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ON RAMIFICATION THEORY IN PROJECTIVE ORDERS

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The ramification theory in commutative rings, as a generalization of the classical one in maximal orders over a Dedekind domain, was established in [1], [13], [15] and etc.. For non-commutative algebras this was also studied in [3], [4], [9], [18] and etc.. However, the different theorem, which is a central part of ramification theory, has not been given in those, except for some special cases (cf. [12], [18]). The main object of this paper is to give the discriminant theorem and the different theorem for projective orders in a (non-commutative) separable algebra, in the most general form.

Let R be a Dedekind domain and K be the quotient field of R. Recently it was proved in [9] that, if R has the perfect residue class fields, then the Noetherian different of a maximal R-order in a central simple K-algebra is a square-free ideal of R. Another object of this paper is, more generally without assuming that R has the perfect residue class fields, to determine completely the structure of the Dedekind different and the Noetherian different for a hereditary R-order in a central simple K-algebra.

In § 2 we discuss some basic properties of the differents of algebras and give criteria on the separability of projective orders. In § 3 we first give the Noetherian different theorem for algebras. Further, using these results, we prove, under some restrictive assumptions, the discriminant theorem and the Dedekind different theorem for projective orders, each of which is a generalization of the classical one for maximal orders over a Dedekind domain.

In § 4 we restrict our attention to hereditary orders over a Dedekind domain. Let R be a Dedekind domain with quotient field K and $\sum$ a central simple K-algebra. We show that, for any hereditary R-order Λ in

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Σ, the image, $Trd_{\Sigma/K}(\Lambda)$, of Λ under the reduced trace $Trd_{\Sigma/K}$ of Σ is an ideal of R which does not depend on Λ and give a characterization of Σ such that, for any hereditary R-order Λ in Σ, $Trd_{\Sigma/K}(\Lambda)$ coincides with R. With these preparations we prove the structure theorem of the differents of hereditary orders over a Dedekind domain.

Finally, in § 5, we introduce the notion of locally symmetric orders and give the Dedekind different theorem for locally symmetric orders. Further we discuss some basic properties of locally symmetric, hereditary orders over a Dedekind domain. It is noted that the study of symmetric maximal orders over a discrete (rank-one) valuation ring is closely related to the study of the Brauer group of a field with discrete (rank-one) valuation whose residue class field is not perfect.

§ 1. Notation and terminology

Throughout this paper we shall only consider rings with identity and modules that are unitary, and we shall denote commutative rings by $R, S, \cdots$ and rings, which are not always commutative, by $\Lambda, \Gamma, \cdots$. All R-algebras considered will be assumed to be faithful as R-modules.

Let Λ be an R-algebra. We denote by Λ° the opposite algebra of Λ and by Λ^* the dual, $\mathrm{Hom}_R(\Lambda, R)$, of Λ. In general there is a $\Lambda \otimes_R \Lambda^\circ$-epimorphism $\Phi_{\Lambda/R}$: $\Lambda \otimes_R \Lambda^\circ \to \Lambda$ defined by $\Phi_{\Lambda/R}(\lambda \otimes_R \mu^\circ) = \lambda\mu$. The kernel of $\Phi_{\Lambda/R}$, which we denote by $J_{\Lambda/R}$, is a left ideal of $\Lambda \otimes_R \Lambda^\circ$, and then the right annihilator of it, which we denote by $A_{\Lambda/R}$, is a right ideal of $\Lambda \otimes_R \Lambda^\circ$. It is easily seen (cf. [3]) that the image $\Phi_{\Lambda/R}(A_{\Lambda/R})$ of $A_{\Lambda/R}$ under $\Phi_{\Lambda/R}$ is an ideal of the center of Λ. We call $\Phi_{\Lambda/R}(A_{\Lambda/R})$ the Noetherian (or homological) different of Λ and denote it by $N_{\Lambda/R}$ (cf. [16], [3]).

Let R be a commutative ring with total quotient ring K and Σ a separable K-algebra which is a finitely generated projective K-module. An R-algebra Λ which is a finitely generated torsion-free R-module such that $K \otimes_R \Lambda = \Sigma$ will be called an R-order in Σ. Especially, an R-order Λ in Σ will be called a projective R-order if it is R-projective. Denote by F the center of Σ and let $t_{\Sigma/K}$ be the composed map of the reduced trace $Trd_{\Sigma/F}: \Sigma \to F$ (cf. [7], where we denote it by Trd_F^Σ) and the trace $T_{F/K}$: $F \to K$. For an R-order Λ in Σ, we define the complementary module, $C_{\Lambda/R}$, and the Dedekind different, $D_{\Lambda/R}$, to be the sets: $C_{\Lambda/R} = \{x \in \Sigma \mid t_{\Sigma/K}(x\Lambda) \subseteq R\}$ and $D_{\Lambda/R} = \{x \in \Sigma \mid C_{\Lambda/R} \cdot x \subseteq \Lambda\}$. Obviously both $C_{\Lambda/R}$ and

$D_{\Lambda/R}$ are two-sided Λ-submodules of Σ. Furthermore we put $\tilde{D}_{\Lambda/R} = [D_{\Lambda/R}]^2$ and call it the weak Dedekind different of Λ. In particular, assume that Λ is a projective R-order in Σ. In case Λ has constant rank n over R, we denote by $d_{\Lambda/R}$ the R-submodule of K which is generated by $\det[t_{\Sigma/K}(u_i u_j)]$ where $\{u_1, u_2, \cdots, u_n\}$ runs over all sets of n elements in Λ. However, in case Λ has not constant rank, there is a unique decomposition, $R = R_1 \oplus R_2 \oplus \cdots \oplus R_l$, such that every $R_i \otimes_R \Lambda$ has constant rank n_i over R_i and $n_1 < n_2 < \cdots < n_l$, and we put $d_{\Lambda/R} = d_{R_1 \otimes_R \Lambda/R_1} \oplus d_{R_2 \otimes_R \Lambda/R_2} \oplus \cdots \oplus d_{R_l \otimes_R \Lambda/R_l}$. We shall call $d_{\Lambda/R}$ the discriminant of Λ. Clearly, $d_{\Lambda/R}$ is a locally free R-submodule of K, and especially, if $t_{\Sigma/K}(\Lambda) \subseteq R$, we have $d_{\Lambda/R} \subseteq D_{\Lambda/R} \subseteq \Lambda \subseteq C_{\Lambda/R}$.

For a ring Λ we shall denote by $J(\Lambda)$ the Jacobson radical of it and by $P(\Lambda)$ the prime radical of it, i.e., the intersection of all prime (two-sided) ideals of Λ, and, more generally, for an ideal $\mathfrak{A}$ of Λ, we shall denote by $P_\Lambda(\mathfrak{A})$ the prime radical of it in Λ, i.e., the intersection of all prime ideals of Λ containing $\mathfrak{A}$.

§ 2. Basic properties

We begin with

Proposition 2.1. *Let Λ be an R-algebra and Z the center of Λ. Then* $N_{\Lambda/Z} \cdot N_{Z/R} \subseteq N_{\Lambda/R} \subseteq N_{\Lambda/Z}$. *If Λ is a finitely generated projective Z-module, then* $N_{\Lambda/R} \subseteq N_{Z/R}$ *and, furthermore, if $N_{Z/R}$ is an invertible ideal of Z, then* $N_{\Lambda/R} = N_{\Lambda/Z} \cdot N_{Z/R}$.

Proof. By tensoring each term of the exact sequence: $0 \longrightarrow J_{Z/R} \longrightarrow Z \otimes_R Z \longrightarrow Z \longrightarrow 0$ with $\Lambda \otimes_R \Lambda^\circ$ over $Z \otimes_R Z$, we obtain the exact sequence:

$$(\Lambda \otimes_R \Lambda^\circ) \otimes_{Z \otimes_R Z} J_{Z/R} \longrightarrow \Lambda \otimes_R \Lambda^\circ \xrightarrow{1 \otimes \Phi_{Z/R}} (\Lambda \otimes_R \Lambda^\circ) \otimes_{Z \otimes_R Z} Z \longrightarrow 0.$$

