ON MAXIMAL SPACELIKE HYPERSURFACES 
IN A LORENTZIAN MANIFOLD

SEIKI NISHIKAWA

ABSTRACT. We prove a Bernstein-type property for maximal spacelike hypersurfaces in a Lorentzian manifold.

§1. Introduction

The object of this note is to prove the following

THEOREM A. Let $N$ be a Lorentzian manifold satisfying the strong energy condition. Let $M$ be a complete maximal spacelike hypersurface in $N$. Suppose that $N$ is locally symmetric and has nonnegative spacelike sectional curvature. Then $M$ is totally geodesic.

For the terminology in the theorem, see Section 2.

It has been proved by Calabi [2] (for $n < 4$) and Cheng-Yau [4] (for all $n$) that a complete maximal spacelike hypersurface in the flat Minkowski $(n + 1)$-space $L^{n+1}$ is totally geodesic. In particular, the only entire nonparametric maximal spacelike hypersurfaces in $L^{n+1}$ are spacelike hyperplanes. This is remarkable since the Euclidean counterpart, the Bernstein theorem, holds only for $n \leq 7$: the entire nonparametric minimal hypersurfaces in the Euclidean space $R^{n+1}$, $n \leq 7$, are hyperplanes (cf. [8]).

Theorem A implies, for instance, that a complete maximal spacelike hypersurface in the Einstein static universe is totally geodesic. In the proof of Theorem A, a refinement of a Bernstein-type theorem of Choquet-Bruhat [5, 6] will be also given.

§2. Definitions

First we set up our terminology and notation. Let $N = (\overline{N}, \overline{g})$ be a

---

S. Nishikawa
Nagoya Math. J.
Vol. 95 (1984), 117-124

On Maximal Spacelike Hypersurfaces
In a Lorentzian Manifold

Seiki Nishikawa

Abstract. We prove a Bernstein-type property for maximal spacelike hypersurfaces in a Lorentzian manifold.

§1. Introduction

The object of this note is to prove the following

Theorem A. Let $N$ be a Lorentzian manifold satisfying the strong energy condition. Let $M$ be a complete maximal spacelike hypersurface in $N$. Suppose that $N$ is locally symmetric and has nonnegative spacelike sectional curvature. Then $M$ is totally geodesic.

For the terminology in the theorem, see Section 2.

It has been proved by Calabi [2] (for $n \leq 4$) and Cheng-Yau [4] (for all $n$) that a complete maximal spacelike hypersurface in the flat Minkowski $(n + 1)$-space $L^{n+1}$ is totally geodesic. In particular, the only entire nonparametric maximal spacelike hypersurfaces in $L^{n+1}$ are spacelike hyperplanes. This is remarkable since the Euclidean counterpart, the Bernstein theorem, holds only for $n \leq 7$: the entire nonparametric minimal hypersurfaces in the Euclidean space $R^{n+1}$, $n \leq 7$, are hyperplanes (cf. [8]).

Theorem A implies, for instance, that a complete maximal spacelike hypersurface in the Einstein static universe is totally geodesic. In the proof of Theorem A, a refinement of a Bernstein-type theorem of Choquet-Bruhat [5, 6] will be also given.

§2. Definitions

First we set up our terminology and notation. Let $N = (\overline{N}, \overline{g})$ be a
Lorentzian manifold with Lorentzian metric \( \bar{g} \) of signature \((- , +, \cdots, +)\). \( N \) has a uniquely defined torsion-free affine connection \( \nabla \) compatible with the metric \( \bar{g} \). \( N \) is said to satisfy the strong energy condition (the timelike convergence condition in Hawking-Ellis [7]) if the Ricci curvature \( \bar{Ric} \) of \( N \) is positive semidefinite for all timelike vectors, that is, if \( \bar{Ric}(u, v) \geq 0 \) for every timelike vector \( v \in TN \) (cf. [1, 6]). \( N \) is called locally symmetric if the curvature tensor \( \bar{R} \) of \( N \) is parallel, that is, \( \bar{R} = 0 \). We say that \( N \) has nonnegative spacelike sectional curvature if the sectional curvature \( \bar{K}(u \wedge v) \) of \( N \) is nonnegative for every nondegenerate tangent 2-plane spanned by spacelike vectors \( u, v \in TN \).

Let \( M \) be a hypersurface immersed in \( N \). \( M \) is said to be spacelike if the Lorentzian metric \( \bar{g} \) of \( N \) induces a Riemannian metric \( g \) on \( M \). For a spacelike \( M \) there is naturally defined the second fundamental form (the extrinsic curvature) \( S \) of \( M \). \( M \) is called maximal spacelike if the mean (extrinsic) curvature \( H = \text{Tr} S \), the trace of \( S \), of \( M \) vanishes identically. \( M \) is maximal spacelike if and only if it is extremal under the variations, with compact support through spacelike hypersurfaces, for the induced volume. \( M \) is said to be totally geodesic (a moment of time symmetry) if the second fundamental form \( S \) vanishes identically.

