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SPECIALISING TREES WITH SMALL APPROXIMATIONS I

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Abstract. Assuming PFA, we shall use internally club ω_1-guessing models as side conditions to show that for every tree T of height ω_2 without cofinal branches, there is a proper and $\aleph_2$-preserving forcing notion with finite conditions which specialises T. Moreover, the forcing has the ω_1-approximation property.

§1. Introduction. By the well-known work of Baumgartner, Malitz, and Reinhardt [2], under Martin's axiom at $\aleph_1$, all trees of height and size ω_1 without cofinal branches are special. Unfortunately, the straightforward generalisations of MA were not thus far capable of specialising ω_2-Aronszajn trees (see [1, 19, 20]). The different behaviour of the specialising problem beyond ω_1 arises from two interconnected factors: the weakness of the current technology of forcing iterations and the nature of trees of height at least ω_2. Thus, the question of finding a legitimate higher version of Martin's axiom, under which every ω_2-Aronszajn tree is special seems challenging (we will say more about this). However, there are still many intriguing results in this research direction. For example, Laver and Shelah [12] showed, assuming the consistency of a weakly compact cardinal, that the ω_2-Suslin Hypothesis is consistent with the Continuum Hypothesis (in fact, they showed that it is consistent that there are ω_2-Aronszajn trees and all of them are special). This result was extended by Golshani and Hayut in [7], where they proved that, modulo the consistency of large cardinals, it is consistent that for every regular cardinal κ, there are κ^+-Aronszajn trees and all of them are special. A more relevant result, where wide trees were involved, was obtained by Golshani and Shelah in [8], that is for a prescribed regular cardinal κ, it is consistent that every tree of height and size κ^+ (with a small number of branches) is weakly special (i.e., there is a colouring with κ colours so that if $s < t, u$ have the same colour, then t and u are comparable). The affinity between these and other similar results is that they rely upon the original technique of Laver and Shelah [12]. Although the main difficulty in proving an iteration theorem for countably closed and $\aleph_2$-c.c forcings is the preservation of $\aleph_2$, it was surmountable by Laver–Shelah's argument due to the particular features of the iterands. The attempts to overcome the difficulty and find a higher analogue of MA have been generally devoted to countably closed forcings until Neeman's discovery [17] of generalised side conditions. His technology allows us to examine

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the connection between the specialisation problem and generalised forms of Martin's axiom, and ask if we still need to consider countably closed forcings in this context. If the consistency of a higher analogue of PFA is achievable, it is then natural to speculate whether such a forcing axiom can imply that all trees in an appropriate subclass of trees of height and size ω_2 are special. As an early application of his method, Neeman [18] attempted to (partially) specialise trees of height ω_2 with finite conditions. To achieve this, he attaches the partial specialising functions to the sequences of models as side conditions. He then demonstrates that the resulting construction belongs to an iterable class which also includes a forcing notion for adding a nonspecial ω_2-Aronszajn tree.

The second factor mentioned above may also lead one to recast the program of finding a generalised MA for the problem of special ω_2-Aronszajn trees, as such trees intrinsically involve a particular compactness phenomenon. One can use some forms of the square principle to construct trees without cofinal branches that cannot be special, even in transitive outer models with the same cardinals. The basic idea goes back to Laver (see [21]) who isolated the concept of an ascending path through a tree and showed that an ω_2-Aronszajn tree with an ascending path is non-special even in any transitive outer model that computes the relevant cardinals correctly. However, the earliest example of a non-special ω_2-Aronszajn tree was constructed by Baumgartner using $\square_{\omega_1}$, which was also independently discovered and generalised by Shelah and Stanley [21]. They showed that $\square_\lambda$ implies the existence of non-specialisable λ^+-Aronszajn trees. The connection between square-like principles and ascending paths through trees or tree-like systems has been studied by several people, just to mention a few: Baumgartner (as mentioned above), Brodsky and Rinot [3], Devlin [6], Cummings [5], Lambie-Hanson [10], Lambie-Hanson and Lücke [11], Laver and Shelah [12], Lücke [13], Neeman [18], Shelah and Stanley [21], and Todorčević [22].

To see why specialising a tree of height beyond ω_1 is subtly different from that of a tree of height ω_1, let us first recall that the standard forcing to specialises a tree T of height κ^+ uses partial specialising functions of size less than κ, and let us denote this forcing by $\mathbb{S}_\kappa(T)$. For a cardinal $\lambda \leq \kappa$, $\mathbb{S}_\lambda(T)$ is defined naturally. Lücke [13] studied the chain condition of $\mathbb{S}_\lambda(T)$, and complete the bridge between the notion of an ascending path and the chain condition of $\mathbb{S}_\lambda(T)$. Under some cardinal arithmetic assumptions, he showed that the nonexistence of a *weak form of ascending paths*[1] of width less than λ through T is equivalent to the κ^+-chain condition of $\mathbb{S}_\lambda(T)$. Note that it is easily seen that $\mathbb{S}_\lambda(T)$ collapses κ^+ if T has a cofinal branch. Observe that also by Baumgartner–Malitz–Reinhardt [2], if T is of height ω_1 without cofinal branches, then $\mathbb{S}_\omega(T)$ has the countable chain condition, as the existence of a cofinal branch through such tree is equivalent to the existence of a (weak) ascending path of finite length. It is also not hard to see that if $\kappa = \omega_1$ and the CH fails, then $\mathbb{S}_{\omega_1}(T)$ collapses the continuum onto ω_1. Thus not only the CH is necessary for preserving $\aleph_2$, but also by Lücke's result, the lack of cofinal branches through T is not enough to ensure that $\mathbb{S}_{\omega_1}(T)$ preserves $\aleph_2$. On the other hand, if T is of height ω_2 and has no cofinal branches, then $\mathbb{S}_\omega(T)$ has the $\aleph_2$-chain condition, but then the question is how to preserve ω_1?

[1] See [13] for the definition.

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Therefore, the behaviour of the continuum function and the existence of ascending paths of width ω can prevent us from specialising trees of height ω_2 merely with countable conditions. Lücke [13] asked the following questions:

(1) Assume PFA. Is every tree of height ω_2 without cofinal branches specialisable?
(2) If T is a tree of height κ^+, for an uncountable regular cardinal κ without ascending paths of width less than κ, is then T specialisable?

Let us end our discussion with a couple of general questions: Do we still need to consider the specialisation of all ω_2-Aronszajn trees in the context of generalised Martin's axiom? If looking for a generalised MA, do we want to have some kinds of compactness at $\aleph_2$ or not?

In this paper, we prove the following theorem.

THEOREM 1.1. *Assume* PFA. *Every tree of height ω_2 without cofinal branches is specialisable via a proper and $\aleph_2$-preserving forcing with finite conditions. Moreover, the forcing has the ω_1-approximation property.*

This theorem answers Lücke's first question in the affirmative.[2] Given a tree T of height ω_2 with no cofinal branches, we shall use internally club ω_1-guessing models to construct a proper forcing notion $\mathbb{P}_T$ similar to Neeman's in [18], so that forcing with $\mathbb{P}_T$ specialises T. Notice that the existence of sufficiently many ω_1-guessing models of size ω_1 implies the failure of certain versions of the square principle. It is also worth mentioning that by an observation due to Lücke, the existence of sufficiently many ω_1-guessing models of size $\aleph_1$ (and hence under PFA) no tree of height ω_2 without cofinal branches contains an ascending path of width ω. Interestingly, we will not use this fact, as the presence of guessing models in our side conditions suffices. By a theorem due to Viale and Weiß [25], under PFA, there are stationarily many internally club guessing models, and by a theorem due to Cox and Krueger [4], this consequence of PFA is consistent with arbitrarily large continuum. Thus essentially, the fact that $2^{\aleph_0} = \aleph_2$ holds under PFA does not play a role in our result and proofs.

We shall also answer the second question above consistently in the affirmative, for trees of height κ^{++} without cofinal branches, in our forthcoming paper [16], which in particular includes a proof of the following theorem.

THEOREM 1.2 [16]. *Assume κ is a regular cardinal, and that $\lambda > \kappa$ is a supercompact cardinal. Then in generic extensions by some $<\kappa$-closed forcing notion, $\kappa^{<\kappa} = \kappa$, $2^\kappa = \lambda = \kappa^{++}$, and every tree of height κ^{++} without cofinal branches is specialisable via some $<\kappa$-closed forcing which preserves κ^+ and κ^{++}.*

Our paper includes four additional sections. We give the preliminaries in Section 2. Section 3 is devoted to the introduction and the basic properties of forcing with pure side conditions. We shall introduce our main forcing and state its basic properties in Section 4. Finally, we establish our main result in Section 5.

§2. Preliminaries. We shall follow standard conventions and notation, but let us recall some of the most important ones. In this paper, by $p \leq q$ in a forcing ordering

[2] To be precise, Lücke's definition of the specialisability of a tree T requires preservation of all cardinal up to the size of T; however in our theorem the size of T will be collapsed to $\aleph_2$.

$\leq$, we mean p is stronger than q; for a cardinal θ, H_θ denotes the collection of sets whose hereditary size is less than θ; for a set X, we let $\mathcal{P}(X)$ denote the power-set of X, and if κ is a cardinal, we let $\mathcal{P}_\kappa(X) := \{A \in \mathcal{P}(X) : |A| < \kappa\}$; recall that a set $\mathcal{S} \subseteq \mathcal{P}_\kappa(H_\theta)$ is stationary, if for every function $F : \mathcal{P}_{\aleph_0}(H_\theta) \to \mathcal{P}_\kappa(H_\theta)$, there is $M \prec H_\theta$ in $\mathcal{S}$ with $M \cap \kappa \in \kappa$ such that M is closed under F.

2.1. Trees. Let us recall the definition of a tree and some related concepts.

Definition 2.1. A *tree* is a partially ordered set $(T, <_T)$ such that for every $t \in T$, $b_t := \{s \in T : s <_T t\}$ is well ordered with respect to $<_T$.

Definition 2.2. Suppose $T = (T, <_T)$ is a tree.

(1) For every $t \in T$, the *height* of t, denoted by $\mathrm{ht}_T(t)$, is the order type of b_t.
(2) The *height* of T, denoted by $\mathrm{ht}(T)$, is $\sup\{\mathrm{ht}_T(t) + 1 : t \in T\}$.
(3) For every $\alpha \leq \mathrm{ht}(T)$, T_α denotes the set of nodes of height α. $T_{\leq\alpha}$ and $T_{<\alpha}$ have the obvious meanings. In particular, $T = T_{<\mathrm{ht}(T)}$ and $T_{\mathrm{ht}(T)} = \varnothing$.
(4) A set $b \subseteq T$ is called a *branch* through T if $(b, <_T)$ is a downward-closed and linearly ordered set. A branch is a *cofinal branch* if its order type is the height of T.
(5) T is called *Hausdorff* if for every limit ordinal $\alpha (\alpha = 0$ is allowed$)$, and every $t \neq s$ in T_α, we have $b_t \neq b_s$.
(6) For every $t \in T$, we let $\overline{b}_t$ denotes $\{s \in T : s \leq_T t\}$.

Observe that a Hausdorff tree is rooted, i.e., it has a unique minimal point.

Definition 2.3. Suppose κ is an infinite cardinal. A tree $(T, <_T)$ of height κ^+ is called *special* if there is a *specialising function* $f : T \to \kappa$, i.e., if $s <_T t$, then $f(s) \neq f(t)$.