It is clear that $(\Lambda \otimes_R \Lambda^\circ) \otimes_{Z \otimes_R Z} Z \cong \Lambda \otimes_{R \otimes_R Z} (\Lambda^\circ \otimes_Z Z) \cong \Lambda \otimes_Z \Lambda^\circ$. If we identify $(\Lambda \otimes_R \Lambda^\circ) \otimes_{Z \otimes_R Z} Z$ with $\Lambda \otimes_Z \Lambda^\circ$, then $1 \otimes \Phi_{Z/R}$ can also be identified with the homomorphism Φ' of $\Lambda \otimes_R \Lambda^\circ$ onto $\Lambda \otimes_Z \Lambda^\circ$ such that $\Phi'(\lambda \otimes_R \mu^\circ) = \lambda \otimes_Z \mu^\circ$ for any $\lambda, \mu \in \Lambda$. Now, denoting by $(\Lambda \otimes_R \Lambda^\circ) J_{Z/R}$ the ideal of $\Lambda \otimes_R \Lambda^\circ$ generated by the image of $J_{Z/R}$ in $\Lambda \otimes_R \Lambda^\circ$, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\Lambda \otimes_R \Lambda^{\circ})J_{Z/R} & \longrightarrow & J_{\Lambda/R} & \longrightarrow & J_{\Lambda/Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\Lambda \otimes_R \Lambda^{\circ})J_{Z/R} & \longrightarrow & \Lambda \otimes_R \Lambda^{\circ} & \xrightarrow{\Phi'} & \Lambda \otimes_Z \Lambda^{\circ} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi_{\Lambda/R}} & & \downarrow{\scriptstyle \Phi_{\Lambda/Z}} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Lambda & \xrightarrow{I_{\Lambda}} & \Lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where all rows and columns are exact. It is easily seen that $\Phi'(A_{\Lambda/R}) \subseteq A_{\Lambda/Z}$, hence, by applying $\Phi_{\Lambda/Z}$ to both sides, we obtain $N_{\Lambda/R} \subseteq N_{\Lambda/Z}$. Let $\bar{\alpha}$ be an element of $A_{\Lambda/Z}$ and denote by α a representative of $\bar{\alpha}$ in $\Lambda \otimes_R \Lambda^{\circ}$. Since $\Phi'(J_{\Lambda/R}\cdot\alpha) = J_{\Lambda/Z}\cdot\bar{\alpha} = 0$, we have $J_{\Lambda/R}\cdot\alpha \subseteq (\Lambda \otimes_R \Lambda^{\circ})J_{Z/R}$ in view of the above diagram, so that $J_{\Lambda/R}\cdot\alpha\cdot A_{Z/R} = 0$, i.e., $\alpha\cdot A_{Z/R} \subseteq A_{\Lambda/R}$, and therefore $\bar{\alpha}\cdot N_{Z/R} = \Phi'(\alpha A_{Z/R}) \subseteq \Phi'(A_{\Lambda/R})$. Accordingly $A_{\Lambda/Z}\cdot N_{Z/R} \subseteq \Phi'(A_{\Lambda/R})$. Applying $\Phi_{\Lambda/Z}$ to both sides, we find that $N_{\Lambda/Z}\cdot N_{Z/R} \subseteq N_{\Lambda/R}$. Now assume that Λ is a finitely generated projective Z-module. Then $\Lambda \otimes_R \Lambda^{\circ}$ is a finitely generated projective, faithful $Z \otimes_R Z$-module, and therefore the annihilator of $(\Lambda \otimes_R \Lambda^{\circ})J_{Z/R}$ coincides with $(\Lambda \otimes_R \Lambda^{\circ})A_{Z/R}$. Since $(\Lambda \otimes_R \Lambda^{\circ})J_{Z/R} \subseteq J_{\Lambda/R}$ by the above diagram, it follows that $A_{\Lambda/R} \subseteq (\Lambda \otimes_R \Lambda^{\circ})A_{Z/R}$, so that, by applying $\Phi_{\Lambda/R}$ to both sides, $N_{\Lambda/R} \subseteq N_{Z/R}\cdot\Lambda$. Consequently we obtain $N_{\Lambda/R} \subseteq N_{Z/R}$, because Z is the direct summand of Λ as Z-modules. Furthermore assume that $N_{Z/R}$ is an invertible ideal of Z. Since $\Phi'(A_{\Lambda/R}) \subseteq (\Lambda \otimes_Z \Lambda^{\circ})N_{Z/R}$, $[N_{Z/R}]^{-1}\cdot \Phi'(A_{\Lambda/R}) \subseteq \Lambda \otimes_Z \Lambda^{\circ}$. However $\Phi'(A_{\Lambda/R}) \subseteq A_{\Lambda/Z}$, so that $J_{\Lambda/Z}\cdot[N_{Z/R}]^{-1}\cdot\Phi'(A_{\Lambda/R}) = 0$. Thus we must have $[N_{Z/R}]^{-1}\cdot\Phi'(A_{\Lambda/R}) \subseteq A_{\Lambda/Z}$, i.e., $\Phi'(A_{\Lambda/R}) \subseteq A_{\Lambda/Z}\cdot N_{Z/R}$. Applying $\Phi_{\Lambda/Z}$ to both sides, we find that $N_{\Lambda/R} \subseteq N_{\Lambda/Z}\cdot N_{Z/R}$, completing the proof of the proposition.

For the Dedekind differents we prove the following proposition which has the similar form to (2. 1).

PROPOSITION 2. 2. *Let Λ be an R-order in a separable K-algebra Σ, and Z the center of Λ. Then $D_{\Lambda/Z}\cdot D_{Z/R} \subseteq D_{\Lambda/R}$. Especially, if $C_{Z/R}$ is an invertible fractional ideal of Z or if Λ is a quasi-Frobenius Z-algebra, then $D_{\Lambda/Z}\cdot D_{Z/R} = D_{\Lambda/R}$.*

Proof. We denote by F the center of Σ. Since $T_{F/K}(Z\cdot Trd_{\Sigma/F}(C_{\Lambda/R})) = t_{\Sigma/K}(C_{\Lambda/R}) \subseteq R$, $Trd_{\Sigma/F}(C_{\Lambda/R}) \subseteq C_{Z/R}$, so that $Trd_{\Sigma/F}(C_{\Lambda/R}\cdot D_{Z/R}) \subseteq Z$, and

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therefore $C_{\Lambda/R}\cdot D_{Z/R}\subseteq C_{\Lambda/Z}$, i.e., $C_{\Lambda/R}\cdot D_{Z/R}\cdot D_{\Lambda/Z}\subseteq\Lambda$. Thus we obtain $D_{Z/R}\cdot D_{\Lambda/Z}\subseteq D_{\Lambda/R}$, which proves our first assertion. Now we see $Trd_{\Sigma/F}(C_{Z/R}\cdot C_{\Lambda/Z})=C_{Z/R}\cdot Trd_{\Sigma/F}(C_{\Lambda/Z})\subseteq C_{Z/R}$, so that $t_{\Sigma/K}(C_{Z/R}\cdot C_{\Lambda/Z})\subseteq R$. Therefore $C_{Z/R}\cdot C_{\Lambda/Z}\subseteq C_{\Lambda/R}$ and so $C_{Z/R}\cdot C_{\Lambda/Z}\cdot D_{\Lambda/R}\subseteq\Lambda$. Accordingly we have $C_{Z/R}\cdot D_{\Lambda/R}\subseteq D_{\Lambda/Z}$. Thus, in case $C_{Z/R}$ is invertible, we find that $D_{\Lambda/R}\subseteq D_{\Lambda/Z}\cdot D_{Z/R}$. Finally, if Λ is a quasi-Frobenius Z-algebra, $C_{\Lambda/Z}$ is invertible by a remark in [18], p. 228. Because Λ is projective over Z, the fact that $C_{Z/R}\cdot C_{\Lambda/Z}\cdot D_{\Lambda/R}\subseteq\Lambda$ implies that $C_{\Lambda/Z}\cdot D_{\Lambda/R}\subseteq D_{Z/R}\cdot\Lambda$. Consequently we must have $D_{\Lambda/R}\subseteq D_{\Lambda/Z}\cdot D_{Z/R}$. This completes the proof of the proposition.

The following result is due to [9] and [18].

Proposition 2. 3. *If Λ is a projective R-order in a central separable K-algebra Σ, then $Trd_{\Sigma/K}(D_{\Lambda/R})=N_{\Lambda/R}$ and $D_{\Lambda/R}\subseteq C_{\Lambda/R}$.*

Proof. See [9], (2. 2. 3) and [18], Th. 7 and 8.

Proposition 2. 4. *Let S be a projective R-order in a commutative separable K-algebra. Then $D_{S/R}=N_{S/R}$.*

Proof. This can be done along the same line as in [1], (3. 1) and therefore we omit this.

From these propositions we derive

Proposition 2. 5. *Let Σ be a separable K-algebra with center F and Λ a projective R-order in Σ with center Z. If Λ is Z-projective and Z is a quasi-Frobenius R-algebra, then $Trd_{\Sigma/F}(D_{\Lambda/R})=N_{\Lambda/R}$.*

Proof. By virtue of (2. 1) and (2. 2) we have $N_{\Lambda/R}=N_{\Lambda/Z}\cdot N_{Z/R}$ and $D_{\Lambda/R}=D_{\Lambda/Z}\cdot D_{Z/R}$. But, according to (2. 3) and (2. 4), $Trd_{\Sigma/F}(D_{\Lambda/Z})=N_{\Lambda/Z}$ and $D_{Z/R}=N_{Z/R}$. Thus we find that $Trd_{\Sigma/F}(D_{\Lambda/R})=N_{\Lambda/R}$.

Now we give criteria on the separability of a projective order. First, for the case where Σ has K as its center, we show

Proposition 2. 6. *For any projective R-order Λ in a central separable K-algebra Σ, the following statements are equivalent:*

(1) $\Lambda^*=\Lambda t$.

(2) $C_{\Lambda/R}=\Lambda$.

(3) $D_{\Lambda/R}=\Lambda$.

