ON MAXIMAL SPACELIKE HYPERSURFACES
IN A LORENTZIAN MANIFOLD

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ABSTRACT. We prove a Bernstein-type property for maximal spacelike hypersurfaces in a Lorentzian manifold.

§1. Introduction

The object of this note is to prove the following

THEOREM A. Let $N$ be a Lorentzian manifold satisfying the strong energy condition. Let $M$ be a complete maximal spacelike hypersurface in $N$. Suppose that $N$ is locally symmetric and has nonnegative spacelike sectional curvature. Then $M$ is totally geodesic.

For the terminology in the theorem, see Section 2.

It has been proved by Calabi [2] (for $n < 4$) and Cheng-Yau [4] (for all $n$) that a complete maximal spacelike hypersurface in the flat Minkowski $(n + 1)$-space $L^{n+1}$ is totally geodesic. In particular, the only entire nonparametric maximal spacelike hypersurfaces in $L^{n+1}$ are spacelike hyperplanes. This is remarkable since the Euclidean counterpart, the Bernstein theorem, holds only for $n \leq 7$: the entire nonparametric minimal hypersurfaces in the Euclidean space $R^{n+1}$, $n \leq 7$, are hyperplanes (cf. [8]).

Theorem A implies, for instance, that a complete maximal spacelike hypersurface in the Einstein static universe is totally geodesic. In the proof of Theorem A, a refinement of a Bernstein-type theorem of Choquet-Bruhat [5, 6] will be also given.

§2. Definitions

First we set up our terminology and notation. Let $N = (N, \bar{g})$ be a

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Lorentzian manifold with Lorentzian metric $g$ of signature $(-, +, \cdots, +)$. $N$ has a uniquely defined torsion-free affine connection $\nabla$ compatible with the metric $g$. $N$ is said to satisfy the strong energy condition (the timelike convergence condition in Hawking-Ellis [7]) if the Ricci curvature $\text{Ric}$ of $N$ is positive semidefinite for all timelike vectors, that is, if $\text{Ric} (v, v) \geq 0$ for every timelike vector $v \in TN$ (cf. [1, 6]). $N$ is called locally symmetric if the curvature tensor $\text{R}$ of $N$ is parallel, that is, $\nabla \text{R} = 0$. We say that $N$ has nonnegative spacelike sectional curvature if the sectional curvature $K(\mathbf{u}, \mathbf{v})$ of $N$ is nonnegative for every nondegenerate tangent 2-plane spanned by spacelike vectors $\mathbf{u}, \mathbf{v} \in TN$.

Let $M$ be a hypersurface immersed in $N$. $M$ is said to be spacelike if the Lorentzian metric $g$ of $N$ induces a Riemannian metric $g$ on $M$. For a spacelike $M$ there is naturally defined the second fundamental form (the extrinsic curvature) $S$ of $M$. $M$ is called maximal spacelike if the mean (extrinsic) curvature $H = \text{Tr} S$, the trace of $S$, of $M$ vanishes identically. $M$ is maximal spacelike if and only if it is extremal under the variations, with compact support through spacelike hypersurfaces, for the induced volume. $M$ is said to be totally geodesic (a moment of time symmetry) if the second fundamental form $S$ vanishes identically.

§3. Local formulas

Let $M$ be a spacelike hypersurface in a Lorentzian $(n + 1)$-manifold $N = (N, g)$. We choose a local field of Lorentz orthonormal frames $e_0, e_1, \cdots, e_n$ in $N$ such that, restricted to $M$, the vectors $e_1, \cdots, e_n$ are tangent to $M$. Let $\omega_0, \omega_1, \cdots, \omega_n$ be its dual frame field so that the Lorentzian metric $g$ can be written as $g = -\omega_0^2 + \sum_i \omega_i^2$. Then the connection forms $\omega_{\alpha \beta}$ of $N$ are characterized by the equations

\[
\begin{align*}
\text{d} \omega_i &= - \sum_k \omega_{ik} \wedge \omega_k + \omega_{i0} \wedge \omega_0, \\
\text{d} \omega_0 &= - \sum_k \omega_{0k} \wedge \omega_k,
\end{align*}
\]

(1)

The curvature forms $\bar{\Omega}_{\alpha \beta}$ of $N$ are given by

\[
\begin{align*}
\bar{\Omega}_{ij} &= \text{d} \omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} - \omega_{ij0} \wedge \omega_{0j}, \\
\bar{\Omega}_{0i} &= \text{d} \omega_{0i} + \sum_k \omega_{0k} \wedge \omega_{ki},
\end{align*}
\]

(2)

\[^{*}\text{We shall use the summation convention with Roman indices in the range } 1 \leq i, j, \cdots \leq n \text{ and Greek in } 0 \leq \alpha, \beta, \cdots \leq n.\]
and we have

\[ \bar{\Omega}_{\alpha\beta} = \frac{1}{2} \sum_{i,j} \bar{R}_{\alpha\beta ij} \omega_i \wedge \omega_j , \]

where \( \bar{R}_{\alpha\beta ij} \) are components of the curvature tensor \( \bar{R} \) of \( N \).