Definition 2.4. Suppose that $\lambda \leq \kappa$ are infinite regular cardinals. Assume that T is a tree of height κ^+. Let $\mathbb{S}_\lambda(T)$ denote the forcing notion consisting of partial specialising functions, of size less than λ, ordered by reversed inclusion, that is $f \in \mathbb{S}_\lambda(T)$ is a partial function from T to κ such that if $s, t \in \mathrm{dom}(f)$ are comparable in T, then $f(t) \neq f(s)$.

Lemma 2.5. *In order to specialise a tree T (of height κ^+, for some infinite cardinal κ), one may assume, without loss of generality, that T is a Hausdorff tree.*

Proof. Recall that a function $f : T_1 \to T_2$ between two trees is called a weak embedding if f respects the strict orders. It is easily seen that if T_1 weakly embeds into T_2 and T_2 is special, then T_1 is special, as the inverse image of an antichain in T_2 under a weak embedding is an antichain in T_1. Thus to prove the lemma, it is enough to show that there is a weak embedding from T into a Hausdorff tree T^* of the same height as T.

Let T^* be the set of all non cofinal branches through T. Then, $(T^*, \subset)$ is a tree of the same height as T. Note that $\varnothing$ is the root of T^*. Moreover, if $a \in T^*$, then the order type of $(a, <_T)$ is exactly $\mathrm{ht}_{T^*}(a)$. Suppose that α is a nonzero limit ordinal and $a, a' \in T^*_\alpha$ with $b_a = b_{a'}$. We claim that $a = a'$. Let $t \in a$. Since the order type of a is a limit ordinal, there is $s \in a$ with $t <_T s$. Let $x = \{u \in T : u <_T s\}$. Now $x <_{T^*} a$. Thus $x \in b_a = b_{a'}$. Then $t \in x \subseteq a'$. So we have $a \subseteq a'$. Similarly, we

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have $a' \subseteq a$, and therefore, $a = a'$. Now, let $f : T \to T^*$ be defined by $f(t) = b_t$. If $s < t$, then b_s is a proper subset of b_t, and hence f is a weak embedding. $\dashv$

2.2. Strong properness and the approximation property. Recall that if $M \prec H_\theta$ contains a forcing $\mathbb{P}$, then a condition $p \in \mathbb{P}$ is called $(M, \mathbb{P})$-generic if for every dense subset D of $\mathbb{P}$ in M, $D \cap M$ is pre-dense below p.

Definition 2.6. Assume that $\mathbb{P}$ is a forcing, and θ is a sufficiently large regular cardinal. Suppose $\mathcal{S} \subseteq \mathcal{P}_\kappa(H_\theta)$ consists of elementary submodels. Then, $\mathbb{P}$ is said to be *proper for* $\mathcal{S}$, if for every $M \in \mathcal{S}$ and every $p \in \mathbb{P} \cap M$, there is an $(M, \mathbb{P})$-generic condition $q \leq p$.

Lemma 2.7. *Let κ be a regular cardinal. Assume that $\mathbb{P}$ is a forcing, and $\theta > \kappa$ is a sufficiently large regular cardinal. Suppose $\mathcal{S} \subseteq \mathcal{P}_\kappa(H_\theta)$ is a stationary set of elementary submodels. If $\mathbb{P}$ is proper for $\mathcal{S}$, then $\mathbb{P}$ preserves the regularity of κ.*

Proof. Let $\gamma < \kappa$ be an ordinal. Assume towards a contraction that some $p \in \mathbb{P}$ forces that $\dot{f}$ is an unbounded function from γ into κ. Pick $M \in \mathcal{S}$ such that $\gamma, \kappa, p, \dot{f} \in M$. Let $q \leq p$ be an $(M, \mathbb{P})$-generic condition. Note that $\gamma \subseteq M$ and $M \cap \kappa \in \kappa$. By our assumption, we can find a condition $q' \leq q$, and ordinal $\zeta < \gamma$ and an ordinal $\delta \geq M \cap \kappa$ such that, $q' \Vdash$ "$\dot{f}(\zeta) = \delta$." Set

$$D = \{r \leq p : r \text{ decides the value } \dot{f}(\zeta)\} \cup \{r \in \mathbb{P} : r \perp p\}.$$

Then D is a dense subset of $\mathbb{P}$ and belongs to M. Since q is $(M, \mathbb{P})$-generic, there is $r \in D \cap M$ such that $r || q'$. Thus r is compatible with p, and hence, by elementarity, there is $\delta' \in M$ such that $r \Vdash$ "$\delta' = \dot{f}(\zeta)$." Now if s is a common extension of q' and r, we have $s \Vdash$ "$\delta' = \delta$." Thus $\delta' = \delta \in M \cap \kappa$, a contradiction! $\dashv$

Let us now recall the following closely related definitions from [9, 14], respectively.

Definition 2.8 (Strong properness). Suppose $\mathbb{P}$ is a forcing notion.

(1) Let X be a set. A condition $p \in \mathbb{P}$ is said to be *strongly* $(X, \mathbb{P})$*-generic*, if for every $q \leq p$, there is some $q \restriction_X \in X \cap \mathbb{P}$ such that every condition $r \in \mathbb{P} \cap X$ extending $q \restriction_X$ is compatible with q.
(2) For a collection of sets $\mathcal{S}$, we say $\mathbb{P}$ is *strongly proper for* $\mathcal{S}$, if for every $X \in \mathcal{S}$ and every $p \in \mathbb{P} \cap X$, there is a strongly $(X, \mathbb{P})$-generic condition extending p.

Remark 2.9. It is easily seen that if p is strongly $(X, \mathbb{P})$-generic and $M \prec H_\theta$ is such that $M \cap \mathbb{P} = X \cap \mathbb{P}$, then p is strongly $(M, \mathbb{P})$-generic, and hence $(M, \mathbb{P})$-generic. It turns out that if a forcing notion is strongly proper for some stationary set $\mathcal{S} \subseteq \mathcal{P}_\kappa(H_\theta)$, then $\mathbb{P}$ is $\mathcal{S}$-proper, and hence it preserves κ, by Lemma 2.7.

Definition 2.10 (κ-approximation property). Suppose κ is an uncountable regular cardinal. A forcing notion $\mathbb{P}$ has the *κ-approximation property*, if for every V-generic filter G, and every $A \in V[G]$ with $A \subseteq V$, the following are equivalent.

(1) $A \in V$.
(2) For every $a \in V$ with $|a|^V < \kappa$, we have $a \cap A \in V$.

Note that it is well-known that if a forcing notion is strongly proper for sufficiently many models in $\mathcal{P}_\kappa(H_\theta)$, then it has the κ-approximation property (see [15]).

2.3. Guessing models. For a set M, we say that a set $x \subseteq M$ is *bounded in* M if there is $y \in M$ such that $x \subseteq y$. Recall that an elementary submodel M of H_θ is called an *internally club model* (or IC-*model* for short) if it is the union of a continuous $\in$-sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of H_θ.

NOTATION 2.11. *For a model $M \prec H_\theta$, let $\kappa_M = \min\{\alpha \in M \cap \theta : \alpha \nsubseteq M\}$. Let κ_M be undefined if the above supremum does not exist.*

DEFINITION 2.12. Suppose M is a set. A set x is *guessed* in M if there is some $x^* \in M$ such that $x^* \cap M = x \cap M$.

We now recall the definition of a guessing model from [25].

DEFINITION 2.13 (γ-guessing model). Assume that θ is an uncountable regular cardinal. Let $M \prec H_\theta$. Suppose that $\gamma \in M$ is a regular cardinal with $\gamma \leq \kappa_M$. Then M is said to be a γ-*guessing model* if the following are equivalent for any x which is bounded in M.

(1) x is γ-approximated in M, i.e., $x \cap a \in M$, for all $a \in M$ of size less than γ.
(2) x is guessed in M.

DEFINITION 2.14 ($\mathrm{GM}^*(\omega_2)$). The principle $\mathrm{GM}^*(\omega_2)$ states that for every sufficiently large regular cardinal θ, the set of ω_1-guessing elementary IC-submodels of H_θ is stationary in $\mathcal{P}_{\omega_2}(H_\theta)$.

The above principle is slightly stronger than Weiß's $\mathrm{ISP}(\omega_2)$ (see [26, 27] for more information on $\mathrm{ISP}(\omega_2)$), which is also equivalent to the principle $\mathrm{GM}(\omega_2)$ that states for every sufficiently large regular cardinal θ, the set of ω_1-guessing elementary submodels of H_θ is stationary in $\mathcal{P}_{\omega_2}(H_\theta)$.

PROPOSITION 2.15 (Viale–Weiß [25]). PFA *implies* $\mathrm{GM}^*(\omega_2)$.

PROOF. The proposition above was mentioned without proof in [25]. A sketch of a proof can be found in [24, Theorem 4.4]. $\dashv$

The following lemma plays a crucial role in our later proofs.

LEMMA 2.16. *Suppose θ is an uncountable regular cardinal. Assume that $M \prec H_\theta$ is countable. Let $Z \in M$ a set. Suppose that $z \mapsto f_z$ is a function on $\mathcal{P}_{\omega_1}(Z)$ in M, where for each $z \in \mathcal{P}_{\omega_1}(Z)$, f_z is a $\{0,1\}$-valued function with $z \subseteq \mathrm{dom}(f_z)$. Assume that $f : Z \cap M \to 2$ is a function that is not guessed in M. Suppose that $B \in M$ is a cofinal subset of $\mathcal{P}_{\omega_1}(Z)$. Then there is $B^* \in M$ cofinal in B such that for every $z \in B^*$, $f_z \nsubseteq f$.*

PROOF. For each $\zeta \in Z$, and $\varepsilon = 0, 1$, let

$$A_\zeta^\varepsilon = \{z \in B : \zeta \in \mathrm{dom}(f_z) \text{ and } f_z(\zeta) = \varepsilon\}.$$

Notice that the sequence

$$\langle A_\zeta^\varepsilon : \zeta \in Z, \varepsilon \in \{0,1\} \rangle$$

belongs to M. We are done if there is some $\zeta \in Z$ such that both A_ζ^0 and A_ζ^1 are cofinal in B, as then by elementarity one can find such $\zeta \in M \cap Z$, and then pick $A_\zeta^{1-f(\zeta)}$. Therefore, let us assume that for every $\zeta \in Z$, there is an $\varepsilon \in \{0,1\}$, which

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is necessarily unique, such that A_ζ^ε is cofinal in B. Now, define h on Z by letting $h(\zeta)$ be ε if and only if A_ζ^ε is cofinal is B. Clearly h is in M, but then $h \restriction_M \neq f$ since f is not guessed in M. Thus, there exists $\zeta \in M \cap Z$ such that $h(\zeta) \neq f(\zeta)$, but it then implies that $A_\zeta^{1-f(\zeta)}$ is cofinal in B and belongs to M. Let B^* be $A_\zeta^{1-f(\zeta)}$. Now if $z \in B^*$, $f_z \nsubseteq f$. ⊣

§3. Pure side conditions. This section is devoted to the forcing with pure side conditions. Such a forcing notion, as well as a finite-support iteration of proper forcings with side conditions, was introduced by Neeman in [17]. However, we cannot use Neeman forcing directly, since we shall work with non-transitive models. Instead, we follow Veličković's presentation [23] of Neeman forcing with finite $\in$-chains of models of two types, where both types of models are non-transitive. We shall sketch some proofs of the necessary facts in this section, and we encourage the reader to consult [23] for more details.

Fix an uncountable regular cardinal θ, and let $x \in H_\theta$ be arbitrary. We let $\mathcal{E}^0 := \mathcal{E}^0(x)$ denote the collection of all countable elementary submodels of $(H_\theta, \in, x)$, and let $\mathcal{E}^1 := \mathcal{E}^1(x)$ denote a collection of elementary IC-submodels of $(H_\theta, \in, x)$. Note that for every $N \in \mathcal{E}^1$ and every $M \in \mathcal{E}^0$, if $N \in M$, then $N \cap M \in \mathcal{E}^0 \cap N$.