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(4) $d_{\Lambda/R} = R$.

(5) $N_{\Lambda/R} = R$, *i.e., Λ is separable over R.*

Here t denotes the restriction of $Trd_{\Sigma/K}$ to Λ.

Proof. The implication $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (5)$ have been verified in [18], Th. 8, and the implication $(2) \Longrightarrow (4)$ is obvious. Hence we have only to show $(4) \Longrightarrow (5)$. By using the similar argument to the proof of [7], (1. 2), it is sufficient to prove this in the case that R is Noetherian. Now assume that $d_{\Lambda/R} = R$. Clearly $\Sigma/P(R)\Sigma$ is a central separable K-algebra and $\Lambda/P(R)\Lambda$ is a projective $R/P(R)$-order in $\Sigma/P(R)\Sigma$. Furthermore we have $d_{\Lambda/P(R)\Lambda/R/P(R)} = d_{\Lambda/R} + P(R)/P(R) = R/P(R)$, and Λ is separable over R whenever $\Lambda/P(R)\Lambda$ is separable over $R/P(R)$. Therefore we may assume that R is a Noetherian ring without non-trivial nilpotent element. Then the integral closure $\bar{R}$ of R in K is expressible as the direct sum of a finite number of Krull domains. Here we have also $d_{\bar{R}\otimes_R \Lambda/\bar{R}} = \bar{R} \otimes_R d_{\Lambda/R} = \bar{R}$, and Λ is separable over R whenever $\bar{R} \otimes_R \Lambda$ is separable over $\bar{R}$. Hence, under the assumption that R is a Krull domain, it is sufficient to prove that Λ is separable over R. Since $Trd_{\Sigma/K}(\Lambda) \subseteq R$ in this case, we have $R = d_{\Lambda/R} \subseteq D_{\Lambda/R} \subseteq \Lambda$ so that $D_{\Lambda/R} = \Lambda$. Because (3) and (5) are equivalent, this implies that Λ is separable over R.

In the general case, some additional hypotheses must be imposed on Λ or R.

PROPOSITION 2. 7. *Let Λ be a projective R-order in a separable K-algebra and suppose that Λ is projective as a module over its center Z. Then we have $D_{\Lambda/R} \subseteq C_{\Lambda/R}$, and the following statements are equivalent for Λ:*

(1) $\Lambda^* = \Lambda t$.

(2) $C_{\Lambda/R} = \Lambda$.

(3) $D_{\Lambda/R} = \Lambda$.

(4) $d_{\Lambda/R} = R$ and $t_{\Sigma/K}(\Lambda) \subseteq R$.

(5) $N_{\Lambda/R} = Z$, i.e., *Λ is separable over R. Here t denotes the restriction of $t_{\Sigma/K}$ to Λ.*

Proof. According to (2. 3), we have $D_{\Lambda/Z} \subseteq C_{\Lambda/Z}$. However, because Λ is Z-projective, it is easily seen that $C_{\Lambda/Z} \subseteq C_{\Lambda/R}$. Thus we obtain $D_{\Lambda/R} \subseteq D_{\Lambda/Z} \subseteq C_{\Lambda/Z} \subseteq C_{\Lambda/R}$. Next we shall prove the second part of the proposition. The implication $(5) \Longrightarrow (1)$ follows immediately from [7], (4. 2), and the implications $(1) \Longleftrightarrow (2) \Longleftrightarrow (3)$ can be shown by the same way as in the proof of [18], Th. 8. Also the implications $(2) \Longleftrightarrow (4)$ are obvious. Hence we need only show $(2), (3) \Longrightarrow (5)$. Assume that $D_{\Lambda/R} = C_{\Lambda/R} = \Lambda$. Since $D_{\Lambda/R} \subseteq D_{\Lambda/Z} \subseteq C_{\Lambda/Z} \subseteq C_{\Lambda/R}$, we have $D_{\Lambda/Z} = C_{\Lambda/Z} = \Lambda$, so that, by (2. 6), Λ is separable over Z. Therefore we have, by (2. 2), $D_{Z/R} = Z$, and hence, from (2. 4), it follows that Z is separable over R. Thus, by [3], (2. 3), we know that Λ is separable over R, which shows $(2), (3) \Longrightarrow (5)$.

PROPOSITION 2. 8. *Assume one of the following conditions:*

(i) *R is a Noetherian ring with Krull dimension* 1.

(ii) *R is a regular domain.*

Let Λ be a projective R-order in a separable K-algebra Σ and Z the center of Λ. Then the following statements are equivalent:

(1) $\Lambda^* = \Lambda t$.

(2) $C_{\Lambda/R} = \Lambda$.

(3) $D_{\Lambda/R} = \Lambda$.

(4) $d_{\Lambda/R} = R$.

(5) $N_{\Lambda/Z} = Z$, *i.e., Λ is separable over R. Here t denotes the restriction of $t_{\Sigma/K}$ to Λ.*

Proof. Evidently we have only to prove $(2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (5)$. Assume that R is a Noetherian ring with Krull dimension 1. By using the same method as in the proof of (2. 6), it is sufficient to prove $(2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (5)$ in case R is a Dedekind domain. If R is a Dedekind domain, then we have $t(\Lambda) \subseteq R$, so that the implications $(2) \Longleftrightarrow (3) \Longleftrightarrow (4)$ are obvious. Now suppose that $C_{\Lambda/R} = \Lambda$. Let Γ be a maximal R-order in Σ which contains Λ. Then we have $\Gamma \subseteq C_{\Gamma/R} \subseteq C_{\Lambda/R} = \Lambda$, and so $\Lambda = \Gamma$. Therefore Λ itself is a maximal R-order in Σ, so that Λ is Z-projective. Thus, by (2. 7), we conclude that Λ is separable over R, which shows that (2) implies (5). Next, assume that R is a regular domain. Since R is integrally closed, we have $t(\Lambda) \subseteq R$. Then we can easily show $(2) \Longleftrightarrow (3) \Longleftrightarrow (4)$, hence we need only show $(2) \Longrightarrow (5)$. If we assume that $C_{\Lambda/R} = \Lambda$, then it can be seen, analogously, that Λ is maximal in Σ. Therefore the center Z is an integrally closed Noetherian ring, and so it is expressible as the direct sum of a finite number of integrally closed integral domains. Hence we may assume that Z is an integrally closed integral domain. Since, for any prime ideal $\mathfrak{p}$ of height 1 in R, $R_{\mathfrak{p}}$ is a Dedekind domain and $C_{\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}} = \Lambda_{\mathfrak{p}}$, $\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}$, and so $Z_{\mathfrak{p}}$ is also separable over $R_{\mathfrak{p}}$. Because R is regular, Z is separable over R by [14], (41. 1). Then, the fact that Λ is R-projective implies that Λ is Z-projective. Thus, by (2. 7), Λ must be separable over R, completing the proof of the proposition.

§ 3. Different theorems

First we give

LEMMA 3. 1. *Let Λ be an R-algebra which is a finitely generated R-module and Z be the center of Λ. Let $\mathfrak{P}$ be a prime ideal of Λ, and put $\mathfrak{q} = \mathfrak{P} \cap Z$ and $\mathfrak{p} = \mathfrak{P} \cap R$. Then the following conditions are equivalent:*

(1) *$\Lambda_{\mathfrak{p}}/\mathfrak{P}\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, and $\mathfrak{P}\Lambda_{\mathfrak{q}} = \mathfrak{p}\Lambda_{\mathfrak{q}}$.*

(2) *$\Lambda_{\mathfrak{p}}/P_{\Lambda}(\mathfrak{q}\Lambda)\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, and $P_{\Lambda}(\mathfrak{q}\Lambda)\Lambda_{\mathfrak{q}} = \mathfrak{p}\Lambda_{\mathfrak{q}}$.*

(3) *$\Lambda_{\mathfrak{p}}/\mathfrak{q}\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\mathfrak{q}\Lambda_{\mathfrak{q}} = \mathfrak{p}\Lambda_{\mathfrak{q}}$.*

(4) *$\Lambda_{\mathfrak{q}}/\mathfrak{p}\Lambda_{\mathfrak{q}}$ is separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.*

Proof. As this is easy, we omit it.

Now let Λ be an R-algebra which is a finitely generated R-module. Then a prime ideal $\mathfrak{P}$ of Λ is said to be unramified over R if it satisfies the equivalent conditions in (3. 1).

THEOREM 3. 2 (*Noetherian different theorem*). *Let Λ be an R-algebra and Z the center of Λ. Assume that both Λ and Z are finitely generated R-modules. For any prime ideal $\mathfrak{P}$ of Λ, the following conditions are equivalent:*

(1) $N_{\Lambda/R} \not\subseteq \mathfrak{P} \cap Z$.