We restrict these forms to \( M \). Then

\[ \omega_0 = 0 , \]

and the induced Riemannian metric \( g \) of \( M \) is written as \( g = \sum_i \omega_i^2 \). From formulas (1)-(4), we obtain the structure equations of \( M \)

\[ d\omega_i = - \sum_k \omega_{ik} \wedge \omega_k , \quad \omega_i + \omega_{ij} = 0 , \]

\[ d\omega_{ij} = - \sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{io} \wedge \omega_{oj} + \bar{\Omega}_{ij} , \]

\[ \Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \frac{1}{2} \sum_{k,l} \bar{R}_{ijkl} \omega_k \wedge \omega_l , \]

where \( \Omega_{ij} \) and \( \bar{R}_{ijkl} \) denote the curvature forms and the components of the curvature tensor \( R \) of \( M \), respectively. We can also write

\[ \omega_{ij} = \sum_k h_{ij} \omega_k , \]

where \( h_{ij} \) are components of the second fundamental form \( S = \sum_{i,j} h_{ij} \omega_i \wedge \omega_j \) of \( M \). Using (6) in (5) then gives the Gauss formula

\[ R_{ijkl} = \bar{R}_{ijkl} - (h_{ik} h_{jl} - h_{il} h_{jk}) . \]

Let \( h_{ijk} \) denote the covariant derivative of \( h_{ij} \) so that

\[ \sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{ik} \omega_{kj} - \sum_k h_{ik} \omega_{kj} . \]

Then, by exterior differentiating (6), we obtain the Coddazi equation

\[ h_{ijk} - h_{ikj} = \bar{R}_{0ijk} . \]

Next, exterior differentiate (8) and define the second covariant derivative of \( h_{ij} \) by

\[ \sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{ijk} \omega_{ik} - \sum_k h_{ik} \omega_{ij} - \sum_k h_{ij} \omega_{ik} . \]

Then we obtain the Ricci formula

\[ h_{ijk} - h_{ijk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl} . \]

Let us now denote the covariant derivative of \( \bar{R}_{\alpha\beta\gamma\delta} \), as a curvature tensor of \( N \), by \( \bar{R}_{\alpha\beta\gamma\delta ; t} \). Then restricting on \( M \), \( \bar{R}_{0ijk; t} \) is given by

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(11) \[ \bar{R}_{ijkl; \ell} = \bar{R}_{ijkl} - \bar{R}_{i0jk}h_{\ell} - \bar{R}_{0ij0}h_{k\ell} - \sum_m \bar{R}_{mj0k}h_{m\ell}, \]

where \( \bar{R}_{ijkl} \) denote the covariant derivative of \( \bar{R}_{ijkl} \) as a tensor on \( M \) so that

\[ \sum_{\ell} \bar{R}_{ijkl; \ell} \omega_{\ell} = d\bar{R}_{ijkl} - \sum_{\ell} \bar{R}_{ijkl} \omega_{i\ell} - \sum_{\ell} \bar{R}_{0ljk} \omega_{l\ell} - \sum_{\ell} \bar{R}_{0ijl} \omega_{k\ell}. \]

The Laplacian \( \Delta h_{ij} \) of the second fundamental form \( h_{ij} \) is defined by

\[ \Delta h_{ij} = \sum_k h_{ij;kk}. \]

From (9) we then obtain

\[ \Delta h_{ij} = \sum_k h_{kij} + \sum_k \bar{R}_{ijkl; kk}, \]

and from (10)

\[ h_{kij} = h_{kij} + \sum_m h_{mi}R_{mkjk} + \sum_m h_{km}R_{mijk}. \]

Replace \( h_{kij} \) in (13) by \( h_{kij} + \bar{R}_{0kij} \) (by (9)) and substitute the right hand side of (13) into \( h_{kij} \) in (12). Then we obtain

\[ \Delta h_{ij} = \sum_k (h_{kij} + \bar{R}_{0kij} + \bar{R}_{ijkl; kk}) + \sum_k (\sum_m h_{mi}R_{mkjk} + \sum_m h_{km}R_{mijk}). \]