§3. Local formulas

Let \( M \) be a spacelike hypersurface in a Lorentzian \((n + 1)\)-manifold \( N = (N, \bar{g}) \). We choose a local field of Lorentz orthonormal frames \( e_0, e_1, \ldots, e_n \) in \( N \) such that, restricted to \( M \), the vectors \( e_0, \ldots, e_n \) are tangent to \( M \). Let \( \omega_0, \omega_1, \ldots, \omega_n \) be its dual frame field so that the Lorentzian metric \( \bar{g} \) can be written as \( \bar{g} = -\omega_0^2 + \sum_{i} \omega_i^2 \).\(^*\) Then the connection forms \( \omega_{\alpha \beta} \) of \( N \) are characterized by the equations

\[
\begin{align*}
\omega_{i0} &= - \sum_k \omega_{ik} \wedge \omega_k + \omega_{i0} \wedge \omega_0, \\
\omega_0 &= - \sum_k \omega_{0k} \wedge \omega_k, \quad \omega_{\alpha \beta} + \omega_{\beta \alpha} = 0.
\end{align*}
\]

The curvature forms \( \bar{\Omega}_{\alpha \beta} \) of \( N \) are given by

\[
\begin{align*}
\bar{\Omega}_{ij} &= \omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} - \omega_{i0} \wedge \omega_{0j}, \\
\bar{\Omega}_{0i} &= \omega_{0i} + \sum_k \omega_{0k} \wedge \omega_{ki}.
\end{align*}
\]

\(^*\) We shall use the summation convention with Roman indices in the range \( 1 \leq i, j, \cdots \leq n \) and Greek in \( 0 \leq \alpha, \beta, \cdots \leq n \).
and we have

\[
\bar{\Omega}_{a\beta} = \frac{1}{2} \sum_{i,\delta} \bar{R}_{a\beta\delta} \omega_i \wedge \omega_\delta,
\]

where \(\bar{R}_{a\beta\delta}\) are components of the curvature tensor \(\bar{R}\) of \(N\).

We restrict these forms to \(M\). Then

\[
\omega_0 = 0,
\]

and the induced Riemannian metric \(g\) of \(M\) is written as \(g = \sum_i \omega_i^2\). From formulas (1)-(4), we obtain the structure equations of \(M\)

\[
\begin{align*}
d\omega_i &= -\sum_k \omega_{ik} \wedge \omega_k, \quad \omega_{ij} + \omega_{ji} = 0, \\
d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{i0} \wedge \omega_{0j} + \bar{\Omega}_{ij}, \\
\Omega_{ij} &= d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \frac{1}{2} \sum_{k,l} \bar{R}_{ljk} \omega_k \wedge \omega_l,
\end{align*}
\]

where \(\Omega_{ij}\) and \(\bar{R}_{ljk}\) denote the curvature forms and the components of the curvature tensor \(\bar{R}\) of \(M\), respectively. We can also write

\[
\omega_{i0} = \sum_j h_{ij} \omega_j,
\]

where \(h_{ij}\) are components of the second fundamental form \(S = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j\) of \(M\). Using (6) in (5) then gives the Gauss formula

\[
R_{ijkl} = \bar{R}_{ijkl} - (h_{ik} h_{lj} - h_{ij} h_{lk}).
\]

Let \(h_{ijk}\) denote the covariant derivative of \(h_{ij}\) so that

\[
\sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{ik} \omega_{kj} - \sum_k h_{ik} \omega_{kj}.
\]

Then, by exterior differentiating (6), we obtain the Coddazi equation

\[
h_{ijk} - h_{ikj} = \bar{R}_{0ijk}.
\]

Next, exterior differentiate (8) and define the second covariant derivative of \(h_{ij}\) by

\[
\sum_{\ell} h_{ij\ell} \omega_{\ell} = dh_{ij} - \sum_{\ell} h_{i\ell j} \omega_{\ell} - \sum_{\ell} h_{i\ell j} \omega_{\ell j} - \sum_{\ell} h_{ij\ell} \omega_{\ell j}.
\]

Then we obtain the Ricci formula

\[
h_{ij\ell} - h_{ij\ell} = \sum_m h_{mj} R_{m\ell j\ell} + \sum_m h_{im} R_{m\ell j\ell}.
\]