Definition 3.1. Assume that $\mathcal{M} \subseteq \mathcal{E}^0 \cup \mathcal{E}^1$.

(1) Suppose that $M, N \in \mathcal{M}$. We say M is below N in $\mathcal{M}$, or equivalently N is above M in $\mathcal{M}$, and denote this by $M \in^* N$ if there is a finite set $\{M_i : i \leq n\} \subseteq \mathcal{M}$ such that $M = M_0 \in \cdots \in M_n = N$.
(2) We say $\mathcal{M}$ is an $\in$-chain, if for every distinct $M, N \in \mathcal{M}$, either $M \in^* N$ in $\mathcal{M}$ or $N \in^* M$ in $\mathcal{M}$.
(3) We say $\mathcal{M}$ is closed under intersections if for every $M \in \mathcal{M} \cap \mathcal{E}^0$, and every $N \in M \cap \mathcal{M}$, $N \cap M$ belongs to $\mathcal{M}$.
(4) If $M, N \in \mathcal{M} \cup \{\varnothing, H_\theta\}$, then by $(M, N)_{\mathcal{M}}$, and intervals of other types, we mean that the interval is considered in the linearly ordered structure $(\mathcal{M}, \in^*)$, e.g., $(M, N)_{\mathcal{M}} = \{P \in \mathcal{M} : M \in^* P \in^* N\}$.

It is easily seen that if $M \in^* N$ holds in an $\in$-chain $\mathcal{M}$, and that $N \in \mathcal{E}^1$, then $M \in N$. We simply write $M \in^* N$, whenever $\mathcal{M}$ is clear from the context.

Remark 3.2. If $M, N \in \mathcal{E}^0$, then $M \subseteq N$ if and only if there is no $P \in \mathcal{E}^1 \cap \mathcal{M}$ with $P \cap N \in^* M \in^* P \in N$.

Definition 3.3 (Forcing with pure side conditions). We let $\mathbb{M}(\mathcal{E}^0, \mathcal{E}^1)$ denote the collection of $\in$-chains $p = \mathcal{M}_p \subseteq \mathcal{E}^0 \cup \mathcal{E}^1$ which are closed under intersections. We consider $\mathbb{M}(\mathcal{E}^0, \mathcal{E}^1)$ as a notion of forcing ordered by reversed inclusion.

We simply denote $\mathbb{M}(\mathcal{E}^0, \mathcal{E}^1)$ by $\mathbb{M}$ whenever there are no confusions. For a condition $p \in \mathbb{M}$, we let also $\mathcal{E}^0_p$ and $\mathcal{E}^1_p$ denote $\mathcal{M}_p \cap \mathcal{E}^0$ and $\mathcal{M}_p \cap \mathcal{E}^1$, respectively. If $p = (\mathcal{M}_p, \dots)$ is a condition in a forcing notion with $\mathcal{M}_p \in \mathbb{M}$, we denote the interval $(M, N)_{\mathcal{M}_p}$ by $(M, N)_p$; such an agreement applies to other types of intervals as well.

Definition 3.4. Let $M \in \mathcal{E}^0 \cup \mathcal{E}^1$, and suppose that $p \in \mathbb{M} \cap M$. We let p^M denote the closure of $\mathcal{M} \cup \{M\}$ under intersections.

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The following is easy and we leave the proof to the reader.

FACT 3.5 [23, Lemma 1.8]. *Let $M \in \mathcal{E}^0 \cup \mathcal{E}^1$, and suppose that $p \in \mathbb{M} \cap M$.*

(1) *If $M \in \mathcal{E}^1$, then $p^M = \mathcal{M} \cup \{M\}$.*
(2) *If $M \in \mathcal{E}^0$, then $p^M = \mathcal{M} \cup \{M\} \cup \{N \cap M : N \in \mathcal{E}^1_p\}$.*
(3) *p^M is a condition in $\mathbb{M}$ and extends p.*

DEFINITION 3.6. For a condition $p \in \mathbb{M}$ and a model $M \in \mathcal{M}_p$, let $p\restriction_M := \mathcal{M}_p \cap M$.

Notice that $p\restriction_M$ is in M, as it is a finite subset of M. If M is in $\mathcal{E}^1$, then $p\restriction_M$ is the interval $(\varnothing, M)_p$ that is an $\in$-chain, but if M is countable, then it is a union of intervals.

FACT 3.7 [23, Fact 1.7]. *Suppose that $p \in \mathbb{M}$. Assume that $M \in \mathcal{M}_p$ is countable. Then*

$$\mathcal{M}_p\restriction_M := \mathcal{M}_p \cap M = \mathcal{M}_p \setminus \bigcup\{[N \cap M, N)_p : N \in (\mathcal{E}^1_p \cap M) \cup \{H_\theta\}\}.$$

PROOF. Let $P \in \mathcal{M}_p\restriction_M$. Thus $P \in M$, which in turn implies that P does not belong to the interval $[M, H_\theta)_p$. Now, let $N \in \mathcal{E}^1_p \cap M$. If $N \in^* P$ or $N = P$, then P does not belong to the interval $[N \cap M, N)_p$. Suppose $P \in^* N$, then $P \in N$, and hence $P \in N \cap M$, which in turn implies that $P \notin [N \cap M, N)_p$. Therefore, the LHS is a subset of RHS. To see the other direction, suppose P does not belong to any interval as described in the above equation. In particular, $P \in^* M$. Now, if $P \notin M$, it then means there are some models in $\mathcal{E}^1_p \cap (P, M)_p$. Let N be the least such model. Then, $N \cap M \in^* P$, since otherwise by the minimality of N, we have $P \in N \cap M \subseteq M$. Thus P belongs to $[N \cap M, N)_p$, which is a contradiction. ⊣

It is not hard to see that $p\restriction_M$ is an $\in$-chain. Now, the following is immediate.

FACT 3.8. *For every condition $p \in \mathbb{M}$ and $M \in \mathcal{M}_p$, $p\restriction_M$ is a condition and $p \leq p\restriction_M$.*

Thus we also have $\mathcal{M}_{p\restriction_M} = \mathcal{M}_p\restriction_M$! This notational equality will be useful later.

FACT 3.9 [23, Fact 1.12]. *Suppose that $p \in \mathbb{M}$ and $M \in \mathcal{E}^1_p$. Then every condition $q \in M$ extending $p\restriction_M$ is compatible with p.*

PROOF. Let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. It is easy to see that $\mathcal{M}_r$ is closed under intersections. To see that it is an $\in$-chain, suppose that $P \in \mathcal{M}_p \setminus \mathcal{M}_q$ and $Q \in \mathcal{M}_q \setminus \mathcal{M}_p$. If $P \neq M$, we then have $Q \in M \in^* P$, and if $P = M$, then obviously $Q \in M$. It is clear that $r \leq p, q$. ⊣

REMARK 3.10. The above condition is the greatest lower bound of p and q, and denoted by $p \wedge q$. Notice that

$$\mathcal{M}_{p \wedge q} = \mathcal{M}_p \cup \mathcal{M}_q.$$

FACT 3.11. *$\mathbb{M}$ is strongly proper for $\mathcal{E}^1$, and hence if $\mathcal{E}^1$ is stationary, then $\mathbb{M}$ preserves $\aleph_2$.*

PROOF. Suppose that $M \in \mathcal{E}^1$. If $p \in M \cap \mathbb{M}$, then by Fact 3.5, p^M is a condition extending p. Let $q \leq p^M$, then $M \in \mathcal{M}_q$. By Fact 3.8, $q\restriction_M$ is a condition in $M \cap \mathbb{M}$.

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Now if $r \in M \cap \mathbb{M}$ extends $q \restriction_M$, then q is compatible with r by Fact 3.9. Thus q is strongly $(M, \mathbb{M})$-generic. By Lemma 2.7 and Remark 2.9, $\mathbb{P}$ perseveres $\aleph_2$. ⊣

LEMMA 3.12 [23, Lemma 1.12]. *Suppose that* $p \in \mathbb{M}$. *Let* $M \in \mathcal{E}^0_p$. *Then every condition* $q \in M$ *extending* $p \restriction_M$ *is compatible with* q. *In fact, the closure of* $\mathcal{M}_p \cup \mathcal{M}_q$ *is a condition in* $\mathbb{M}$, *which is also the greatest lower bound of* p *and* q.

REMARK 3.13. As before we again denote the above common extension by $p \wedge q$. Notice that

$$\mathcal{M}_{p\wedge q} = \mathcal{M}_p \cup \mathcal{M}_q \cup \{N \cap M : N \in \mathcal{E}^1_q,\ M \in \mathcal{E}^0_p, \text{ and } \mathrm{N} \in M\}.$$

The following is similar to Fact 3.11 in light of Lemma 3.12.

FACT 3.14. $\mathbb{M}$ *is strongly proper for* $\mathcal{E}^0$.

§4. The forcing construction.

In this section, we first present the phenomenon of *overlapping* that was introduced by Neeman in his paper [18] regarding (partial) specialisation of trees of height and size ω_2. Neeman's strategy is to attach $\mathbb{S}_\omega(T)$ to side conditions consisting of models of two types: countable and transitive, where he also requires several constraints describing the interaction of the working parts, which are elements of $\mathbb{S}_\omega(T)$, and the models as side conditions. He then analyses this interaction. Our approach is similar to Neeman's, and we still need to require one of the fundamental constraints, though our forcing is simpler than Neeman's. His definition of overlapping reads as follows: A model M overlaps a node $t \in T \setminus M$, if there is no non-cofinal branch $b \in M$ with $t \in b$. Our terminology is different from Neeman's; we say a node $t \in T$ is guessed in M if t belongs to some (non-cofinal) branch $b \in M$.

Throughout this section, we fix a Hausdorff tree $(T, <_T)$ of height ω_2 without cofinal branches. We also fix a regular cardinal θ such that $\mathcal{P}(T) \in H_\theta$. We let $\mathcal{E}^0 := \mathcal{E}^0(T)$ and $\mathcal{E}^1 := \mathcal{E}^1(T)$ consist, respectively, of countable elementary submodels, and ω_1-guessing elementary IC-submodels of $(H_\theta, \in, T)$. We reserve the symbols p, q, r for forcing conditions, and s, t, u for nodes in T.

4.1. Overlaps between models and nodes.

CONVENTION 4.1. *A branch through* T *is called a* T*-branch.*

DEFINITION 4.2. Suppose that $t \in T$ and $M \in \mathcal{E}^0 \cup \mathcal{E}^1$. We abuse language and say t is *guessed* in M if and only if there is a T-branch $b \in M$ with $t \in b$.

Thus every $t \in M$ is already guessed in M, and that no node t with $\mathrm{ht}(t) \geq \sup(M \cap \omega_2)$ is guessed in M, since M has no cofinal branches. We shall often use the following without mentioning.

LEMMA 4.3. *Suppose that* $t \in T$ *and* $M \in \mathcal{E}^0 \cup \mathcal{E}^1$. *If there is* $s \in M$ *with* $t \leq_T s$, *then* t *is guessed in* M.

PROOF. Pick $s \in T \cap M$ with $t \leq_T s$. Then $\bar{b}_s \in M$ is a T-branch and $t \in \bar{b}_s$. ⊣

NOTATION 4.4. *Assume that* $t \in T$ *and* $M \in \mathcal{E}^0 \cup \mathcal{E}^1$. *Then*

- $\eta_M(t)$ *denotes* $\sup\{\mathrm{ht}(s) : s \in T \cap M$ *and* $s \leq_T t\}$.