From (7), (11) and (14) we then obtain

\[ \Delta h_{ij} = \sum_k h_{kij} + \sum_k \bar{R}_{0kij} + \sum_k \bar{R}_{ijkl; kk} \]
\[ + \sum_k (h_{kij} \bar{R}_{0ij; j} + h_{ij} \bar{R}_{0ij; k}) \]
\[ + \sum_m (h_{mj} \bar{R}_{mkjk} + 2h_{mk} \bar{R}_{mijk} + h_{mk} \bar{R}_{mkjk}) \]
\[ - \sum_{m,k} (h_{mk} h_{mk} + h_{mk} h_{mk} h_{k} - h_{mk} h_{mk} h_{k} - h_{mk} h_{mk} h_{k}). \]

Now we assume that \( N \) is locally symmetric, that is, \( \bar{R}_{ij; k} = 0 \) and that \( M \) is maximal in \( N \), so that \( \sum_k h_{kk} = 0 \). Then, from (15) we obtain

\[ \sum_{i,j} h_{ij} \Delta h_{ij} = \sum_{i,j,k} h_{ij}^2 \bar{R}_{0ijk} + \sum_{i,j,k,m} 2(h_{ij} h_{mj} \bar{R}_{mkjk} + h_{ij} h_{mk} \bar{R}_{mijk}) \]
\[ + (\sum_{i,j} h_{ij}^2)^2. \]

\[^{10} \text{This is the Lorentzian version of the well-known formula established, for example, in [8].}\]
§ 4. Proof of Theorem A

Theorem A is an immediate consequence of the following

**Theorem B.** Let \(N = (N, g)\) be a locally symmetric Lorentzian \((n + 1)\)-manifold and \(M\) be a complete maximal spacelike hypersurface in \(N\). Assume that there exist constants \(c_1, c_2\) such that

1. \(\text{Ric}(v, v) \geq c_1\) for all timelike vectors \(v \in TN\),
2. \(K(u \wedge v) \geq c_2\) for all nondegenerate tangent 2-planes spanned by spacelike vectors \(u, v \in TN\), and
3. \(c_1 + 2nc_2 \geq 0\).

Then \(M\) is totally geodesic.

To prove Theorem B, we first note

**Lemma 1.** Under the assumptions of Theorem B,

\[
\frac{1}{2}A(\sum_{i,j} h^2_{ij}) \geq (\sum_{i,j} h^2_{ij})^2.
\]

**Proof.** For any point \(p \in M\), we may choose our frame \(\{e_i, \ldots, e_n\}\) at \(p\) so that \(h_{ij} = \lambda_i \delta_{ij}\). Then, by assumption (ii) of Theorem B, we have at \(p\)

\[
\sum_{i,j,k,m} 2(h_{ij}h_{km}R_{mek} + h_{ij}h_{mk}R_{mik}) = \sum_{i,k} 2(\lambda_i R_{ike} + \lambda_k R_{kle}) = \sum_{i,k} (\lambda_i - \lambda_k)^2 R_{ike} \geq c_2 \sum_{i,k} (\lambda_i - \lambda_k)^2 = 2c_2(n \sum_i \lambda_i^2 - \sum_i \lambda_i^2) = 2nc_2 \sum_{i,j} h^2_{ij}.
\]

Also we have by assumption (i)

\[
\sum_k R_{kkkk} \geq c_1.
\]

It then follows from (16) and assumption (iii) that

\[
\frac{1}{2}A(\sum_{i,j} h^2_{ij}) = \sum_{i,j,k} h^2_{ijk} + \sum_{i,j} h_{ij} \Delta h_{ij} \geq (c_1 + 2nc_2)(\sum_{i,j} h^2_{ij}) + (\sum_{i,j} h^2_{ij})^2 \geq (\sum_{i,j} h^2_{ij})^2.
\]

Let \(u = \sum_{i,j} h^2_{ij}\) be the squared of the length of the second fundamental form of \(M\). The proof of Theorem B is complete if we show that \(u\) vanishes identically. Recall that from (17), \(u\) satisfies
\( \Delta u \geq 2u^2. \)

Then, by the maximum principle, the result is immediate provided \( M \) is compact.

We now assume that \( M \) is noncompact and complete. We will modify the maximum principle argument as in [4]. Take a point \( p \in M \), and let \( r \) denote the geodesic distance on \( M \) from \( p \) with respect to the induced Riemannian metric. For \( a > 0 \), let \( B_a(p) = \{ x \in M \mid r(x) < a \} \) be the geodesic ball of radius \( a \) and center \( p \).

