda_0)^{k_0} I&=\Big(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\frac{\Delta^{(n)}(\lambda_0)}{n!}\Big)\Big(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\Big)\\&=\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}\end{aligned}$$

and

$$(\lambda-\lambda_0)^{k_0} I=\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}.$$

By the uniqueness of Taylor's expansion for analytic maps, we obtain that for $n\in\{0,\dots,k_0-1\}$,

$$0=\sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}=\sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}.$$

Therefore, the result follows. ∎

Now we look for an explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated with λ_0. Set

$$\begin{aligned}\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}&:=(\lambda I-B)\Big(L_1\Big(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\Big)+\alpha\Big)\\&\qquad+L_2\Big(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\Big)+y,\\ \Psi_2(\lambda)(\varphi)(\theta)&:=\int_\theta^0 e^{\lambda(\theta-l)}\varphi(l)\,dl,\end{aligned}$$

and

$$\Psi_3(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}:=L_1\Big(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\Big)+\alpha.$$

Then all maps are analytic and

$$
(3.11)\quad (\lambda I - (A + L + \widehat{L}))^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}
= \begin{pmatrix} 0_{\mathbb{R}^n} \\ e^{\lambda\,\cdot}\,\Delta(\lambda)^{-1}\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} + \Psi_2(\lambda)(\varphi)(\,\cdot\,) \\ (I - L_1(e^{\lambda\,\cdot}\, I))\left(\Delta(\lambda)^{-1}\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\right) - \Psi_3(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\end{pmatrix}.
$$

We observe that the only singularity in (3.11) is $\Delta(\lambda)^{-1}$. Since Ψ_1, Ψ_2, and Ψ_3 are analytic, we have for $j = 1, 2, 3$ that

$$
(3.12)\qquad \Psi_j(\lambda) = \sum_{n=0}^{+\infty} \frac{(\lambda - \lambda_0)^n}{n!} L_n^j(\lambda_0),
$$

where $|\lambda - \lambda_0|$ is small enough and $L_n^j(\,\cdot\,) := \frac{d^n\Psi_j(\,\cdot\,)}{d\lambda^n}, \forall n \geq 0, \forall j = 1, 2, 3$. Hence we get

$$
\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}[(\lambda-\lambda_0)^{k_0}\Psi_i(\lambda)]
= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\sum_{n=0}^{+\infty}\frac{(n+k_0)!}{(n+1)!}\frac{(\lambda-\lambda_0)^{n+1}}{n!}L_n^i(\lambda_0) = 0, i = 2, 3.
$$

From (3.10) and (3.12) we obtain

$$
\begin{aligned}
&\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}[(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\Psi_1(\lambda)]\\
&\quad= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\Big[\Big(\sum_{n=-k_0}^{+\infty}(\lambda-\lambda_0)^{n+k_0}\Delta_n\Big)\Big(\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^1(\lambda_0)\Big)\Big]\\
&\quad= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\Big[\Big(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^{n}\Delta_{n-k_0}\Big)\Big(\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^1(\lambda_0)\Big)\Big]\\
&\quad= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\Big[\sum_{n=0}^{+\infty}\sum_{j=0}^{n}(\lambda-\lambda_0)^{n-j}\Delta_{n-j-k_0}\frac{(\lambda-\lambda_0)^j}{j!}L_j^1(\lambda_0)\Big]
\end{aligned}
$$

$$= \lim_{\lambda \to \lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \Big[\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^n \sum_{j=0}^{n} \Delta_{n-j-k_0} \frac{1}{j!} L_j^1(\lambda_0) \Big]$$

$$= \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^1(\lambda_0)$$

and

$$\lim_{\lambda \to \lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} [e^{\lambda\theta} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1} \Psi_1(\lambda)]$$

$$= \lim_{\lambda \to \lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \Big[e^{\lambda\theta} \Big(\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^n \sum_{j=0}^{n} \Delta_{n-j-k_0} \frac{1}{j!} L_j^1(\lambda_0) \Big) \Big]$$

$$= \sum_{i=0}^{k_0-1} \frac{1}{i!} \theta^i e^{\lambda_0 \theta} \sum_{j=0}^{k_0-1-i} \frac{1}{j!} \Delta_{-1-j-i} L_j^1(\lambda_0).$$

From the above results we can obtain the explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated with λ_0, which is given in the following proposition.

Proposition 3.4 *Each $\lambda_0 \in \sigma(A+L+\widehat{L})$ is a pole of $(\lambda I - (A+L+\widehat{L}))^{-1}$ of order $k_0 \geq 1$. Moreover, k_0 is the only integer such that there exists $\Delta_{-k_0} \in M_n(\mathbb{R})$ with $\Delta_{-k_0} \neq 0$, such that*

$$\Delta_{-k_0} = \lim_{\lambda \to \lambda_0} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1}.$$