- *$O_M(t)$ denotes the unique node $s \in T_{\eta_M(t)}$ such that $s \leq_T t$.*
- *$b_M(t)$ denotes $b_{O_M(t)}$.*

Observe that $O_M(t)$ is always well-defined as T is a rooted tree belonging to every model in $\mathcal{E}^0 \cup \mathcal{E}^1$. By definition, we have $\eta_M(t) \leq \sup(M \cap \omega_2)$. In our analysis, we shall focus on $O_M(t)$ rather than t itself. It would be useful to have this intuition that if $t \notin M$, then the node $O_M(t)$ is where b_t detaches from M. We shall see that if $M \in \mathcal{E}^1$, then not only $\eta_M(t)$ is less than $M \cap \omega_2$, but also if its cofinality is uncountable, then $O_M(t)$ is in M. Moreover, if $M \in \mathcal{E}^1$, then t is guessed in M if and only if $t = O_M(t) \in M$. The situation is different for countable models, as if $M \in \mathcal{E}^0$ and $t \in M$ is of uncountable height in T, then one can find some $s \in b_t \setminus M$. Such an s is necessarily guessed in M though it does not belong to M.

LEMMA 4.5. *Suppose that $t \in T$ and $M \in \mathcal{E}^0 \cup \mathcal{E}^1$.*

(1) *If t is guessed in M and $\eta_M(t) \in M$, then $t \in M$.*
(2) *If t is guessed in M, but $\eta_M(t) \notin M$, then* $\mathrm{ht}(t) \leq \min(M \cap \omega_2 \setminus \eta_M(t))$.

PROOF. Of course, the first item follows from the proof of the second one, but we prefer to give independent proofs.

(1) Assume that $b \in M$ is a T-branch containing t. Pick $s \in b \cap M$ of height $\eta_M(t)$, which is possible as $t \in b$ implies that the order-type of b is at least $\eta_M(t) + 1$. Thus $s \leq_T t$. On the other hand, if $s <_T t$, then there is $u \in b \cap M$ of height $\eta_M(t) + 1$, but then $u \leq_T t$, which is impossible by the definition of $\eta_M(t)$. Thus $t = s \in M$.
(2) We may assume that M is in $\mathcal{E}^0$ as otherwise it is trivial. One easily observes that $\eta_M(t)$ is below $\sup(M \cap \omega_2)$ since T does not have cofinal branches. Now $\eta^* := \min(M \cap \omega_2 \setminus \eta_M(t))$ is an ordinal below ω_2, but above $\eta_M(t)$. Let $b \in M$ be a branch containing t. Assume towards a contradiction that $\mathrm{ht}(t) > \eta^*$, then there is some node $s \in b$ of height η^*, and thus $s <_T t$. It then follows that $\eta_M(t) \geq \eta^* > \eta_M(t)$, a contradiction. ⊣

The following is too easy, and we leave the proof to the reader.

LEMMA 4.6. *Suppose that $t \in T$ and $M \in \mathcal{E}^0 \cup \mathcal{E}^1$. If $\eta_M(t)$ is a successor ordinal, then $O_M(t)$ is in M.*

In general, if the supremum in the definition of $\eta_M(t)$ is attained by an element in $T \cap M$, then that element is $O_M(t)$, which belongs to M. The above lemma essentially means that it does happen if $\eta_M(t)$ is a successor ordinal. We now turn our attention to the situation where the overlaps are more complicated as $\eta_M(t)$ is a limit ordinal.

LEMMA 4.7. *Suppose that $t \in T$ and $M \in \mathcal{E}^1$. If $\mathrm{cof}(\eta_M(t))$ is not countable, then $O_M(t) \in M$.*

PROOF. By Lemma 4.6, we may assume that $\eta_M(t)$ is a limit ordinal, and thus of cofinality ω_1. Let $\eta = \eta_M(t)$. Since M is of size $\aleph_1$ and $\omega_1 \subseteq M$, we have $b_M(t) \subseteq M$. For every countable $a \in M$, the height of nodes in $a \cap b_M(t)$ is bounded below η due to the fact that $\eta_M(t)$ has uncountable cofinality. Thus it is easily seen that $b_M(t)$ is countably approximated in M. Since M is an ω_1-guessing model, there is

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$b \in M$ such that $b \cap M = b_M(t)$. By elementarity, b is a T-branch, and hence it is of size $\aleph_1$ (in particular, $\eta < M \cap \omega_2$.) Thus $b \subseteq M$, which in turn implies that $b_M(t) = b \in M$. But then $O_M(t) \in M$ as it can be read off from $b_M(t)$ due to the fact that T is Hausdorff. ⊣

COROLLARY 4.8. *Suppose that $t \in T$ and $M \in \mathcal{E}^1$. Then $\eta_M(t)$ is in M.*

PROOF. By definition $\eta_M(t) \leq M \cap \omega_2$. Since M is an IC-model with $\omega_1 \subseteq M$, the ordinal $M \cap \omega_2$ is of uncountable cofinality. If $\eta_M(t) = M \cap \omega_2$, then by Lemma 4.7, $O_M(t) \in M$. This is a contradiction, as $M \cap \omega_2 = \eta_M(t) = \mathrm{ht}(O_M(t)) \in M$! Thus $\eta_M(t) < M \cap \omega_2$, and hence $\eta_M(t) \in M$. ⊣

The following is key for us.

LEMMA 4.9. *Assume that $N \in \mathcal{E}^1$ and $M \in \mathcal{E}^0$ with $N \in M$. Let $t \in T \cap N$. If t is guessed in M, then t is guessed in $N \cap M$.*

PROOF. Let $b \in M$ be a T-branch containing t. Let $\gamma = \sup\{\mathrm{ht}(s) : s \in N \cap b\}$. Then γ exists as $t \in N$ and $\mathrm{ht}(t) \leq \gamma$. Note that $\gamma \in M \cap \omega_2$ by elementarity. Observe that if $\gamma = \mathrm{ht}(s)$, for some $s \in N \cap b$, then by elementarity, $s \in N \cap M$. We then have $t \in \overline{b}_s \in N \cap M$. Thus let us assume that the supremum γ is not obtained by any element of $N \cap b$. In particular, $\mathrm{ht}(t) < \gamma$ and the cofinality of γ is either ω or ω_1. We consider two cases:

Case 1: $\mathrm{cof}(\gamma) = \omega$.
By elementarity, there is a strictly $<_T$-increasing sequence $\langle s_n : n \in \omega \rangle \in M$ of nodes in $b \cap N$ such that $\sup\{\mathrm{ht}(s_n) : n \in \omega\} = \gamma$. Since we assumed $\mathrm{ht}(t) < \gamma$, there is n such that $t \leq_T s_n$. Note that $s_n \in N \cap M$, and hence $t \in \overline{b}_{s_n} \in N \cap M$. Therefore, t is guessed in $N \cap M$.

Case 2: $\mathrm{cof}(\gamma) = \omega_1$.
We claim that $b \cap T_{\leq \gamma}$ is guessed in N. To see this, observe that $b \cap T_{\leq \gamma}$ is ω_1-approximated in N, since if $a \in N$ is a countable set, then there is $s \in N \cap b \cap T_{\leq \gamma}$ such that $a \cap b \cap T_{\leq \gamma} = a \cap b_s$ (as the cofinality of γ is ω_1.) But $a \cap b_s \in N$. As N is an ω_1-guessing model, we have $b \cap T_{\leq \gamma}$ is guessed in N. By the elementarity of M, there is $b^* \in N \cap M$ such that $b^* \cap N = b \cap T_{\leq \gamma} \cap N$. Now $t \in N \cap b \cap T_{\leq \gamma} = b^* \cap N$. Notice that, by elementarity, b^* is a T-branch. Thus $b^* \in N \cap M$ witnesses that t is guessed in $N \cap M$. ⊣

LEMMA 4.10. *Assume that $N \in \mathcal{E}^1$ and $M \in \mathcal{E}^0$ with $N \in M$. Let $t \in T \cap N$. Then $\eta_{N \cap M}(t) = \eta_M(t)$, and hence $O_{N \cap M}(t) = O_M(t)$.*

PROOF. Since $N \cap M \subseteq M$, $\eta_{N \cap M}(t) \leq \eta_M(t)$. Assume towards a contradiction that the equality fails. Thus, there is some $s \in M$ whose height is above $\eta_{N \cap M}(t)$ such that $s \leq_T O_M(t) \leq_T t$. Then $s \in N$ as $\omega_1 \cup \{t\} \subseteq N$. Therefore, $s \in N \cap M$, and hence $\mathrm{ht}(s) \leq \eta_{N \cap M}(t)$, a contradiction. Since both $O_{N \cap M}(t)$ and $O_M(t)$ are below t and of the same height, they are equal. ⊣

4.2. The forcing construction and its basic properties. We are now ready to define our forcing notion $\mathbb{P}_T$ to specialise T in generic extensions.

DEFINITION 4.11 ($\mathbb{P}_T$). A condition in $\mathbb{P}_T$ is a pair $p = (\mathcal{M}_p, f_p)$ satisfying the following items.

(1) $\mathcal{M}_p \in \mathbb{M} := \mathbb{M}(\mathcal{E}^0, \mathcal{E}^1)$.
(2) $f_p \in \mathbb{S}_\omega(T)$.
(3) For every $M \in \mathcal{E}^0_p$, if $t \in \mathrm{dom}(f_p) \cap M$, then $f_p(t) \in M$.
(4) For every $M \in \mathcal{E}^0_p$ and every $t \in \mathrm{dom}(f_p)$ with $f_p(t) \in M$, if t is guessed in M, then $t \in M$.

We say p is stronger than q if and only if the following are satisfied.

(1) $\mathcal{M}_p \supseteq \mathcal{M}_q$.
(2) $f_p \supseteq f_q$.

Given a condition p in $\mathbb{P}_T$ and a model $M \in \mathcal{E}^0 \cup \mathcal{E}^1$ containing p, we define an extension of p that will turn later to be generic for the relevant models.

Definition 4.12. Suppose that $M \in \mathcal{E}^0 \cup \mathcal{E}^1$ and $p \in M \cap \mathbb{P}_T$. We let p^M be defined by $(\mathcal{M}^M_p, f_p)$.

Recall that $\mathcal{M}^M_p$ is the closure of $\mathcal{M}_p \cup \{M\}$ under intersections (see Fact 3.5).

Proposition 4.13. *Suppose that $M \in \mathcal{E}^0 \cup \mathcal{E}^1$ and $p \in M \cap \mathbb{P}_T$. Then p^M is a condition extending p such that $M \in \mathcal{M}_{p^M}$.*

Proof. We check Definition 4.11 item by item. Item 1 is essentially Fact 3.5. Item 2 is obvious of course. To see Items 3 and 4 hold true, let $N \in \mathcal{E}^0_{p^M}$. We may assume that $N \notin \mathcal{M}_p$. Therefore, the only interesting case is $M \in \mathcal{E}^0$ and $N = P \cap M$, for some $P \in \mathcal{E}^1_p$. Thus fix such models.

Item 3: Let $t \in \mathrm{dom}(f_{p^M}) \cap N$. We have $f_p(t) \in M$, as $p \in M$, and also we have $f_p(t) \in P$, as $\omega_1 \subseteq P$. Thus $f_p(t) \in P \cap M = N$.

Item 4: Let $t \in \mathrm{dom}(f_p)$ be such that $f_p(t) \in N$. If there is a T-branch $b \in N$ with $t \in b$, then $t \in P$ (since $b \subseteq P$), and hence $t \in P \cap M = N$.