Let us now denote the covariant derivative of \(\bar{R}_{a\beta\delta}\), as a curvature tensor of \(N\), by \(\bar{R}_{a\beta\delta;\ell}\). Then restricting on \(M\), \(\bar{R}_{0ijk\ell}\) is given by
where \( R_{\alpha jk} \) denote the covariant derivative of \( R_{\alpha jk} \) as a tensor on \( M \) so that
\[
\sum_\ell \frac{\partial}{\partial x^\ell} R_{\alpha jk \ell} \omega_\ell = dR_{\alpha jk} - \sum_\ell \frac{\partial}{\partial x^\ell} R_{\alpha jk \ell} \omega_\ell - \sum_\ell \frac{\partial}{\partial x^\ell} R_{\alpha jk \ell} \omega_\ell - \sum_\ell \frac{\partial}{\partial x^\ell} R_{\alpha jk \ell} \omega_\ell.
\]

The Laplacian \( \Delta h_{ij} \) of the second fundamental form \( h_{ij} \) is defined by
\[
\Delta h_{ij} = \sum_k h_{ijkk}.
\]

From (9) we then obtain
\[
\Delta h_{ij} = \sum_k h_{kij} + \sum_k R_{0ijk},
\]
and from (10)
\[
h_{kij} = h_{kij} + \sum_m h_{mi} R_{mkj} + \sum_m h_{km} R_{mijk}.
\]
Replace \( h_{kij} \) in (13) by \( h_{kij} + R_{0kij} \) (by (9)) and substitute the right hand side of (13) into \( h_{kij} \) in (12). Then we obtain
\[
\Delta h_{ij} = \sum_k (h_{kij} + R_{0kij} + R_{0ijk})
\]
\[
+ \sum_k (\sum_m h_{mi} R_{mkj} + \sum_m h_{km} R_{mijk}).
\]

From (7), (11) and (14) we then obtain
\[
\Delta h_{ij} = \sum_k h_{kij} + \sum_k R_{0kij} + \sum_k R_{0ij};
\]
\[
+ \sum_k (h_{kij} R_{0ij} + h_{ij} R_{0ijke})
\]
\[
+ \sum_{m,k} (h_{mij} R_{mkj} + 2h_{mkj} R_{mijk} + h_{mij} R_{mij})
\]
\[
- \sum_{m,k} (h_{mij} h_{mkj} + h_{kmj} h_{ijk} - h_{kjm} h_{ijk} - h_{km} h_{mkj})
\]

Now we assume that \( N \) is locally symmetric, that is, \( R_{\alpha jk} = 0 \) and that \( M \) is maximal in \( N \), so that \( \sum_k h_{kij} = 0 \). Then, from (15) we obtain
\[
\sum_i h_{ij} \Delta h_{ij} = \sum_i h_{ij}^2 R_{0ij} + \sum_i 2(h_{ij} h_{ij} R_{mijk} + h_{ij} h_{kij} R_{mijk})
\]
\[
+ (\sum_i h_{ij})^2.
\]

This is the Lorentzian version of the well-known formula established, for example, in [8].
§ 4. Proof of Theorem A

Theorem A is an immediate consequence of the following

THEOREM B. Let \( N = (N, g) \) be a locally symmetric Lorentzian \((n + 1)\)-manifold and \( M \) be a complete maximal spacelike hypersurface in \( N \). Assume that there exist constants \( c_1, c_2 \) such that

(i) \( \text{Ric} \,(u, v) \geq c_1 \) for all timelike vectors \( v \in TN \),

(ii) \( \text{K}(u \wedge v) \geq c_2 \) for all nondegenerate tangent 2-planes spanned by spacelike vectors \( u, v \in TN \), and

(iii) \( c_1 + 2nc_2 \geq 0 \).

Then \( M \) is totally geodesic.

To prove Theorem B, we first note

LEMMA 1. Under the assumptions of Theorem B,

\[
\frac{1}{2} \Delta (\sum_{i,j} h^2_{ij}) \geq (\sum_{i,j} h^2_{ij})^2.
\]

Proof. For any point \( p \in M \), we may choose our frame \( \{ e_1, \cdots, e_n \} \) at \( p \) so that \( h_{ij} = \lambda_i \delta_{ij} \). Then, by assumption (ii) of Theorem B, we have at \( p \)

\[
\sum_{i,j,k,m} 2(h_{ij} h_{km} R_{ik\ell k} + h_{ij} h_{mk} R_{i\ell k}) = \sum_{i,k} 2(\lambda_i \lambda_k R_{i\ell k}) + \lambda_i \lambda_k R_{ik\ell k} \geq c_2 \sum_{i,k} (\lambda_i - \lambda_k)^2 = 2c_2 (n \sum_i \lambda_i^2 - (\sum_i \lambda_i)^2) = 2nc_2 \sum_{i,j} h^2_{ij}.
\]

Also we have by assumption (i)

\[
\sum_k R_{kk} \geq c_1.
\]