(2) *$\mathfrak{P}$ is unramified.*

Proof. Put $\mathfrak{q} = \mathfrak{P} \cap Z$ and $\mathfrak{p} = \mathfrak{P} \cap R$ as in (3. 1). If we suppose $N_{\Lambda/R} \subseteq \mathfrak{q}$, then, by (2. 1), we have $N_{Z/R} \subseteq \mathfrak{q}$ or $N_{\Lambda/Z} \subseteq \mathfrak{q}$. In case $N_{Z/R} \subseteq \mathfrak{q}$, it can be shown that $Z_{\mathfrak{q}}/\mathfrak{p}Z_{\mathfrak{q}}$ is not separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, by using the same method as in the proof of [1], (2. 7). On the other hand, in case $N_{\Lambda/Z} \subseteq \mathfrak{q}$, $\Lambda_{\mathfrak{q}}$ is not separable over $Z_{\mathfrak{q}}$. Therefore, in both cases, $\Lambda_{\mathfrak{q}}/\mathfrak{p}\Lambda_{\mathfrak{q}}$ is not separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Conversely suppose that $N_{\Lambda/R} \not\subseteq \mathfrak{q}$. Then we have, by (2. 1), $N_{\Lambda/Z} \not\subseteq \mathfrak{q}$, so that $\Lambda_{\mathfrak{q}}$ is separable over $Z_{\mathfrak{q}}$. Again, by

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applying (2. 1) to $\Lambda_{\mathfrak{q}}$, we obtain $N_{\Lambda_{\mathfrak{q}}/R_{\mathfrak{p}}} = N_{Z_{\mathfrak{q}}/R_{\mathfrak{p}}}$. Because $N_{\Lambda_{\mathfrak{q}}/R_{\mathfrak{p}}} \supseteq [N_{\Lambda/R}]_{\mathfrak{q}}$, we have $N_{Z_{\mathfrak{q}}/R_{\mathfrak{p}}} \not\subseteq \mathfrak{q}Z_{\mathfrak{q}}$, and so we can show, again by using the same method as in the proof of [1], (2. 7), that $Z_{\mathfrak{q}}/\mathfrak{p}Z_{\mathfrak{q}}$ is separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Consequently $\Lambda_{\mathfrak{q}}/\mathfrak{p}\Lambda_{\mathfrak{q}}$ must be separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. This completes the proof of the theorem.

In the rest of this section, we assume that R is a commutative ring with total quotient ring K and that Σ is a separable K-algebra which is a finitely generated projective K-module.

THEOREM 3. 3. (*Discriminant theorem*). *Let Λ be a projective R-order in a separable K-algebra Σ, and assume one of the following conditions:*

(i) *R is the center of Λ and $d_{\Lambda/R} \subseteq R$.*

(ii) *Λ is projective as a module over its center and $t_{\Sigma/K}(\Lambda) \subseteq R$.*

(iii) *R is a Noetherian ring with Krull dimension 1 and $t_{\Sigma/K}(\Lambda) \subseteq R$.*

(iv) *R is a regular domain.*

Then, for any prime ideal $\mathfrak{p}$ of R, the following statements are equivalent:

(1) $d_{\Lambda/R} \not\subseteq \mathfrak{p}$.

(2) *Any prime ideal $\mathfrak{P}$ of Λ such that $\mathfrak{p} = \mathfrak{P} \cap R$ is unramified.*

Proof. In each case of (i), (ii), (iii) and (iv) we have $d_{\Lambda/R} \subseteq R$, hence the condition that $d_{\Lambda/R} \not\subseteq \mathfrak{p}$ is equivalent to that $d_{\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}} = R_{\mathfrak{p}}$. By virtue of (2. 6), (2. 7) or (2. 8), this is also equivalent to the condition that $N_{\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}}$ coincides with the center of $\Lambda_{\mathfrak{p}}$, i.e., $\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}$. Thus we can see the equivalence of (1) and (2).

PROPOSITION 3. 4. *Let R be a Noetherian ring and Λ a projective R-order in a central separable K-algebra Σ. Suppose that $d_{\Lambda/R} \subseteq R$, or that any principal ideal of R, generated by a non-zero divisor in R, is unmixed. Then any minimal prime divisor of $N_{\Lambda/R}$ has height 1 in R.*

Proof. Now assume that $d_{\Lambda/R} \subseteq R$. Because $d_{\Lambda/R}$ is an invertible ideal of R, any minimal prime divisor of $d_{\Lambda/R}$ has height 1 in R. According to (2. 6), for a prime ideal $\mathfrak{p}$ of R, we have $d_{\Lambda/R} \subseteq \mathfrak{p}$ if and only if $N_{\Lambda/R} \subseteq \mathfrak{p}$. So any minimal prime divisor of $N_{\Lambda/R}$ has height 1 in R. Secondly suppose that any principal ideal of R, generated by a non-zero divisor in R, is unmixed and that there is a minimal prime divisor $\mathfrak{p}$ of $N_{\Lambda/R}$ with height $\geqq 2$. Then we may further assume that R is a local ring with a maximal ideal $\mathfrak{p}$. Since $d_{\Lambda/R}$ is invertible in R, we can put $d_{\Lambda/R} = \frac{b}{a}R$ for some non-zero divisors a, b in R. For any prime ideal $\mathfrak{p}'$ of height 1 in R, we have $N_{\Lambda_{\mathfrak{p}'}/R_{\mathfrak{p}'}} = R_{\mathfrak{p}'}$ and so, by (2. 6), $d_{\Lambda_{\mathfrak{p}'}/R_{\mathfrak{p}'}} = R_{\mathfrak{p}'}$, i.e., $aR_{\mathfrak{p}'} = bR_{\mathfrak{p}'}$. However, both aR and bR are unmixed. Therefore we obtain $aR = bR$, hence $d_{\Lambda/R} = R$. From (2. 6) it follows that $N_{\Lambda/R} = R$, which is obviously a contradiction.

COROLLARY 3. 5. *Under the same assumptions in* (3. 4), *Λ is separable over R, if and only if, for any prime ideal $\mathfrak{p}$ of height* 1 *in R, $\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}$.*

This corollary is a generalization of [3], (4. 6).

We now prove, as a generalization of the different theorem for maximal orders over a Dedekind domain, the following

THEOREM 3. 6. (*Dedekind different theorem*). *Let R be a Noetherian ring and Λ a projective R-order in a separable K-algebra Σ. Let Z be the center of Λ and assume that Λ is Z-projective. Then the weak Dedekind different $\tilde{D}_{\Lambda/R}$ is contained in Λ, and, for any prime ideal $\mathfrak{P}$ of Λ, the following statements are equivalent:*

(1) $\tilde{D}_{\Lambda/R} \nsubseteq \mathfrak{P}$.

(2) $\tilde{D}_{\Lambda/R} \nsubseteq (\mathfrak{P} \cap Z)\Lambda$.

(3) $\tilde{D}_{\Lambda/Z} \nsubseteq (\mathfrak{P} \cap Z)\Lambda$ *and* $D_{Z/R} \nsubseteq \mathfrak{P} \cap Z$.

(4) $\mathfrak{P}$ *is unramified.*

Especially, if $t_{\Sigma/K}(\Lambda) \subseteq R$, these are also equivalent to

(1)′ $D_{\Lambda/R} \nsubseteq \mathfrak{P}$.

Proof. We have shown $D_{\Lambda/R} \subseteq C_{\Lambda/R}$ in (2. 7), so that $\tilde{D}_{\Lambda/R} = [D_{\Lambda/R}]^2 \subseteq C_{\Lambda/R} \cdot D_{\Lambda/R} \subseteq \Lambda$. Therefore we need only prove the second part. However the implications (1)′ $\Longleftrightarrow$ (1) $\Longrightarrow$ (2) are obvious, and so it suffices to show the rest of the implications. To simplify our notation we put $\mathfrak{q} = \mathfrak{P} \cap Z$ and $\mathfrak{p} = \mathfrak{P} \cap R$.

The case $Z = R$. In this case (2) $\Longleftrightarrow$ (3) is also evident. If $\mathfrak{P}$ is unramified, then we have, by (3. 2), $N_{\Lambda/R} \nsubseteq \mathfrak{p}$, and so $N_{\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}} = R_{\mathfrak{p}}$. Hence,