Finally, by the construction of p^M, we have $M \in \mathcal{M}_{p^M}$, and by Fact 3.5, $p^M \leq p$. ⊣

We now define the restriction of a condition to a model in the side conditions coordinate.

Definition 4.14 (Restriction). Suppose that $p \in \mathbb{P}_T$ and $M \in \mathcal{M}_p$. We let the *restriction* of p to M be $p\restriction_M = (\mathcal{M}_{p\restriction_M}, f_p\restriction_M)$, where $f_p\restriction_M$ is the restriction of the function f_p to $\mathrm{dom}(f_p) \cap M$.

Recall that $\mathcal{M}_{p\restriction_M} = \mathcal{M}_p \cap M$. Observe that if M is in $\mathcal{E}^0$, then by Item 3 of Definition 4.11, $f_{p\restriction_M} = f_p \cap M$. This is trivial for models in $\mathcal{E}^1$.

Proposition 4.15. *Suppose that $p \in \mathbb{P}_T$ and $M \in \mathcal{M}_p$. Then $p\restriction_M \in \mathbb{P}_T \cap M$ and $p \leq p\restriction_M$.*

Proof. We check Definition 4.11 item by item. By Fact 3.8, $\mathcal{M}_{p\restriction_M}$ is an $\in$-chain and closed under intersections, and hence it is in $\mathbb{M}$. By Item 3 of Definition 4.11, $f_p \cap M$ is in $\mathbb{S}_\omega(T)$. Observe that M contains $p\restriction_M$, as it is a finite subset of M. Items 3 and 4 remain valid since all models in $\mathcal{M}_{p\restriction_M}$ and all nodes in $\mathrm{dom}(f_{p\restriction_M})$ are, respectively, in $\mathcal{M}_p$ and $\mathrm{dom}(f_p)$. It is easy to see that p extends $p\restriction_M$. ⊣

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NOTATION 4.16. *For a condition $p \in \mathbb{P}_T$, a model $M \in \mathcal{M}_p$, and a condition $q \in M \cap \mathbb{P}_T$ with $q \leq p \restriction_M$, we let $p \wedge q$ denote the pair $(\mathcal{M}_p \wedge \mathcal{M}_q, f_p \cup f_q)$.*

Note that $p \wedge q$ is not necessarily a condition; however we shall use it as a pair of objects. Notice that $\mathcal{M}_{p\wedge q}$ is the closure of $\mathcal{M}_p \cup \mathcal{M}_q$ under intersections, and belongs to $\mathbb{M}$ (see Remark 3.10 and Remark 3.13) and that also $f_{p\wedge q}$ is a well-defined function due to the fact that p satisfies Item 3 of Definition 4.11.

LEMMA 4.17. *Suppose p is a condition in $\mathbb{P}_T$ and M is a model in $\mathcal{M}_p$. Assume that $q \in M \cap \mathbb{P}_T$ extends $p \restriction_M$. Then $p \wedge q$ satisfies Item 3 of Definition 4.11.*

PROOF. Fix $N \in \mathcal{E}^0_{p\wedge q}$ and $t \in \operatorname{dom}(f_p) \cup \operatorname{dom}(f_q)$. Assume that t is in N. We shall show that $f_{p\wedge q}(t) \in N$. We split the proof into two cases.

Case 1: M is in $\mathcal{E}^1$.
In this case, $\mathcal{M}_{p\wedge q} = \mathcal{M}_p \cup \mathcal{M}_q$, by Remark 3.10. If $N \in \mathcal{M}_q$, then $t \in N \subseteq M$, and hence $t \in \operatorname{dom}(f_q)$. Thus $f_{p\wedge q}(t) = f_q(t) \in N$. Now suppose that $N \in \mathcal{M}_p \setminus \mathcal{M}_q$. We may assume $t \in \operatorname{dom}(f_q)$. Therefore, in $\mathcal{M}_p$, we have $M \in^* N$, which in turn implies that there is $M' \in \mathcal{E}^1_p$ such that $M \subseteq M' \in N$ and $M' \cap N \in M$. Then, $M' \cap N \in \mathcal{M}_q$ and $t \in M' \cap N$. As q is a condition, we have $f_{p\wedge q}(t) = f_q(t) \in M' \cap N \subseteq N$.

Case 2: M is in $\mathcal{E}^0$.
Observe that it is enough to assume $N \in \mathcal{M}_p \cup \mathcal{M}_q$: if $N \in \mathcal{M}_p \wedge \mathcal{M}_q$, then $N = P \cap N'$, for some $P' \in \mathcal{M}_p \cup \mathcal{M}_q$, and some $N' \in \mathcal{M}_p \cup \mathcal{M}_q$. By our assumption, $f_{p\wedge q}(t)$ belongs to N', and hence, $f_{p\wedge q}(t) \in P' \cap N' = N$, as $\omega_1 \subseteq P'$.
As in the previous case, we may assume $t \in \operatorname{dom}(f_q)$ and $N \in \mathcal{M}_p \setminus \mathcal{M}_q$. Let us first assume that $N \in^* M$. Suppose that N is the minimal counter-example with the above properties. Thus there is $P \in \mathcal{E}^1_p \cap M$ such that $N \in [P \cap M, P)_p$. Now $P \cap M \not\subseteq N$, as otherwise $f_q(t) \in N$, since $t \in P \in \mathcal{M}_q$ and $f_q(t) \in P \cap M$. Therefore, there is some $Q \in N$ such that $Q \cap N \in^* P \cap M \in Q$. Notice that $t \in P$, and hence $t \in P \cap M \subseteq Q$. Thus $t \in Q \cap N$. Now $Q \cap N$ is also a counter-example to our claim, since $t \in Q \cap N \subseteq N$, $Q \cap N \in \mathcal{M}_p \setminus \mathcal{M}_q$ (as otherwise, we would have $f_q(t) \in Q \cap N \subseteq N$), and $Q \cap N \in^* M$. This contradicts our minimality assumption.
Two cases remain. The case $N = M$ is trivial, and thus we only need to assume that $M \in^* N$. If $M \subseteq N$, then $f_q(t) \in N$. And if $M \not\subseteq N$, then there is some $P \in \mathcal{E}^1_p$ such that $P \cap N \in^* M \in P \in N$ (see Remark 3.2). Notice that $t \in P \cap N$. Thus by the previous paragraph, $f_q(t) \in P \cap N \subseteq N$. ⊣

4.3. Preserving $\aleph_2$. In this subsection, we prove that $\mathbb{P}_T$ preserves the regularity of $\aleph_2$. With a similar idea, we shall establish the properness of $\mathbb{P}_T$ in the subsequent subsection.

LEMMA 4.18. *Suppose p is a condition in $\mathbb{P}_T$ and that $M \in \mathcal{E}^1_p$. Assume that $q \in M$ is a condition extending $p \restriction_M$. Then $p \wedge q$ satisfies Item 4 of Definition 4.11.*

PROOF. Set $r = p \wedge q$. Notice that f_r is well-defined as a function. Now fix $t \in \operatorname{dom}(f_r)$ and $N \in \mathcal{E}^0 \cap \mathcal{M}_r$ so that $f_r(t) \in N$. We shall show that if t is guessed

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in N, then $t \in N$. Notice that by Remark 3.10, we have $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. We shall consider the nontrivial cases:

Case 1: $t \in \mathrm{dom}(f_p)$ and $N \in \mathcal{M}_q \setminus \mathcal{M}_p$.

Assume that t is guessed in N. Thus there is a T-branch $b \in N \subseteq M$ with $t \in b$. As b is of size $\leq \aleph_1$ and $\omega_1 \subseteq M$, we have $t \in b \subseteq M$. Thus $t \in M$, which in turn implies that $t \in \mathrm{dom}(f_q)$ and $f_q(t) = f_p(t) = f_r(t) \in N$. But then $t \in N$, as q is a condition.

Case 2: $t \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$ and $N \in \mathcal{M}_p \setminus \mathcal{M}_q$.

In this situation, N is not in M since $\mathcal{M}_q \supseteq \mathcal{M}_p \cap M$, and hence there is some $M' \in \mathcal{E}^1_p$ with $M \subseteq M' \in N$ such that $M' \cap N \in M$. Note that $t \in M'$. Assume that t is guessed in N. By Lemma 4.9, t is guessed in $M' \cap N$. On the one hand, $f_q(t) = f_r(t)$ belongs to $M' \cap N$, and that $M' \cap N \in M \cap \mathcal{M}_p \subseteq \mathcal{M}_q$. Since q is a condition, we have $t \in M' \cap N \subseteq N$.

Thus far, we have shown that $p \wedge q$ satisfies all items in Definition 4.11, possibly except Item 2. We shall show that under appropriate circumstances, $p \wedge q$ is indeed a condition. We now prepare the ground for this. ⊣

Definition 4.19. For a condition $p \in \mathbb{P}_T$ and a model $M \in \mathcal{E}^1_p$, we let

$$\mathcal{D}(p, M) = \{t \in \mathrm{dom}(f_p) : t \notin M\}.$$

Definition 4.20 (M-support). Suppose p is a condition in $\mathbb{P}_T$ and that $M \in \mathcal{E}^1_p$. We say that a function $\sigma : \mathcal{D}(p, M) \to T \cap M$ is an *M-support* for p if the following hold, for every $t \in \mathrm{dom}(\sigma)$.

(1) If $O_M(t) \in M$, then $\sigma(t) = O_M(t)$.
(2) If $O_M(t) \notin M$, then $\sigma(t) <_T O_M(t)$ is such that there is no node in $\mathrm{dom}(f_p)$ whose height belongs to the interval $[\mathrm{ht}(\sigma(t)), \eta_M(t))$.

Lemma 4.21. *Suppose p is a condition in $\mathbb{P}_T$. Assume that $M \in \mathcal{E}^1_p$. Then, there is an M-support σ for p.*

Proof. Fix $p \in \mathbb{P}_T$. It is enough to define σ for $t \in \mathcal{D}(p, M)$ with $O_M(t) \notin M$. Thus fix such a t. Notice that $\mathrm{dom}(f_p)$ is finite, and that, by Lemma 4.6, $\eta_M(t)$ is a limit ordinal. Thus one may easily find a node $\sigma(t)$ with the above properties. ⊣

Definition 4.22 (M-reflection). Suppose that $p \in \mathbb{P}_T$ and $M \in \mathcal{E}^1_p$. A condition q is called an *(M, σ)-reflection* of p, where σ is an M-support for p, if the following properties are satisfied.

(1) $q \leq p \restriction_M$.
(2) For every $t \in \mathrm{dom}(\sigma)$, the following hold:
    (a) There is no node in $\mathrm{dom}(f_q)$ whose height is the interval $[\mathrm{ht}(\sigma(t)), \eta_M(t))$.
    (b) For every $s \in \mathrm{dom}(f_q)$, if $s <_T \sigma(t)$, then $f_q(s) \neq f_p(t)$.

Let $R_p(M, \sigma)$ be the set of (M, σ)-reflections of p with support σ.

Remark 4.23. Notice that if $M^* \prec H_{\theta^*}$, for some sufficiently large regular cardinal θ^*, which contains T and H_θ, and that p is a condition in $\mathbb{P}_T$ with $M := M^* \cap H_\theta \in \mathcal{E}^1_p$, then $R_p(M, \sigma) \in M^*$, whenever σ is an M-support for p.