**Lemma 2.** For any \( a > 0 \), there exists a constant \( c \) depending only on \( n \) such that

\[
(19) \quad u(x) \leq \frac{ca^2(1 + |c|^{1/2})}{(a^2 - r(x)^2)^{1/2}}
\]

for all \( x \in B_a(p) \).

**Proof.** Assuming that \( u \) is not identically zero on \( B_a(p) \), we consider the function

\[
f(x) = (a^2 - r(x)^2)u(x), \quad x \in B_a(p).
\]

Then \( f \) attains a nonzero maximum at some point \( q \in B_a(p) \), for the closure of \( B_a(p) \) is compact since \( M \) is complete. As in [§2, 3], we may assume that \( f \) is \( C^2 \) around \( q \). Then we have

\[
\nabla f(q) = 0, \quad \Delta f(q) \leq 0.
\]

Hence at \( q \)

\[
\frac{\nabla u}{u} = \frac{4rVr}{a^2 - r^2},
\]

\[
\frac{\Delta u}{u} \leq \frac{\nabla u^2}{u^2} + \frac{8r^2}{(a^2 - r^2)^2} + \frac{4(1 + r\Delta r)}{a^2 - r^2},
\]

from which we obtain

\[
(20) \quad \frac{\Delta u}{u}(q) \leq \frac{24r^2}{(a^2 - r^2)^2}(q) + \frac{4(1 + r\Delta r)}{a^2 - r^2}(q).
\]

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\[\text{(20)}\]

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from which we obtain

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(20) \quad \frac{\Delta u}{u}(q) \leq \frac{24r^2}{(a^2 - r^2)^2}(q) + \frac{4(1 + r\Delta r)}{a^2 - r^2}(q).
\]
(21) \[ \Delta r(q) \leq \min_{\sigma \in \Gamma(p,q)} \left[ \frac{n-1}{r(q) - k} - \frac{1}{(r(q) - k)^2} \int_{\sigma(t)}^{r(q)} \left( t - k \right)^2 \text{Ric} \left( \dot{\sigma}(t), \dot{\sigma}(t) \right) dt \right], \]

where \( \dot{\sigma}(t) \) is the tangent vector of the minimizing geodesic \( \sigma : [0, r(q)] \to M \) from \( p \) to \( q \) and \( \text{Ric} \) denote the Ricci curvature of \( M \). Also, from (7) and assumption (ii) of Theorem B, \( \text{Ric} \left( \dot{\sigma}(t), \dot{\sigma}(t) \right) \) is bounded from below by

(22) \[ \text{Ric} \left( \dot{\sigma}(t), \dot{\sigma}(t) \right) \geq (n - 1)c, \]

since \( M \) is maximal spacelike. From (21) and (22) we then obtain

(23) \[ r\Delta r(q) \leq (n - 1) + 2(n - 1)|c_i|^2 r(q). \]

It follows from (20) and (23) that

\[ (a^2 - r(q)^2)u^{-1} du(q) \leq 24a^2 + 8na^2(1 + |c_i|^2 a). \]

From (18) we then obtain

\[ f(q) = (a^2 - r(q)^2)u(q) \leq ca^2(1 + |c_i|^2 a), \]

c being a constant depending only on \( n \). Since \( q \) is the maximum point of \( f \) in \( B_a(p) \), this implies that

(24) \[ (a^2 - r(x)^2)u(x) \leq ca^2(1 + |c_i|^2 a) \]

for all \( x \in B_a(p) \).

Since \( M \) is complete, we may fix \( x \) in Inequality (19) and let \( a \) tend to infinity. Then we obtain \( u(x) = 0 \) for all \( x \in M \). This completes the proof of Theorem B.

**Remark.** Let \( N = L^{k+1} \times S^{n-k} \) be the product Lorentzian manifold of the flat Minkowski \((k+1)\)-space \( L^{k+1}, 1 \leq k \leq n \), and \( S^{n-k}, \) a Riemannian \((n-k)\)-manifold of positive constant curvature. Then \( N \) satisfies the assumptions of Theorem A. The Einstein static space \( N = (R, -dt^2) \times S^n \) also satisfies these assumptions.

The Lorentzian \((n+1)\)-manifold \( S_i^{n+1} \) of constant curvature \( c > 0 \), called the de Sitter space, satisfies the assumptions of Theorem B (with \( c_i = -cn, c_i = c \)). Theorem B then gives a refinement of a theorem of Choquet-Bruhat [Theorem 4.6, 6].
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