It then follows from (16) and assumption (iii) that

\[
\frac{1}{2} \Delta (\sum_{i,j} h^2_{ij}) = \sum_{i,j,k} h_{ij}^2 R_{ik\ell k} + \sum_{i,j} h_{ij} \partial h_{ij} \geq (c_1 + 2nc_2)(\sum_{i,j} h^2_{ij}) + (\sum_{i,j} h^2_{ij})^2 \geq (\sum_{i,j} h^2_{ij})^2.
\]

Let \( u = \sum_{i,j} h^2_{ij} \) be the squared of the length of the second fundamental form of \( M \). The proof of Theorem B is complete if we show that \( u \) vanishes identically. Recall that from (17), \( u \) satisfies
\( \Delta u \geq 2u^2. \)

Then, by the maximum principle, the result is immediate provided \( M \) is compact.

We now assume that \( M \) is noncompact and complete. We will modify the maximum principle argument as in [4]. Take a point \( p \in M \), and let \( r \) denote the geodesic distance on \( M \) from \( p \) with respect to the induced Riemannian metric. For \( a > 0 \), let \( B_a(p) = \{ x \in M \mid r(x) < a \} \) be the geodesic ball of radius \( a \) and center \( p \).

**Lemma 2.** For any \( a > 0 \), there exists a constant \( c \) depending only on \( n \) such that

\[
\begin{equation}
 u(x) \leq \frac{ca^2(1 + |c_1|^n a)}{(a^2 - r(x))^2}
\end{equation}
\]

for all \( x \in B_a(p) \).

*Proof.* Assuming that \( u \) is not identically zero on \( B_a(p) \), we consider the function

\[
f(x) = (a^2 - r(x))^2 u(x), \quad x \in B_a(p).
\]

Then \( f \) attains a nonzero maximum at some point \( q \in B_a(p) \), for the closure of \( B_a(p) \) is compact since \( M \) is complete. As in \([2, 3]\), we may assume that \( f \) is \( C^2 \) around \( q \). Then we have

\[
\nabla f(q) = 0, \quad \Delta f(q) \leq 0.
\]

Hence at \( q^* \)

\[
\begin{align*}
\frac{\nabla u}{u} &= \frac{4r \nabla r}{a^2 - r^2}, \\
\frac{\Delta u}{u} &\leq \frac{|\nabla u|^2}{u^2} + \frac{8r^2}{(a^2 - r^2)^2} + \frac{4(1 + r \Delta r)}{a^2 - r^2},
\end{align*}
\]

from which we obtain

\[
\begin{equation}
\frac{\Delta u}{u}(q) \leq \frac{24r^2}{(a^2 - r^2)^2}(q) + \frac{4(1 + r \Delta r)}{a^2 - r^2}(q).
\end{equation}
\]

On the other hand, according to [Lemma 1, 9], \( \Delta r(q) \) is bounded from above by

\footnote{We may concentrate on the case of \( q = p \) for the proof become simpler when \( q = p \).}
where $a$ is the tangent vector of the minimizing geodesic $\sigma: [0, r(q)] \to M$ from $p$ to $q$ and $\text{Ric}$ denotes the Ricci curvature of $M$. Also, from (7) and assumption (ii) of Theorem B, $\text{Ric}(\sigma(t), \sigma(t))$ is bounded from below by $R(q)$.

It follows from (20) and (23) that

$$f(q) = (a^2 - r(q))u(q) \leq c_1a + c_2$$

for all $x \in B(p)$.

Since $M$ is maximal spacelike, we may fix $x$ in Inequality (19) and let $a$ tend to infinity. Then we obtain $u(x) = 0$ for all $x \in M$. This completes the proof of Theorem B.

Remark. Let $N = \mathbb{L}^{k+1} \times S^n$, be the product Lorentzian manifold of the flat Minkowski $(k+1)$-space $\mathbb{L}^{k+1}$, $1 \leq k \leq n$, and $S^n$, a Riemannian $(n-k)$-manifold of positive constant curvature. Then $N$ satisfies the assumptions of Theorem A. The Einstein static space $\mathbb{R} \times S^n$, a Riemannian $(n+1)$-manifold of constant curvature $c > 0$, called the de Sitter space, satisfies the assumptions of Theorem B (with $c = -c_1$, $c_2 = 0$). Theorem B then gives a refinement of a theorem of Choquet-Bruhat [Theorem 4.6, 6].

The Lorentzian $(n+1)$-manifold $S^{n+1}$ of constant curvature $c > 0$, called the de Sitter space, satisfies the assumptions of Theorem B (with $c = -c_1$, $c_2 = 0$). Theorem B then gives a refinement of a theorem of Choquet-Bruhat [Theorem 4.6, 6].
REFERENCES


Department of Mathematics
Faculty of Science
Kyushu University
Fukuoka 812
Japan