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LEMMA 4.24. *Let $p \in \mathbb{P}_T$. Assume that $M \in \mathcal{E}^1_p$, and let σ be an M-support for p. Then $p \in R_p(M, \sigma)$.*

PROOF. We check the items in Definition 4.22. Item 1 is essentially Proposition 4.15. Item 2a follows from the definition of σ. Item 2b follows from the fact that p is a condition, and that $\sigma(t) <_T t$. ⊣

LEMMA 4.25. *Suppose p is a condition in $\mathbb{P}_T$. Let $M \in \mathcal{E}^1_p$, and let $q \in M$ be an (M, σ)-reflection of p, for some M-support σ for p. Let $r = p \wedge q$. Then $f_r \in B_\omega(T)$.*

PROOF. Since $q \leq p\restriction_M$, f_r is well-defined as a function. We shall show that it satisfies the specialising property. To do this, we only discuss the nontrivial case by considering two arbitrary comparable nodes $t \in \mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q)$ and $s \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$. We claim that $f_r(t) \neq f_r(s)$. Observe that $s \in M$. The fact that $M \cap \omega_2$ is an ordinal imply that if $t \leq_T s$, then $t \in M$, which is a contradiction as $t \notin \mathrm{dom}(f_q)$. Thus, the only possibility is $s <_T t$. Since $q \in R_p(M, \sigma) \cap M$, the height of s is not in the interval $\big[\mathrm{ht}(\sigma(t)), \eta_M(t)\big)$. Thus $s <_T \sigma(t)$. Then Item 2b of Definition 4.22 implies that $f_q(s) \neq f_p(t)$. Therefore, $f_r(t) \neq f_r(s)$. ⊣

We have now all the necessary tools to prove the preservation of $\aleph_2$ by $\mathbb{P}_T$.

LEMMA 4.26. *Suppose p is a condition in $\mathbb{P}_T$. Assume that θ^* is a sufficiently large regular cardinal, and that $M^* \prec H_{\theta^*}$ contains the relevant objects. Suppose that $M := M^* \cap H_\theta$ is in $\mathcal{E}^1_p$. Then, p is $(M^*, \mathbb{P}_T)$-generic.*

PROOF. Fix $p' \leq p$. Then $M \in \mathcal{M}_{p'}$. Thus we may assume that $p = p'$. Let $D \in M^*$ be a dense subset of $\mathbb{P}_T$. We may also assume that $p \in D$. By Lemmas 4.21 and 4.24, there exists an M-support of p, say σ, such that $p \in R_p(M, \sigma)$. Notice that $R_p(M, \sigma)$ is in M^*. Thus by elementarity, there is some $q \in D \cap R_p(M, \sigma) \cap M$. Set $r = p \wedge q$. Now, Fact 3.9 and Lemmas 4.17, 4.18, and 4.25 imply that r satisfies Items 1–4 of Definition 4.11, respectively. It is clear that $p \wedge q$ extends both p and q. ⊣

COROLLARY 4.27. *Assume $\mathrm{GM}^*(\omega_2)$. Then $\mathbb{P}_T$ preserves $\aleph_2$.*

PROOF. Let θ^* be a sufficiently large regular cardinal. By Lemma 2.7, it is enough to show that for stationary many models M in H_{θ^*}, of size $\aleph_1$, every condition in M can be extended to an $(M, \mathbb{P}_T)$-generic condition. Let

$$\mathcal{S} = \{M \prec H_{\theta^*} : \mathcal{E}^1, \mathcal{E}^0, T, \theta \in M \text{ and } M \cap H_\theta \in \mathcal{E}^1\}.$$

By $\mathrm{GM}^*(\omega_2)$, $\mathcal{S}$ is stationary in $\mathcal{P}_{\omega_2}(H_{\theta^*})$. Now let $M^* \in \mathcal{S}$ and $p \in \mathbb{P}_T \cap M^*$. Set $M = M^* \cap H_\theta$. By Proposition 4.13, p^M is a condition with $p^M \leq p$, and by Lemma 4.26 it is $(M^*, \mathbb{P}_T)$-generic. ⊣

4.4. Properness. This subsection is devoted to the proof of the properness of $\mathbb{P}_T$. We will closely follow our strategy in the previous subsection. Notice that our notation and definition related to models in $\mathcal{E}^0$ are similar to the ones we used for the preservation of $\aleph_2$, but hopefully there will be no confusion, since these two parts are completely independent.

LEMMA 4.28. *Suppose p is a condition in $\mathbb{P}_T$ and that $M \in \mathcal{E}^0_p$. Assume that $q \in M$ is a condition extending $p\restriction_M$. Then $p \wedge q$ satisfies Item 4 of Definition 4.11.*

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PROOF. Set $r = p \wedge q$. Notice that f_r is well-defined as a function. Fix $t \in \mathrm{dom}(f_r)$ and $N \in \mathcal{E}^0 \cap \mathcal{M}_r$ so that t is guessed in N and $f_r(t) \in N$. We shall show that $t \in N$. As in Lemma 4.18, we shall study the nontrivial cases, and thus we may assume that either $t \in \mathrm{dom}(f_q)$ and $N \notin \mathcal{M}_q$, or $t \in \mathrm{dom}(f_p)$ and $N \notin \mathcal{M}_p$. Since M is in $\mathcal{E}^0$, the proof consists of three cases as $\mathcal{M}_r \setminus (\mathcal{M}_p \cup \mathcal{M}_q)$ may be nonempty. Recall that by Remark 3.13, $\mathcal{M}_r$ is the union of $\mathcal{M}_p \cup \mathcal{M}_q$ and the set of models of the form $P \cap Q$, where $P \in Q$ are in $\mathcal{E}^1_q$ and $\mathcal{E}^0_p$, respectively.

Case 1: $t \in \mathrm{dom}(f_q)$ and $N \in \mathcal{M}_p \setminus \mathcal{M}_q$.

In this situation, we have $N \in (P \cap M, P]_p$ for some $P \in (\mathcal{E}^1_p \cap M) \cup \{H_\theta\}$. Since t is guessed in $N \subseteq P$ and $\omega_1 \subseteq P$, we have $t \in P$. Assume towards a contraction that $t \notin N$. We may assume that N is the least model in $\mathcal{M}_p$ with the above properties. This implies that $P \cap M \nsubseteq N$, since $t \in P \cap M$. Therefore, by Remark 3.2, there is a model $Q \in \mathcal{E}^1_p$ such that $P \cap M \in Q \in N \in P$ and $Q \cap N \in^* P \cap M$. Observe that $t \in Q$. By Lemma 4.9, t is guessed in $Q \cap N$. On the other hand $f_q(t) \in Q \cap N$. Since $t \notin Q \cap N$, our minimality assumption implies that $Q \cap N$ is in $\mathcal{M}_q$, but then since q is a condition, t is an element of $Q \cap N \subseteq N$, a contradiction!

Case 2: $t \in \mathrm{dom}(f_p)$ and $N \in \mathcal{M}_q$.

We have $f_p(t) \in N \subseteq M$. Observe that t is also guessed in M, since $N \subseteq M$. As p is a condition, Item 4 of Definition 4.11 implies that $t \in M \cap \mathrm{dom}(f_p) \subseteq \mathrm{dom}(f_q)$. On the other hand, q is a condition and $N \in \mathcal{M}_q$, and hence, by Item 4 of Definition 4.11, $t \in N$.

Case 3: $t \in \mathrm{dom}(f_r)$ and $N \in \mathcal{M}_r \setminus (\mathcal{M}_p \cup \mathcal{M}_q)$.

There are $P \in \mathcal{E}^1_q$ and $Q \in \mathcal{E}^0_p$ with $P \in Q$ such that $N = P \cap Q$. Let $b \in N$ be a T-branch with $t \in b$. Then t is guessed in Q, as $b \in Q$. We have also $f_p(t) \in Q$. Thus by the two previous cases, $t \in Q$. On the other hand, $b \in P$ and $b \subseteq P$, as T has no cofinal branches, and $P \cap \omega_2$ is an ordinal. Thus $t \in P$. Therefore, $t \in P \cap Q = N$. $\dashv$

NOTATION 4.29. *Assume that p is a condition in $\mathbb{P}_T$, and that $M \in \mathcal{E}^0_p$.*

(1) *We let $\mathcal{D}(p, M)$ denote the set of $t \in \mathrm{dom}(f_p)$ such that $t \notin M$, but $f_p(t) \in M$.*
(2) $\mathcal{O}(p, M) := \{t \in \mathcal{D}(p, M) : O_M(t)$ *is not guessed in M and* $\eta_M(t) \notin M\}$.

DEFINITION 4.30 (M-support). Suppose p is a condition in $\mathbb{P}_T$ and $M \in \mathcal{E}^0_p$. We say a function $\sigma : \mathcal{D}(p, M) \to M$ is an *M-support* for p if the following hold, for every $t \in \mathrm{dom}(\sigma)$.

(1) If $O_M(t)$ is guessed in M, then $\sigma(t) \in M$ is such that $M \cap \sigma(t) = M \cap b_M(t)$.
(2) If $O_M(t)$ is not guessed in M, then $\sigma(t) \subseteq b_M(t)$ is a T-branch in M such that no node in $\mathrm{dom}(f_p)$ has height in the interval $\big[\mathrm{ht}(\sup(\sigma(t))), \eta_M(t)\big)$.

Note that if $t \in \mathrm{dom}(\sigma)$ and $O_M(t)$ is guessed in M, then by elementarity, $\sigma(t)$ is a T-branch, in fact it is a cofinal branch through $T_{<\eta^*_M(t)}$, where $\eta^*_M(t) = \min(M \cap \omega_2 \setminus \eta_M(t))$. Moreover, $\sigma(t)$ is unique.

LEMMA 4.31. *Let $p \in \mathbb{P}_T$, and let $M \in \mathcal{E}^0_p$. Then, there is an M-support for p.*

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Proof. Suppose that $t \in \mathcal{D}(p, M)$. If $O_M(t)$ is guessed in M, then there is a T-branch $b \in M$ such that $O_M(t) \in b$. Let $\eta^*_M(t) = \min(M \cap \omega_2 \setminus \eta_M(t))$, and set $\sigma(t) := b \cap T_{<\eta^*_M(t)}$. It is easily seen that $M \cap \sigma(t) = M \cap b_M(t)$.

If $O_M(t)$ is not guessed in M, then $\eta_M(t)$ is a limit ordinal by Lemma 4.6. Since $\mathrm{dom}(f_p)$ is finite, there is a sequence of nodes in M cofinal in $O_M(t)$. Thus one can find an ordinal $\gamma \in M$, such that there is no node in $\mathrm{dom}(f_p)$ whose height is in the interval $[\gamma, \eta_M(t))$. Choose a node s of height γ below $O_M(t)$ and set $\sigma(t) := \overline{b}_s$. We have $s \in M$, since $\gamma \in M$. Thus $\sigma(t) \in M$. Observe that $\mathrm{ht}(\sup(\sigma(t))) = \mathrm{ht}(s) = \gamma$. ⊣

Definition 4.32 (M-reflection). Suppose p is a condition in $\mathbb{P}_T$. Assume that $M \in \mathcal{E}^0_p$. Let σ be an M-support for p. A condition q is called an (M, σ)-*reflection of p* if the following properties are satisfied.

(1) $q \leq p\restriction_M$.
(2) The following hold for every $t \in \mathrm{dom}(\sigma)$.
   (a) If $\eta_M(t) \in M$, then there is no node in $\mathrm{dom}(f_q)$ whose height belongs to the interval $\big[\mathrm{ht}(\sup(\sigma(t))), \eta_M(t)\big)$.
   (b) For every $s \in \mathrm{dom}(f_q)$ with $s \in \sigma(t)$, $f_q(s) \neq f_p(t)$.

Let $R_p(M, \sigma)$ denote the set of (M, σ)-reflections of p.