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by (2. 6), $D_{\Lambda_\mathfrak{p}/R_\mathfrak{p}} = \Lambda_\mathfrak{p}$, i.e., $\tilde{D}_{\Lambda_\mathfrak{p}/R_\mathfrak{p}} = \Lambda_\mathfrak{p} \not\subseteq \mathfrak{P}\Lambda_\mathfrak{p}$. Since $D_{\Lambda_\mathfrak{p}/R_\mathfrak{p}} = [D_{\Lambda/R}]_\mathfrak{p}$, this implies $\tilde{D}_{\Lambda/R} \not\subseteq \mathfrak{P}$, which proves (4) $\Longrightarrow$ (1). Hence we have only to prove (2) $\Longrightarrow$ (4). Now suppose that $\mathfrak{P}$ is ramified. Then it suffices to show that $\tilde{D}_{\Lambda/R} \subseteq \mathfrak{p}\Lambda$. However, by (3. 2), $\mathfrak{P}$ is ramified if and only if $N_{\Lambda/R} \subseteq \mathfrak{p}$. Therefore it suffices to show this under the assumptions that R is a local ring with a maximal ideal $\mathfrak{p}$ and $N_{\Lambda/R}$ is $\mathfrak{p}$-primary. Since $K \underset{R}{\otimes} N_{\Lambda/R} = K$, $\mathfrak{p}$ is not a prime divisor of 0, hence we have height$_R\mathfrak{p} \geqq 1$. If height$_R\mathfrak{p}$ $= 1$, denoting by a a non-zero divisor of R contained in $\mathfrak{p}$, we find a non negative integer l such that $[N_{\Lambda/R}]^l \not\subseteq aR$ but $[N_{\Lambda/R}]^{l+1} \subseteq aR$. Because $Trd_{\Sigma/K}(D_{\Lambda/R}) = N_{\Lambda/R}$ by (2. 3), we see $Trd_{\Sigma/K}([N_{\Lambda/R}]^l D_{\Lambda/R}) = [N_{\Lambda/R}]^{l+1} \subseteq aR$, and so $a^{-1}[N_{\Lambda/R}]^l \cdot D_{\Lambda/R} \subseteq C_{\Lambda/R}$. Hence we have $[N_{\Lambda/R}]^l \tilde{D}_{\Lambda/R} \subseteq a\Lambda$. Since R is a local ring, this shows $\tilde{D}_{\Lambda/R} \subseteq \mathfrak{p}\Lambda$. On the other hand, if height$_R\mathfrak{p} > 1$, we have, by (3. 4), $d_{\Lambda/R} \not\subseteq R$. However, since $d_{\Lambda/R}$ is an invertible fractional ideal of R, we have $d_{\Lambda/R} = \frac{b}{a}R$ for some non-zero divisors a, b of R. If we put $\mathfrak{c} = \{c \in R \mid cd_{\Lambda/R} \subseteq R\}$, then $\mathfrak{c}$ is the $\mathfrak{p}$-primary ideal of R according to (2. 6), because $N_{\Lambda/R}$ is $\mathfrak{p}$-primary. Therefore there is a non-negative integer l such that $[N_{\Lambda/R}]^l \not\subseteq \mathfrak{c}$ but $[N_{\Lambda/R}]^{l+1} \subseteq \mathfrak{c}$. Then $d_{\Lambda/R} \cdot [N_{\Lambda/R}]^l \cdot Trd_{\Sigma/K}(D_{\Lambda/R}) = d_{\Lambda/R}[N_{\Lambda/R}]^{l+1} \subseteq R$, and so $d_{\Lambda/R} \cdot [N_{\Lambda/R}]^l \cdot \tilde{D}_{\Lambda/R} \subseteq \Lambda$. Thus we obtain $b \cdot [N_{\Lambda/R}]^l \cdot \tilde{D}_{\Lambda/R} \subseteq a\Lambda$. Since $b \cdot [N_{\Lambda/R}]^l \not\subseteq aR$, this shows $\tilde{D}_{\Lambda/R} \subseteq \mathfrak{p}\Lambda$, which completes the proof of (2) $\Longrightarrow$ (4).

The general case. By the preceding proof in case $Z = R$, $\mathfrak{P}$ is unramified over Z if and only if $\tilde{D}_{\Lambda/Z} \not\subseteq \mathfrak{q}\Lambda$, and, by (3. 2), $\mathfrak{q}$ is unramified if and only if $N_{Z/R} \not\subseteq \mathfrak{q}$. Then we see easily (3) $\Longleftrightarrow$ (4), because $D_{Z/R} = N_{Z/R}$ by (2. 4). If $\tilde{D}_{\Lambda/R} \subseteq \mathfrak{P}$, then we have, by (2. 2), $\tilde{D}_{\Lambda/Z} \subseteq \mathfrak{P}$ or $D_{Z/R} \subseteq \mathfrak{q}$. Again by the proof in case $Z = R$, then, we have $\tilde{D}_{\Lambda/Z} \subseteq \mathfrak{q}\Lambda$ or $D_{Z/R} \subseteq \mathfrak{q}$. Therefore (3) implies (1). Now it is sufficient to show (2) $\Longrightarrow$ (3). Assume that $\tilde{D}_{\Lambda/Z} \subseteq \mathfrak{q}\Lambda$ or $D_{Z/R} \subseteq \mathfrak{q}$. If $\tilde{D}_{\Lambda/Z} \subseteq \mathfrak{q}\Lambda$, then $\tilde{D}_{\Lambda/R} \subseteq \tilde{D}_{\Lambda/Z} \subseteq \mathfrak{q}\Lambda$. Hence we have only to consider the case that $\tilde{D}_{\Lambda/Z} \not\subseteq \mathfrak{q}\Lambda$ but $D_{Z/R} \subseteq \mathfrak{q}$. To show $\tilde{D}_{\Lambda/R} \subseteq \mathfrak{q}\Lambda$, we may assume that R is a local ring with a maximal ideal $\mathfrak{p}$. If we suppose that R is complete, Z can be expressed as the direct sum of a finite number of complete local rings $Z_1, Z_2, \cdots, Z_t$. Since $Z_\mathfrak{q}$ coincides with one of Z_i's and $\tilde{D}_{\Lambda/R} = \tilde{D}_{Z_1 \underset{R}{\otimes} \Lambda/R} \oplus \tilde{D}_{Z_2 \underset{R}{\otimes} \Lambda/R} \oplus \cdots \oplus \tilde{D}_{Z_t \underset{R}{\otimes} \Lambda/R}$, we may further assume $Z = Z_\mathfrak{q}$. Because $\tilde{D}_{\Lambda/Z} \not\subseteq \mathfrak{q}\Lambda$, Λ is separable over Z. Hence, by (2. 2), we have $D_{\Lambda/R} = D_{Z/R} \cdot \Lambda$, so that $\tilde{D}_{\Lambda/R} = [D_{Z/R}]^2\Lambda \subseteq \mathfrak{q}\Lambda$. Generally, let R^* be the completion of R, and put $Z^* = R^* \underset{R}{\otimes} Z$ and $\Lambda^* = R^* \underset{R}{\otimes} \Lambda$.

Then we have $D_{Z^*/R^*} = R^* \underset{R}{\otimes} D_{Z/R}$, and therefore $D_{Z^*/R^*} \subseteq \mathfrak{q}Z^*$. Since $\Lambda_{\mathfrak{q}}$ is separable over $Z_{\mathfrak{q}}$, $\Lambda^*_{\mathfrak{q}Z^*}$ is also separable over $Z^*_{\mathfrak{q}Z^*}$. According to the preceding argument, we must have $\tilde{D}_{\Lambda^*/R^*} \subseteq \mathfrak{q}\Lambda^*$. From the fact that $\tilde{D}_{\Lambda/R} = \tilde{D}_{\Lambda^*/R^*} \cap \Lambda$, it follows immediately that $\tilde{D}_{\Lambda/R} \subseteq \mathfrak{q}\Lambda$. This completes the proof of the theorem.

It is clear that the different theorem for hereditary orders in [12] is a special case of (3. 6).

§ 4. Differents of hereditary orders

In this section we shall determine completely the structure of the differents of a hereditary order over a Dedekind domain.

LEMMA 4. 1. *Let R be a Dedekind domain with quotient field K and Σ a central simple K-algebra. Let Λ be a hereditary R-order in Σ and put $I(\Sigma) = Trd_{\Sigma/K}(\Lambda)$. Then $I(\Sigma)$ is an ideal of R which depends only on Σ, but which does not depend on Λ.*

Proof. In order to prove this, we may assume that R is a complete discrete valuation ring. Then this follows immediately from [11], (6. 2).

The following lemma is a slight generalization of [3], (8. 4).

LEMMA 4. 2. *Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$, K the quotient field of R and Λ a maximal R-order in a central simple K-algebra. Then Λ is separable over R if and only if $R/\mathfrak{p}$ is the center of $\Lambda/J(\Lambda)$.*

Proof. The only if part of the lemma is well known (cf. [3]). Hence it is sufficient to prove the if part. If we suppose that the center of $\Lambda/J(\Lambda)$ coincides with $R/\mathfrak{p}$, then there exists a splitting field $\bar{L}$ of $\Lambda/J(\Lambda)$ which is a finite separable extension of $R/\mathfrak{p}$ such that $\bar{L} \underset{R/\mathfrak{p}}{\otimes} \Lambda/J(\Lambda)$ is isomorphic to a full matrix algebra over $\bar{L}$. Now we can find a discrete valuation ring S which is finitely generated, and separable over R such that $S/\mathfrak{p}S \cong \bar{L}$. If we denote by L the quotient field of S, $S \underset{R}{\otimes} \Lambda$ is a hereditary S-order in $L \underset{K}{\otimes} \Sigma$ and we have $S \underset{R}{\otimes} \Lambda/J(S \underset{R}{\otimes} \Lambda) \cong \bar{L} \underset{R/\mathfrak{p}}{\otimes} \Lambda/J(\Lambda)$. Then, by [10], (3. 5), $S \underset{R}{\otimes} \Lambda$ is a maximal S-order in $L \underset{K}{\otimes} \Sigma$. Hence we may assume that $\Lambda/J(\Lambda)$ is a full matrix algebra over $R/\mathfrak{p}$, because, in order to prove that Λ is separable, it suffices to show that $S \underset{R}{\otimes} \Lambda$ is separable over S. Furthermore, without loss of generality, we may assume that R is complete. Then, as is well known, both Σ and $\Lambda/J(\Lambda)$ can be considered as division rings.