Furthermore, the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace of $(A+L+\widehat{L})$ associated with λ_0 is defined by the following formula

$$B_{-1}^{\lambda_0} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^1(\lambda_0) \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} - L_1(\widehat{\varphi}) \end{pmatrix},$$

where

$$\widehat{\varphi}(\theta) = \sum_{i=0}^{k_0-1} \theta^i e^{\lambda_0\theta} \frac{1}{i!} \sum_{j=0}^{k_0-1-i} \frac{1}{j!} \Delta_{-1-j-i} L_j^1(\lambda_0) \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix},$$

$$\Delta_{-j} = \lim_{\lambda \to \lambda_0} \frac{1}{(k_0-j)!} \frac{d^{k_0-j}}{d\lambda^{k_0-j}} \big((\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1} \big), \; j = 1, \dots, k_0,$$

$$L_0^1(\lambda) \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = (\lambda I - B) \Big(L_1 \Big(\int_{\cdot}^{0} e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + \alpha \Big) + L_2 \Big(\int_{\cdot}^{0} e^{\lambda(\cdot - l)} \varphi(l) dl \Big) + y,$$

and

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$$
\begin{aligned}
L_j^1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} &= \frac{d^j}{d\lambda^j}\left[L_0^1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\right]\\
&= (\lambda I - B)L_1\Big(\int_\cdot^0 (\cdot - l)^j e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big)\\
&\quad + j\frac{d^{j-1}}{d\lambda^{j-1}}\Big[L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big) + \alpha\Big]\\
&\quad + L_2\Big(\int_\cdot^0 (\cdot - l)^j e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big), \qquad j \geq 1.
\end{aligned}
$$

Here

$$
\frac{d^i}{d\lambda^i}\Big[L_1\Big(\int_\cdot^0 e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big) + \alpha\Big] = L_1\Big(\int_\cdot^0 (\cdot - l)^i e^{\lambda(\cdot - l)}\varphi(l)\,dl\Big), \quad i \geq 1.
$$

In studying Hopf bifurcation it is usually necessary to consider the projector for a simple eigenvalue. In the following we consider the case when λ_0 is a simple eigenvalue of $(A+L+\widehat{L})$. That is, λ_0 is a pole of order 1 of the resolvent of $(A+L+\widehat{L})$ and the dimension of the eigenspace of $(A+L+\widehat{L})$ associated with the eigenvalue λ_0 is 1.

We know that λ_0 is a pole of order 1 of the resolvent of $(A+L+\widehat{L})$ if and only if there exists $\Delta_{-1} \neq 0$, such that $\Delta_{-1} = \lim_{\lambda\to\lambda_0} (\lambda - \lambda_0)\,\Delta(\lambda)^{-1}$. From Lemma 3.3, we have $\Delta_{-1}\Delta(\lambda_0) = \Delta(\lambda_0)\Delta_{-1} = 0$. From the proof of Lemma 2.9, it can be checked that λ_0 is simple if and only if $\dim[N(\Delta(\lambda_0))] = 1$. In that case, there exist $V_{\lambda_0}, W_{\lambda_0} \in \mathbb{C}^n \setminus \{0\}$ such that

$$
W_{\lambda_0}^T\Delta(\lambda_0) = 0 \quad \text{and} \quad \Delta(\lambda_0)V_{\lambda_0} = 0. \tag{3.13}
$$

Hence, we can always assume that (replacing $V_{\lambda_0}W_{\lambda_0}^T$ by $\delta V_{\lambda_0}W_{\lambda_0}^T$ for some $\delta \neq 0$ if necessary) $\Delta_{-1} = V_{\lambda_0}W_{\lambda_0}^T$. We can see that $B_{-1}^{\lambda_0}B_{-1}^{\lambda_0} = B_{-1}^{\lambda_0}$ if and only if

$$
\Delta_{-1} = \Delta_{-1}\Big[I - L_1(e^{\lambda_0\cdot}) + (\lambda_0 I - B)L_1\Big(\int_\cdot^0 e^{\lambda_0\cdot}\,dl\Big) + L_2\Big(\int_\cdot^0 e^{\lambda_0\cdot}\,dl\Big)\Big]\Delta_{-1}.
$$

Therefore, we obtain the following corollary.

Corollary 3.5 *$\lambda_0 \in \sigma(A+L+\widehat{L})$ is a simple eigenvalue of $(A+L+\widehat{L})$ if and only if*

$$
\lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^2\Delta(\lambda)^{-1} = 0 \quad \textit{and} \quad \dim[N(\Delta(\lambda_0))] = 1.
$$

Moreover, the projector on the eigenspace associated with λ_0 is

$$
B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n}\\ e^{\lambda_0\cdot}\Delta_{-1}L_0^1(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\\ \Delta_{-1}L_0^1(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} - L_1\left(e^{\lambda_0\cdot}\Delta_{-1}L_0^1(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\right)\end{pmatrix},
$$

where

$$L_0^1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = (\lambda I - B)\Big(L_1\Big(\int_{\cdot}^{0} e^{\lambda(\,\cdot\,-l)}\varphi(l)dl\Big) + \alpha\Big) + L_2\Big(\int_{\cdot}^{0} e^{\lambda(\cdot-l)}\varphi(l)\,dl\Big) + y$$

and $\Delta_{-1} = V_{\lambda_0}W_{\lambda_0}^T$ *in which* $V_{\lambda_0}, W_{\lambda_0} \in \mathbb{C}^n \setminus \{0\}$ *are two vectors satisfying* (3.13) *and*

$$\Delta_{-1} = \Delta_{-1}\Big[I - L_1\left(e^{\lambda_0\cdot}\right) + (\lambda_0 I - B)L_1\Big(\int_{\cdot}^{0} e^{\lambda_0\cdot}\,dl\Big) + L_2\Big(\int_{\cdot}^{0} e^{\lambda_0\cdot}\,dl\Big)\Big]\Delta_{-1}.$$

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