Notice that as before, if $M^* \prec H_{\theta^*}$, for some sufficiently large regular cardinal θ^* which contains T and H_θ, and p is a condition in $\mathbb{P}_T$ with $M := M^* \cap H_\theta \in \mathcal{E}^0_p$, then $R_p(M, \sigma) \in M^*$, whenever σ is an M-support for p.

Lemma 4.33. *Suppose p is a condition in $\mathbb{P}_T$, and that $M \in \mathcal{E}^0_p$. Let σ be an M-support set for p. Then $p \in R_p(M, \sigma)$.*

Proof. Let us check the items in Definition 4.32. Item 1 is essentially Proposition 4.15. To verify Item 2, let us fix $t \in \mathrm{dom}(\sigma)$.

Item 2a: Assume that $\eta_M(t) \in M$. If $O_M(t)$ is not guessed in M, then by the Item 2 of Definition 4.30, there is no node in $\mathrm{dom}(f_p)$ with height in the interval $\big[\mathrm{ht}(\sup(\sigma(t))), \eta_M(t)\big)$. Thus let us assume that $O_M(t)$ is guessed in M. We show that $\sigma(t) = b_M(t)$, which in turn implies that the interval $\big[\mathrm{ht}(\sup(\sigma(t))), \eta_M(t)\big)$ is empty. To show that $\sigma(t) = b_M(t)$, it is enough to show that $b_M(t) \in M$. Suppose $b \in M$ is a T-branch with $O_M(t) \in b$. Then the order type of b is at least $\eta_M(t) + 1$ and $O_M(t)$ is the $\eta_M(t)$-th element of b. Since $\eta_M(t) \in M$, we have $O_M(t) \in M$, and hence $b_M(t) \in M$.

Item 2b: Suppose that $s \in \sigma(t)$ and $f_p(s) = f_p(t)$. Then s is guessed in M. As $f_p(t) \in M$ and p is a condition, we have $s \in M$. This implies that $s \leq_T O_M(t) \leq_T t$. Since p is a condition, we $t = s \in M$, which is a contradiction (as $t \notin M$)! ⊣

Lemma 4.34. *Suppose $p \in \mathbb{P}_T$, and that $M \in \mathcal{E}^0_p$. Assume that $q \in M \cap R_p(M, \sigma)$. Let $r := p \wedge q$. Then $r' = (\mathcal{M}_r, f_r \setminus \{(t, f_p(t)) : t \notin \mathcal{O}(p, M)\})$ is a condition.*

Proof. Lemmas 3.12, 4.17, and 4.28 imply that r' satisfies Items 1, 3, and 4 of Definition 4.11, respectively. Therefore, it remains to show that the well-defined function $f_{r'} := f_r \setminus \{(t, f_p(t)) : t \notin \mathcal{O}(p, M)\}$ is a condition in $\mathbb{S}_\omega(T)$. To see this, let $s \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$ and $t \in \mathrm{dom}(f_{r'}) \setminus \mathrm{dom}(f_q)$. Assume that s and t are

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comparable in T, we shall show that $f_q(s) \neq f_p(t)$. We may assume that $f_p(t) \in M$. Thus $t <_T s$ is impossible, as otherwise t is guessed in M, and hence $t \in M$, which is a contradiction! Consequently, the only possible case is $s <_T t$. In this case, $s <_T O_M(t)$. We claim that $s \in \sigma(t)$. This is clear if $O_M(t)$ is guessed in M. If $O_M(t)$ is not guessed in M, then $\eta_M(t) \in M$ as $t \notin \mathcal{O}(p, M)$. Therefore, by Item 2a of Definition 4.32, the height of s avoids the interval $\left[\mathrm{ht}(\sup(\sigma(t))), \eta_M(t)\right)$. Thus $s <_T \sup(\sigma(t))$, and hence $s \in \sigma(t)$. In either case, $s \in \sigma(t)$, but then Item 2b of Definition 4.32 implies that $f_p(t) \neq f_q(s)$. ⊣

Proposition 4.35. *Suppose that $p \in \mathbb{P}_T$. Let θ^* be a sufficiently large regular cardinal. Assume that $M^* \prec H_{\theta^*}$ is countable and contains T and θ. If $M := M^* \cap H_\theta \in \mathcal{M}_p$. Then p is $(M^*, \mathbb{P}_T)$-generic.*

Proof. Assume that $p' \leq p$. Since $M \in \mathcal{M}_{p'}$, we may assume without loss of generality that $p' = p$. Let $D \in M^*$ be a dense subset of $\mathbb{P}_T$. We may also assume, without loss of generality, that $p \in D$. Since M^* is fixed throughout proof, we simply denote $\eta_M(t)$ by η_t. By Lemmas 4.31 and 4.33, there is an M-support σ for p so that $p \in R_p(M, \sigma)$. Observe that $R_p(M, \sigma) \in M^*$. Let $\langle t_i : i < m \rangle$ enumerate $\mathcal{O}(p, M)$ so that $\eta_{t_i} \leq \eta_{t_{i+1}}$, for every $i < m - 1$. Let $\langle \eta_i : i < m' \rangle$ be the strictly increasing enumeration of $\{\eta_{t_i} : i < m\}$. To reduce the amount of notation, we may assume that $m = m'$. For every $i < m$, set

$$\eta_i^* = \min(M \cap (\omega_2 + 1) \setminus \eta_i).$$

Notice that $\eta_i^* < \eta_{i+1}$, for every $i < m - 1$. For every $i < m$, we let also $\hat{t}_i$ denote $\sup(\sigma(t_i))$. Note that $\hat{t}_i$ exists, as $t_i \in \mathcal{O}(p, M)$. Let us call a map $x \mapsto p_x$ from $\mathcal{P}_{\omega_1}(T)$ into $\mathbb{P}_T$, a T-assignment if the following properties are satisfied for every $x \in \mathcal{P}_{\omega_1}(T)$.

(1) $p_x \in R_p(M, \sigma) \cap D$.
(2) $|\mathrm{dom}(f_{p_x})| = |\mathrm{dom}(f_p)|$.
(3) For every $s \in \mathrm{dom}(f_{p_x})$ and every $i < m$, if $\mathrm{ht}(s) \in \left[\mathrm{ht}(\hat{t}_i), \eta_i^*\right)$, then

$$\sup\{\mathrm{ht}(u) : u \in x \cap T_{<\eta_i^*}\} < \mathrm{ht}(s).$$

We first show that there are T-assignments in M^*. ⊣

Claim 4.36. *There is a T-assignment in M^*.*

Proof. We observe that all the parameters in the above properties are in M^*. By elementarity and the Axiom of Choice, it is enough to show that for every $x \in M^*$, there is such $p_x \in H_{\theta^*}$. Thus fix $x \in M^*$. We claim that p is such a witness. The first item is clear by Lemma 4.33 and that the second one is trivial. To see the third one holds true, fix $i < m$ and observe that

- $\{\mathrm{ht}(u) : u \in x \cap T_{<\eta_i^*}\}$ is bounded below η_i (as the cofinality of η_i^* is uncountable, x is countable and $M \cap \eta_i^* = M \cap \eta_i$) and
- there is no node in $\mathrm{dom}(f_p)$ whose height lies in the interval $\left[\mathrm{ht}(\hat{t}_i), \eta_i\right)$ (by the construction of $\sigma(t_i)$, see Item 2 of Definition 4.30).

Thus if $s \in \mathrm{dom}(f_p)$ is of height at least $\mathrm{ht}(\hat{t}_i)$, then $\mathrm{ht}(s) \geq \eta_i$, and thus

$$\sup\{\mathrm{ht}(u) : u \in x \cap T_{<\eta_i^*}\} < \eta_i \leq \mathrm{ht}(s).$$ ⊣

Fix a T-assignment $x \mapsto p_x$ in M^*. We shall show that there is a set $B^* \in M^*$ cofinal in $\mathcal{P}_{\omega_1}(T)$ such that for every $x \in M^* \cap B^*$, p_x and p are compatible. Let $n := |\mathrm{dom}(f_p)|$. For each $x \in \mathcal{P}_{\omega_1}(T)$, fix an enumeration of $\mathrm{dom}(f_{p_x})$, say $\langle t^x_j : j < n\rangle$. For every $B \subseteq \mathcal{P}_{\omega_1}(T)$, let

$$B(i, j) := \{x \in B : \mathrm{ht}(t^x_j) \geq \mathrm{ht}(\hat{t}_i)\}.$$

Note that if $B \in M^*$, then $B(i, j) \in M^*$.

Claim 4.37. *Let $i < m$ and $j < n$. Suppose that $B \in M^*$ is an unbounded subset of $\mathcal{P}_{\omega_1}(T)$. Assume that $B(i, j)$ is cofinal in B. Then, there is a cofinal subset $B_{i,j}$ of $B(i, j)$ in M^* such that for every $x \in M^* \cap B_{i,j}$, $t^x_j \not<_T O_M(t_i)$.*

Proof. Let Ψ_i be the characteristic function of $b_M(t_i)$ on T. Note that Ψ_i is not guessed in M. For every $x \subseteq T$, we let $\psi^x_j : x \to 2$ be defined by $\psi^x_j(s) = 1$ if and only if $s <_T t^x_j$. Now consider the mapping $x \mapsto \psi^x_j$. Since Ψ_i is not guessed in M, Lemma 2.16 implies that there is a set $B_{i,j} \in M^*$ cofinal in $B(i, j)$ such that for every $x \in B_{i,j}$, $\psi^x_j \nsubseteq \Psi_i$.

Assume towards a contradiction that there is $x \in M^* \cap B_{i,j}$ with $t^x_j <_T O_M(t_i)$. Then $t^x_j \in M \cap T_{<\eta_i}$, and for every $s \in x$ of height at least η^*_i, we have $\psi^x_j(s) = 0 = \Psi_i(s)$. Thus $\psi^x_j \nsubseteq \Psi_i$ implies that there is some $s \in T_{<\eta^*_i} \cap M$ such that $\psi^x_j(s) \neq \Psi_i(s)$. Since $x \in B(i, j)$, we have $\mathrm{ht}(t^x_j) \in \left[\mathrm{ht}(\hat{t}_i), \eta^*_i\right)$. On the other hand, by Item 3 in the definition of a T-assignment, we have $\mathrm{ht}(s) < \mathrm{ht}(t^x_j)$. Thus $s <_T t^x_j$ if and only if $s \not<_T O_M(t_i)$, which contradicts $t^x_j <_T O_M(t_i)$. ⊣

Returning to our main proof, let e be a bijection between mn and $m \times n$. For every $k < mn$, set $e(k) := (e_0(k), e_1(k))$. We build a descending sequence $\langle B_k : -1 \leq k < mn\rangle$ of cofinal subsets of $P_{\omega_1}(T)$ with $B_k \in M^*$ as follows. Let also $B_{-1} := \mathcal{P}_{\omega_1}(T)$. Suppose that B_k, for $k \geq -1$, is constructed. Set $C^k := B_k(e_0(k), e_1(k))$ and ask the following question:

- Is C^k cofinal in B_k?

Then proceed as follows:

- If the answer to the above question is YES, then apply Claim 4.37 to C^k, $e_0(k+1)$ and $e_1(k+1)$ to obtain $C^k_{e_0(k+1),e_1(k+1)} \in M^*$ as in the claim, and then set $B_{k+1} := C^k_{e_0(k+1),e_1(k+1)}$.
- If the answer to the above question is NO, then let $B_{k+1} = B_k \setminus C^k$.