Therefore we have only to prove $\Lambda = R$ under the assumptions that $\Lambda/J(\Lambda) = R/\mathfrak{p}$ and that Σ is a division K-algebra. In this case, putting $n^2 = \dim_K \Sigma$ and denoting by e the integer such that $\mathfrak{p}\Lambda = [J(\Lambda)]^e$, we have $n^2 = e$. But, on the other hand, we have $n \mid e$ (cf. [17], Chap. 2, Cor. 1 to Th. 11), so that $n = 1$, i.e., $\Sigma = K$. Thus we must have $\Lambda = R$, which completes the proof of the lemma.

Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$, K the quotient field of R and Σ be a central simple K-algebra. Then, by virtue of [10], (4. 6), there exists a division $R/\mathfrak{p}$-algebra Δ such that, for any hereditary R-order Λ in Σ, $\Lambda/J(\Lambda)$ can be expressed as the direct sum of a finite number of full matrix algebras over Δ. Now we denote such a division $R/\mathfrak{p}$-algebra Δ by $\Delta(\Sigma)$.

Let R be a discrete valuation ring and K the quotient field of R. Then a finite separable extension L of K is said to be unramified, if the integral closure of R in L is separable over R. The set of all classes of central simple K-algebras which have splitting fields, unramified over K, is obviously the subgroup of the Brauer group, $Br(K)$, of K. We now denote this subgroup by $V(R)$. More generally, if R is a Dedekind domain with quotient field K, then we put $V(R) = \bigcap_{\mathfrak{p}} V(R_{\mathfrak{p}})$ where $\mathfrak{p}$ runs over all maximal ideals of R. According to [3], (6. 3), we have $Br(R) \subseteq V(R)$.

The following theorem is known partially in case R is complete.

THEOREM 4. 3. *Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$ and K the quotient field. Then, for any central simple K-algebra Σ, the following statements are equivalent:*

(1) *The class, $\tilde{\Sigma}$, of Σ is contained in $V(R)$.*

(2) *$\Delta(\Sigma)$ is separable over $R/\mathfrak{p}$.*

(3) *$I(\Sigma) = R$.*

Proof. In order to prove (2) $\Longleftrightarrow$ (3) we may suppose that R is complete. However, in this case, the implications (2) $\Longleftrightarrow$ (3) are shown in [17], p. 148. Hence we have only to prove (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (2). If we assume $\tilde{\Sigma} \in V(R)$, there exists an unramified extension L of K which is a splitting field of Σ. Let S be the integral closure of R in L. Then, for any hereditary R-order Λ in Σ, $S \otimes_R \Lambda$ is also a hereditary S-order in $L \otimes_K \Sigma$. Since $L \otimes_K \Sigma$ is a full matrix algebra over

L, any maximal S-order Γ in $L \underset{K}{\otimes} \Sigma$ is also a full matrix algebra over S, and therefore $\Gamma/J(\Gamma)$ is separable over $S/\mathfrak{p}S$. Then, by [10], (4. 6), $S \underset{R}{\otimes} \Lambda/J(S \underset{R}{\otimes} \Lambda)$ is also separable over $S/\mathfrak{p}S$. Because $S \underset{R}{\otimes} \Lambda/J(S \underset{R}{\otimes} \Lambda) \cong S/\mathfrak{p}S \underset{R/\mathfrak{p}}{\otimes} \Lambda/J(\Lambda)$, we can conclude that $\Lambda/J(\Lambda)$ is separable over $R/\mathfrak{p}$, i.e., that $\Delta(\Sigma)$ is separable over $R/\mathfrak{p}$.

(2) $\Longrightarrow$ (1). Suppose that $\Delta(\Sigma)$ is separable over $R/\mathfrak{p}$. Then, for any maximal R-order Λ in Σ, $\Lambda/J(\Lambda)$ is separable over $R/\mathfrak{p}$. If we denote by $\bar{L}'$ a maximal commutative separable subfield of $\Delta(\Sigma)$, $\bar{L}'$ is a splitting field of $\Lambda/J(\Lambda)$. Let S' be a discrete valuation ring which is separable over R such that $S'/\mathfrak{p}S' \cong \bar{L}'$ and L' the quotient field of S'. Then $S' \underset{R}{\otimes} \Lambda$ is a hereditary S'-order in $L' \underset{K}{\otimes} \Sigma$ and we have $S' \underset{R}{\otimes} \Lambda/J(S' \underset{R}{\otimes} \Lambda) \cong \bar{L}' \underset{R/\mathfrak{p}}{\otimes} \Lambda/J(\Lambda)$. Therefore $S' \underset{R}{\otimes} \Lambda/J(S' \underset{R}{\otimes} \Lambda)$ is expressible as the direct sum of a finite number of full matrix algebras over $\bar{L}'$. Hence, by the preceding remark, for a maximal S'-order Γ' in $L' \underset{K}{\otimes} \Sigma$, $\Gamma'/J(\Gamma')$ is a full matrix algebra over $\bar{L}'$. So, by virtue of (4. 2), Γ' must be separable over S'. By [3], (6. 3), $L' \underset{K}{\otimes} \Sigma$ has a splitting field L which is an unramified extension of L'. Since L is an unramified extension of K, $\tilde{\Sigma}$ must be contained in $V(R)$.

Lemma 4. 4. *Let R be a Dedekind domain with quotient field K, and Λ a hereditary R-order in a central simple K-algebra. Then, for any maximal ideal $\mathfrak{p}$ of R, $P_\Lambda(\mathfrak{p}\Lambda)$ is an invertible ideal of Λ, and any invertible ideal of Λ can be expressed uniquely as the product of a finite number of $P_\Lambda(\mathfrak{p}\Lambda)$'s. Furthermore the Dedekind different $D_{\Lambda/R}$ of Λ is invertible in Λ.*

Proof. The first part of the lemma was proved in [10], (7. 6), and the second part follows immediately from [6], (4. 3) and a remark in [18], p. 228.

If Λ is a hereditary R-order in a central simple K-algebra, then, by (4. 4), for any maximal ideal $\mathfrak{p}$ of R, there is a positive integer e such that $\mathfrak{p}\Lambda = [P_\Lambda(\mathfrak{p}\Lambda)]^e$. We denote this integer by $e(\mathfrak{p})$ and call it the ramification index of $P_\Lambda(\mathfrak{p}\Lambda)$.

Now, we give, as our main theorem in this section,

Theorem 4. 5. *Let R be a Dedekind domain with quotient field K and Σ a central simple K-algebra. Let Λ be a hereditary R-order in Σ. If we denote by $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ all of the prime ideals $\mathfrak{p}_i$ of R such that $\tilde{\Sigma} \in V(R_{\mathfrak{p}_i})$ and $P_\Lambda(\mathfrak{p}_i\Lambda)$*

$\neq \mathfrak{p}_i\Lambda$ *and by* $\mathfrak{p}'_1, \mathfrak{p}'_2, \cdots, \mathfrak{p}'_s$ *all of the prime ideals* $\mathfrak{p}'_j$ *of* R *such that* $\widetilde{\sum} \notin V(R_{\mathfrak{p}'_j})$, *then we have*

$$D_{\Lambda/R} = \left[\prod_{i=1}^{t} P_\Lambda(\mathfrak{p}_i\Lambda)^{e(\mathfrak{p}_i)-1}\right] \cdot \left[\prod_{j=1}^{s} P_\Lambda(\mathfrak{p}'_j\Lambda)^{e(\mathfrak{p}'_j)-f(\mathfrak{p}'_j)}\right] \cdot [I(\sum)\Lambda],$$

where each $f(\mathfrak{p}'_j)$ *is an integer which depends on* $P_\Lambda(\mathfrak{p}'_j\Lambda)$ *such that* $1 \leqq f(\mathfrak{p}'_j) \leqq e(\mathfrak{p}'_j)$, *and*

$$N_{\Lambda/R} = \left[\prod_{i=1}^{t} \mathfrak{p}_i\right] \cdot \left[\prod_{j=1}^{s} \mathfrak{p}'^{\,\delta(\mathfrak{p}_j)}_j\right] \cdot [I(\sum)]^2,$$

where each $\delta(\mathfrak{p}'_j)$ *means* 0 *or* 1. *Especially* $N_{\Lambda/R}$ *is square-free if and only if* $\widetilde{\sum} \in V(R)$.

Proof. By localizing Λ with respect to every maximal ideal of R, it is sufficient to prove the theorem under the assumption that R is a discrete valuation ring with a maximal ideal $\mathfrak{p}$. Then we have, clearly, $J(\Lambda) = P_\Lambda(\mathfrak{p}\Lambda)$. Since $Trd_{\Sigma/K}(\Lambda) = I(\sum)$, we have $I(\sum)^{-1}\Lambda \subseteq C_{\Lambda/R}$, hence $D_{\Lambda/R} \subseteq I(\sum)\Lambda$. But $D_{\Lambda/R} \nsubseteq \mathfrak{p}I(\sum)\Lambda$. In fact, if $D_{\Lambda/R} \subseteq \mathfrak{p}I(\sum)\Lambda$, then $\mathfrak{p}^{-1}I(\sum)^{-1}\Lambda \subseteq C_{\Lambda/R}$ and so $R = Trd_{\Sigma/K}(C_{\Lambda/R}) \supseteq \mathfrak{p}^{-1}$, which is obviously a contradiction. Therefore, by (4. 4), we have $D_{\Lambda/R} = [J(\Lambda)]^{e(\mathfrak{p})-f} \cdot [I(\sum)\Lambda]$ for an integer f such that $1 \leqq f \leqq e(\mathfrak{p})$. If we suppose $\widetilde{\sum} \in V(R)$, then we have, by (4. 3), $I(\sum) = R$, and therefore $D_{\Lambda/R} = [J(\Lambda)]^{e(\mathfrak{p})-f}$. Because $Trd_{\Sigma/K}(J(\Lambda)) \subseteq \mathfrak{p}$, $Trd_{\Sigma/K}([J(\Lambda)]^{1-e(\mathfrak{p})}) = \mathfrak{p}^{-1}Trd_{\Sigma/K}(J(\Lambda)) \subseteq R$, so that $D_{\Lambda/R} \subseteq [J(\Lambda)]^{e(\mathfrak{p})-1}$. Consequently, in case $\widetilde{\sum} \in V(R)$, we observe $f = 1$. This completes the proof of the assertion for $D_{\Lambda/R}$ in the theorem. According to (2. 3) we have $Trd_{\Sigma/K}(D_{\Lambda/R}) = N_{\Lambda/R}$, so that $N_{\Lambda/R} = I(\sum)Trd_{\Sigma/K}([J(\Lambda)]^{e(\mathfrak{p})-f}) \subseteq [I(\sum)]^2$. However, if we assume $N_{\Lambda/R} \subseteq \mathfrak{p}^2[I(\sum)]^2$, then we have $[D_{\Lambda/R}]^2 \subseteq \mathfrak{p}^2[I(\sum)]^2\Lambda$, and hence, by (4. 4), $D_{\Lambda/R} \subseteq \mathfrak{p}I(\sum)\Lambda$, which is a contradiction. Thus we must have $N_{\Lambda/R} = [I(\sum)]^2$ or $\mathfrak{p}[I(\sum)]^2$. Here we have $N_{\Lambda/R} = \mathfrak{p}[I(\sum)]^2$ if and only if $[D_{\Lambda/R}]^2 \subseteq \mathfrak{p}[I(\sum)]^2\Lambda$, i.e., if and only if $e(\mathfrak{p}) \geqq 2f$. Furthermore, by virtue of (4. 3), we have $\widetilde{\sum} \in V(R)$ if and only if $I(\sum) = R$, i.e., if and only if $N_{\Lambda/R} = R$ or $\mathfrak{p}$. Thus the proof of the theorem is completed.

Evidently, (4. 5) is a generalization of [9], (3. 3. 2).

§ 5. Locally symmetric orders

Let R be a commutative ring with total quotient ring K and $\sum$ a separable K-algebra. A projective R-order Λ in $\sum$ is said to be symmetric

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if Λ is a symmetric R-algebra, and, more generally, Λ is said to be locally symmetric if, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda_{\mathfrak{m}}$ is a symmetric $R_{\mathfrak{m}}$-algebra. In [7] it was proved that any separable, projective R-order in Σ is symmetric.

LEMMA 5. 1. *A projective R-order Λ in Σ is locally symmetric if and only if $C_{\Lambda/R} = \mathfrak{a}\Lambda$ for some invertible fractional ideal $\mathfrak{a}$ of the center of Λ. Furthermore a quasi-Frobenius R-order Λ in Σ is locally symmetric if and only if $D_{\Lambda/R} = \mathfrak{a}\Lambda$ for some invertible fractional ideal $\mathfrak{a}$ of the center of Λ.*

Proof. This follows from the fact that $\Lambda \cong \Lambda^* \cong C_{\Lambda/R}$ as two-sided Λ-modules (cf. [18], p. 228.)

The Dedekind different theorem for locally symmetric orders can be described as follows.

THEOREM 5. 2. *Let R be a regular domain with quotient field K, and Σ a separable K-algebra. Let Λ be a locally symmetric, projective R-order in Σ and denote by Z the center of Λ. Then, for any prime ideal $\mathfrak{P}$ of Λ, the following conditions are equivalent:*

(1) $D_{\Lambda/R} \nsubseteq \mathfrak{P}$.

(2) $D_{\Lambda/R} \nsubseteq (\mathfrak{P} \cap Z)\Lambda$.

(3) $\mathfrak{P}$ *is unramified.*

Proof. Because R is regular, we have $t_{\Sigma/K}(\Lambda) \subseteq R$. By (5. 1), then, there is an invertible ideal $\mathfrak{a}$ of Z such that $D_{\Lambda/R} = \mathfrak{a}\Lambda$. From this the equivalence of (1) and (2) follows directly. Let $\mathfrak{P}$ be a prime ideal of Λ, and put $\mathfrak{q} = \mathfrak{P} \cap Z$ and $\mathfrak{p} = \mathfrak{P} \cap R$. Now, to prove the equivalence of (2) and (3) we may assume that R is a complete regular local ring with a maximal ideal $\mathfrak{p}$ and further that Z is a local ring with a maximal ideal $\mathfrak{q}$. Then $\mathfrak{P}$ is unramified if and only if Λ is separable, i.e., if and only if $D_{\Lambda/R} = \Lambda$ according to (2. 8). Therefore $\mathfrak{P}$ is unramified if and only if $\mathfrak{a} = Z \nsubseteq \mathfrak{q}$, which shows the equivalence of (2) and (3). This concludes the proof of the theorem.

It should be remarked that (5. 2) is not included in (3. 6). (5. 2) can be applied to the group algebra $R\Pi$ of a finite group Π over a Dedekind domain R of characteristic 0.

In the rest of this section we shall be concerned with locally symmetric orders over a Dedekind domain R in a central simple K-algebra.

First we give, as an additional remark to (4. 5),

PROPOSITION 5. 3. *Let R be a Dedekind domain and Σ a central simple K-algebra. Then a hereditary R-order Λ in Σ is locally symmetric if and only if $D_{\Lambda/R} = I(\Sigma)\Lambda$. Especially, if $\tilde{\Sigma} \in V(R)$, a hereditary R-order in Σ is locally symmetric if and only if it is separable.*

Proof. This follows immediately from (4. 4), (4. 5) and (5. 1).

PROPOSITION 5. 4. *Let R be a Dedekind domain and Σ a central simple K-algebra. Let Λ be a locally symmetric R-order in Σ. If Ω is an R-order in Σ such that $\Lambda \subseteq \Omega$ and $Trd_{\Sigma/K}(\Lambda) = Trd_{\Sigma/K}(\Omega)$, then Ω coincides with Λ. In particular, a locally symmetric, hereditary R-order in Σ is always maximal.*

Proof. Without loss of generality we may assume that R is a discrete valuation ring with a maximal ideal uR. Now we can put $Trd_{\Sigma/K}(\Lambda) = Trd_{\Sigma/K}(\Omega) = u^l R$ for some non-negative integer l. From this it follows that $C_{\Lambda/R} = u^{-l}\Lambda$, since Λ is symmetric over R. It is clear that $\Lambda \subseteq \Omega \subseteq C_{\Omega/R} \subseteq C_{\Lambda/R}$, and so we obtain $C_{\Omega/R} \subseteq u^{-l}\Lambda$. However, $Trd_{\Sigma/K}(u^{-l}\Omega) = R$, so that $u^{-l}\Omega \subseteq C_{\Omega/R}$. Thus $u^{-l}\Omega \subseteq u^{-l}\Lambda$, i.e., $\Omega \subseteq \Lambda$. This proves the first part of the proposition. From the first part and (4. 3) we can easily show the second part.

LEMMA 5. 5. *Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$ and Σ a central simple K-algebra. For any maximal R-order Λ in Σ, the following conditions are equivalent:*

(1) *For any unramified extension L of K, denoting by S the integral closure of R in L, $S \otimes_R \Lambda$ is a maximal S-order in $L \otimes_K \Sigma$.*

(2) *The center of $\Lambda/J(\Lambda)$ is a purely inseparable extension of $R/\mathfrak{p}$.*

Proof. We denote the center of $\Lambda/J(\Lambda)$ by $\bar{Z}$. For any separable extension $\bar{S}$ of $R/\mathfrak{p}$, we can find a discrete valuation ring S, separable over R, such that $S/\mathfrak{p}S \cong \bar{S}$. Now assume (1). Then, denoting by L the quotient field of S, $S \otimes_R \Lambda$ is maximal in $L \otimes_K \Sigma$. But we have $S \otimes_R \Lambda/J(S \otimes_R \Lambda) \cong \bar{S} \otimes_{R/\mathfrak{p}} \Lambda/J(\Lambda)$, so that the center of $S \otimes_R \Lambda/J(S \otimes_K \Lambda)$ is isomorphic to $\bar{S} \otimes_{R/\mathfrak{p}} \bar{Z}$. Hence $\bar{S} \otimes_{R/\mathfrak{p}} \bar{Z}$ is a field by [2], (2. 1). Thus $\bar{Z}$ must be purely inseparable over $R/\mathfrak{p}$. Conversely assume (2). Let L be an unramified extension of

K and S the integral closure of R in L. Then $S \underset{R}{\otimes} \Lambda$ is a hereditary S-order in $L \underset{K}{\otimes} \Sigma$, because S is separable over R. Since $S \underset{R}{\otimes} \Lambda / J(S \underset{R}{\otimes} \Lambda) \cong S/\mathfrak{p}S \underset{R/\mathfrak{p}}{\otimes} \Lambda / J(\Lambda)$, we obtain $S_{\mathfrak{q}} \underset{R}{\otimes} \Lambda / J(S_{\mathfrak{q}} \underset{R}{\otimes} \Lambda) \cong S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \underset{R/\mathfrak{p}}{\otimes} \Lambda / J(\Lambda)$ for any maximal ideal $\mathfrak{q}$ of S. From the assumption, then, we can observe that, for each $\mathfrak{q}$, $S_{\mathfrak{q}} \underset{R}{\otimes} \Lambda / J(S_{\mathfrak{q}} \underset{R}{\otimes} \Lambda)$ has a field as its center. Therefore, again by [10], (3. 5), $S \underset{R}{\otimes} \Lambda$ must be maximal in $L \underset{K}{\otimes} \Sigma$.

PROPOSITION 5. 6. *Let R be a Dedekind domain and Σ a central simple K-algebra. If a maximal R-order Λ in Σ is locally symmetric, then, for any maximal ideal $\mathfrak{p}$ of R, the center of $\Lambda / P_{\Lambda}(\mathfrak{p}\Lambda)$ is a purely inseparable extension of $R/\mathfrak{p}$.*

Proof. In order to prove this we may assume that R is a discrete valuation ring with a maximal ideal $\mathfrak{p}$. Let L be an unramified extension of K and S the integral closure of R in L. Then $S \underset{R}{\otimes} \Lambda$ is a hereditary S-order in $L \underset{K}{\otimes} \Sigma$. Furthermore $S \underset{R}{\otimes} \Lambda$ is also a symmetric S-algebra. Hence $S \underset{R}{\otimes} \Lambda$ is maximal in $L \underset{K}{\otimes} \Sigma$ by (5. 4). Thus, by virtue of (5. 5), the center of $\Lambda / J(\Lambda)$ is purely inseparable over $R/\mathfrak{p}$.

By (5. 3), only in case $\widetilde{\Sigma} \notin V(R)$, it may happen that there exists a locally symmetric, non-separable, maximal R-order in Σ. In fact, we shall prove

THEOREM 5. 7. *Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$ and K the quotient field of R. Suppose that R is of characteristic $p > 0$. Then the following conditions are equivalent:*

(1) *The residue class field $R/\mathfrak{p}$ is not perfect.*

(2) *There exists a central simple K-algebra Σ which contains a symmetric, non-separable, maximal R-order.*

Proof. The implication $(2) \Longrightarrow (1)$ was proved in (5. 3). Hence we have only to show $(1) \Longrightarrow (2)$. Suppose that $R/\mathfrak{p}$ is not perfect. Here we shall construct a central simple K-algebra Σ which contains a symmetric, non-separable, maximal R-order. Let u be a prime element of R, i.e., an element of R such that $\mathfrak{p} = uR$, and let X, Y be two indeterminates. Now put $f(X) = X^p - u^{p-1}X - u$. Then $f(X)$ is an Eisenstein polynomial of the Artin-Schreier type. Therefore, if we put $L = K[X]/f(X)K[X]$, L is a

Galois extension of K whose group, G, is a cyclic group of order p, and, furthermore, putting $S = R[X]/f(X)R[X]$ and denoting by x the residue of X in S, S is a discrete valuation ring with a maximal ideal $\mathfrak{q} = xS$ whose quotient field coincides with L, so that any element of G operates on S as an automorphism over R. Since $R/\mathfrak{p}$ is not perfect, there exists an element a of R such that, denoting by $\bar{a}$ the residue of a in $R/\mathfrak{p}$, the polynomial $Y^p - \bar{a}$ is irreducible in $R/\mathfrak{p}[Y]$. We denote by σ the generator of G. Let $S[Y]$ be the non-commutative polynomial ring over S such that $s^\sigma Y = Ys$ for any $s \in S$, and put $\Lambda = S[Y]/(Y^p - a)S[Y]$ and $\sum = L \otimes_S \Lambda$. Obviously $\sum$ is a crossed product. So $\sum$ is a central simple K-algebra and Λ is an R-order in $\sum$. It can be seen easily that $J(\Lambda) = \mathfrak{q}\Lambda \supsetneqq \mathfrak{p}\Lambda$ and $\Lambda/J(\Lambda) = (R/\mathfrak{p})[Y]/(Y^p - \bar{a})(R/\mathfrak{p})[Y]$. Hence Λ is a non-separable, maximal R-order in $\sum$. However, by a direct computation, we obtain $I(\sum) = \mathfrak{p}^{p-1}$ and $C_{\Lambda/R} = \mathfrak{p}^{1-p}\Lambda$, and so Λ is symmetric over R by (5. 3). Thus $\sum$ is a central simple K-algebra as required.

We did not succeed in omitting from (5. 7) the assumption that R is of characteristic $p > 0$ (cf. [8]). However, quite analogously, we obtain

PROPOSITION 5. 8. *Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$ and K the quotient field of R. Suppose that R is of characteristic* 0, *that the residue class field $R/\mathfrak{p}$ is of characteristic $p > 0$ and that R contains the primitive p-th root ζ of* 1. *Then the following conditions are equivalent:*

(1) *$R/\mathfrak{p}$ is not perfect.*

(2) *There exists a central simple K-algebra $\sum$ which contains a non-symmetric, maximal R-order Λ such that $\Lambda/\mathfrak{p}\Lambda$ is symmetric over $R/\mathfrak{p}$ and the center of $\Lambda/J(\Lambda)$ is purely inseparable over $R/\mathfrak{p}$.*

Proof. (2) $\Longrightarrow$ (1) follows immediately from (4. 2) and (4. 3). So it is sufficient to show (1) $\Longrightarrow$ (2). By using $f(X) = X^p - u$ instead of $f(X) = X^p - u^{p-1}X - u$ in the proof of (5. 7), we can construct a crossed product $\sum$ and a maximal R-order Λ in $\sum$ such that $J(\Lambda) = x\Lambda$. Then we have $\Lambda/\mathfrak{p}\Lambda = (S/\mathfrak{p}S)[Y]/(Y^p - \bar{a})(S/\mathfrak{p}S)[Y]$ and $\Lambda/J(\Lambda) = (R/\mathfrak{p})[Y]/(Y^p - \bar{a})(R/\mathfrak{p})[Y]$. Since the residue of ζ in $R/\mathfrak{p}$ coincides with a unit element of $R/\mathfrak{p}$, σ operates trivially on $S/\mathfrak{p}S$, and therefore $\Lambda/\mathfrak{p}\Lambda$ is a commutative local ring. But $\Lambda/\mathfrak{p}\Lambda$ is uniserial $R/\mathfrak{p}$-algebra, because Λ is maximal in $\sum$. Hence $\Lambda/\mathfrak{p}\Lambda$ is a commutative Frobenius $R/\mathfrak{p}$-algebra, so that it is symmetric over $R/\mathfrak{p}$.

However, as is easily seen, $I(\sum) = pR$ and $D_{\Lambda/R} = p\cdot[J(\Lambda)]^{p-1}$. By virtue of (5. 3) this shows that Λ is not symmetric over R. Thus $\sum$ is as required.

From (5. 7) and (5. 8) the following corollary follows directly.

COROLLARY 5. 9. *Let R be a discrete valuation ring with a maximal ideal $\mathfrak{p}$ which satisfies the assumptions in* (5. 7) *or* (5. 8) *and K the quotient field of R. Then the residue class field $R/\mathfrak{p}$ is perfect if and only if $V(R) = Br(K)$.*

Finally we give a remark on symmetric algebra. Let R be a local ring with a maximal ideal $\mathfrak{m}$ and Λ an R-algebra which is a finitely generated free R-module. As is well known (cf. [7]), Λ is separable over R if $\Lambda/\mathfrak{m}\Lambda$ is separable over $R/\mathfrak{m}$. It has been proved in [6], (3. 3) and (3. 4) that Λ is (quasi-) Frobenius over R if $\Lambda/\mathfrak{m}\Lambda$ is (quasi-) Frobenius over $R/\mathfrak{m}$. However (5. 8) shows that the similar assertion for symmetricity is false.

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