It is clear that $\langle B_k : -1 \leq k < mn\rangle$ is descending and each B_k is in M^*. Set $B^* := B_{mn-1}$. Note that if $x \in C^k_{e_0(k+1),e_1(k+1)}$, then $t^x_{e_1(k+1)} \not<_T O_M(t_{e_0(k+1)})$, by Claim 4.37.

Claim 4.38. *For every $x \in B^* \cap M^*$, p_x and p are compatible.*

Proof. Fix $x \in B^* \cap M^*$. Then $p_x \in M^* \cap D$. Let $r = p_x \wedge p$. We claim that r is a condition. By Lemma 4.34, we only need to check if there are comparable $s \in \mathrm{dom}(f_{p_x}) \setminus \mathrm{dom}(f_p)$ and $t \in \mathcal{O}(p, M)$ such that $f_{p_x}(s) = f_p(t)$. We shall see that it does not happen. Thus assume towards a contradiction that there are such t and s. Then $t = t_i$ and $s = t^x_j$, for some $i < m$ and $j < n$. Note that $f_{p_x}(s), t^x_j \in M$, as $x \in M^*$. Observe that if $t_i \leq_T t^x_j$, then t_i is guessed in M, and hence it belongs to

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M by Item 4 of Definition 4.11, which is a contradiction. Thus $t_j^x <_T t_i$, which in turn implies that $t_j^x \in b_M(t_i)$ (recall that $O_M(t_i)$ is not guessed in M.) Since $f_{p_x}(s) = f_p(t)$ and $p_x \in R_p(M, \sigma)$, Item 2b in Definition 4.32 implies that $\mathrm{ht}(t_j^x) \not< \mathrm{ht}(\hat{t}_i)$. Thus $\mathrm{ht}(t_j^x) \geq \mathrm{ht}(\hat{t}_i)$. Let $k \geq 0$ be such that $e(k) = (i, j)$. Since $x \in B^* \subseteq B_k \subseteq B_{k-1}$ and that $\mathrm{ht}(t_j^x) \geq \mathrm{ht}(\hat{t}_i)$, we have $B_k = C_{i,j}^{k-1}$, but then $t_j^x \not<_T O_M(t_i)$ by Claim 4.37, which is a contradiction since $t_j^x \in b_M(t_i)$ implies that $t_j^x <_T O_M(t_i)$. ⊣

REMARK 4.39. Note that to find the cofinal set B^* in the above proof, we could start with any set which is cofinal in $\mathcal{P}_{\omega_1}(T)$.

COROLLARY 4.40. *$\mathbb{P}_T$ is proper.*

PROOF. Let θ^* be a sufficiently large regular cardinal. Assume that $M^* \prec H_{\theta^*}$ is countable and contains $H_\theta, T, \mathcal{E}^0$, and $\mathcal{E}^1$. Set $M = M^* \cap H_\theta$, and let $p \in M^*$ be a condition. Notice that the set of such models is a club in $\mathcal{P}_{\omega_1}(H_{\theta^*})$. By Proposition 4.13, p^M is a condition with $p^M \leq p$ such that $M \in \mathcal{M}_{p^M}$. Now, Proposition 4.35 guarantees that p^M is $(M^*, \mathbb{P}_T)$-generic. Thus $\mathbb{P}_T$ is proper. ⊣

We shall use the above strategy and Lemma 2.16 to show that $\mathbb{P}_T$ has the ω_1-approximation property.

PROPOSITION 4.41. *$\mathbb{P}_T$ has the ω_1-approximation property.*

PROOF. Assume towards a contradiction that $\dot{A}$ is a $\mathbb{P}_T$-name such that for some $p \in \mathbb{P}_T$ and some $X \in V$, we have

- $p \Vdash$ "$\dot{A} \subseteq \check{X}$,"
- $p \Vdash$ "$\dot{A} \notin V$," and
- $p \Vdash$ "$\dot{A}$ is countable approximated in V," i.e., for every countable set $a \in V$, $p \Vdash$ "$\dot{A} \cap \check{a} \in V$."

Without loss of generality, we may work with a $\mathbb{P}_T$-name for the characteristic function of $\dot{A}$, say $\dot{f}$. We may also, without loss of generality, assume that either $T \subseteq X$ or $X \subseteq T$. To see this, observe that by passing to an isomorphic copy of T, we may assume that the underlying set of T is $|T|$. On the other hand, using a bijection between X and $|X|$, we can assume that the domain of $\dot{f}$ is forced to be $|X|$. As $|X|$ and $|T|$ are comparable, we may assume that either $T \subseteq X$ or $X \subseteq T$.

Let us assume that $T \subseteq X$, the other case is proved similarly. Let θ^* be a sufficiently large regular cardinal. Let $M^* \prec H_{\theta^*}$ be a countable model containing all the relevant objects, including p. Set $M = M^* \cap H_\theta$. We can extend p^M to a condition q such that q decides $\dot{f} \restriction_{M^*}$, i.e., for some function $g : M^* \cap X \to 2$ in V, $q \Vdash$ "$\dot{f} \restriction_{M^*} = \check{g}$."

CLAIM 4.42. *g is not guessed in M^*.*

PROOF. Suppose that g is guessed in M^*. Let $g^* \in M^*$ be such that $g^* \cap M^* = g$. Set

$$D = \{r \leq p : \exists x \in X \;\; r \Vdash \text{“}g^*(x) \neq \dot{f}(x)\text{”}\} \cup \{r \in \mathbb{P}_T : r \perp p\}.$$

Obviously $D \in M^*$. We use elementarity to show that D is dense in $\mathbb{P}_T$. Thus let $r \in M^* \cap \mathbb{P}_T$. We may assume that r is compatible with p. Thus, there is $s \in M \cap \mathbb{P}_T$

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such that $s \leq p, r$. Since $p \Vdash$ "$\dot{f} \notin V$," there is $x \in M^* \cap X$ and there is $s' \leq s$ in M^* such that $s' \Vdash$ "$g^*(x) \neq \dot{f}(x)$." Thus $s' \in D \cap M$.

On the other hand, by Proposition 4.35, q is $(M^*, \mathbb{P}_T)$-generic. Thus, there is $u \in D \cap M^*$ such that $u || q$. But then $u || p$, and thus there is $x \in M^* \cap X$ such that $u \Vdash$ "$g^*(x) \neq \dot{f}(x)$." This is impossible, as $q \Vdash g^*(x) = g(x) = \dot{f}(x)$. ⊣

Fix an M-support set σ for q. As in the proof of Proposition 4.35, we can find, in M^*, a function $x \mapsto (q_x, g_x)$ on $\mathcal{P}_{\omega_1}(X)$ such that:

(1) $q_x \in R_p(M, \sigma)$.
(2) $|\mathrm{dom}(f_{q_x})| = |\mathrm{dom}(f_q)|$.
(3) For every $s \in \mathrm{dom}(f_{q_x})$ and every $i < m$, if $\mathrm{ht}(s) \in \left[\mathrm{ht}(\hat{t}_i), \eta_i^*\right)$, then

$$\sup\{\mathrm{ht}(u) : u \in x \cap T_{<\eta_i^*}\} < \mathrm{ht}(s).$$

(4) $g_x : \mathrm{dom}(g_x) \to 2$ is a function with countable domain containing x as a subset.
(5) $q_x \Vdash g_x \restriction_x = \dot{f} \restriction_x$.

Here, η_i, η_i^*, and $\hat{t}_i$ are as in the proof of Proposition 4.35. Note that to find an assignment in M^*, observe that if $x \in M^*$, then $x \subseteq \mathrm{dom}(g)$, and thus we can use (q, g) as a witness. Since we assumed $T \subseteq X$ and by the above claim g is not guessed in M^*, we first apply Lemma 2.16 to find a set $B \in M^*$, cofinal in $\mathcal{P}_{\omega_1}(X)$, such that for every $x \in B$, $g_x \nsubseteq g$. Now let C be the restriction of B to T, i.e., $C = \{x \cap T : x \in B\}$. Then C is cofinal in $\mathcal{P}_{\omega_1}(T)$. Using the Axiom of Choice, for each $c \in C$, pick $x_c \in B$ such that $x_c \cap T = c$. Fix such a choice function $c \mapsto x_c$ in M^* and consider the assignment $c \mapsto q_{x_c}$. By the above properties, $c \mapsto q_c = q_{x_c}$ is a T-assignment in M^*. Thus, as in Proposition 4.35, there is some $c \in C \cap M^*$ such that q_c is compatible with q. There exists $x \in B \cap M^*$ with $x_c = c$, but this is a contradiction, as $g_x \nsubseteq g$ implies that $q_{x_c} = q_c$ is not compatible with q! ⊣

LEMMA 4.43. *Suppose that $p \in \mathbb{P}_T$ and $t \in T$. Then there is some $q \leq p$ such that $t \in \mathrm{dom}(f_q)$.*

PROOF. Assume that t is not in $\mathrm{dom}(f_p)$. If t is not in any model belonging to $\mathcal{E}_p^0$, then pick ν below ω_1 and different from the values of f_p such that

$$\nu > \max\{M \cap \omega_1 : M \in \mathcal{E}_p^0\},$$

and then set $q = (\mathcal{M}_p, f_p \cup \{(t, \nu)\})$. Then Item 1 of Definition 4.11 is easily fulfilled, Item 2 holds true as $\nu \notin \mathrm{rang}(f_p)$. Item 3 is obvious as t does not belong to any model in $\mathcal{M}_q = \mathcal{M}_p$. Finally, Item 4 is fulfilled, since $f_q(t) = \nu$ belongs to no model in $\mathcal{E}_q^0 = \mathcal{E}_p^0$.

Now assume that there are some models in $\mathcal{E}_p^0$ containing t. Let M be the least countable model in $\mathcal{M}_p$ with $t \in M$. Let $\nu \in M \cap \omega_1 \setminus \mathrm{ran}(f_p)$ be such that

$$\nu > \max\{N \cap \omega_1 : N \in \mathcal{E}_p^0 \cap M\}.$$

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Set $q = (\mathcal{M}, f_p \cup \{(t, v)\})$. We claim that q is a condition. As in the previous case, Items 1 and 2 of Definition 4.11 hold true, and thus we only need to check Items 3 and 4.

Item 3: Assume that $N \in \mathcal{E}_p^0$ contains t. By the minimality of M, $M \in^* N$. We claim that $M \subseteq N$. Suppose this is not the case. Thus there is some $P \in \mathcal{E}_p^1$ such that $P \cap N \in^* M \in P \in N$, but then $t \in P \cap N$, which contradicts the minimality of M. Thus $M \subseteq N$, and hence $v \in M \subseteq N$.

Item 4: Suppose that $N \in \mathcal{E}_p^0$ is such that $v \in N$ and t is guessed in N. We shall show that $M \subseteq N$, and hence $t \in N$. We first show that $N \in^* M$ is impossible. To see this, observe that $N \notin M$ by our choice of v. Thus if $N \in^* M$, then there is some $P \in \mathcal{E}_p^1 \cap M$ such that $N \in [P \cap M, P)_p$. Now t belongs to P as it is guessed in $N \subseteq P$, and thus $t \in P \cap M$, which contradicts the minimality of M.

Now if $M \nsubseteq N$, there is $P \in \mathcal{M}_p$ such that $P \cap N \in^* M \in P \in N$. Then since $t \in P$ is guessed in N, by Lemma 4.9, t is guessed in $P \cap N$. Notice that $v \in P \cap N \in^* M$, which is a contradiction as $P \cap N \in^* M$, as is was shown in the previous paragraph. ⊣

Remark 4.44. Notice that $\mathbb{P}_T$ forces $|H_\theta| = |T| = \aleph_2$.

§5. Conclusion.

In this section, we prove our main theorem.

Theorem 5.1. *Assume that* $\mathrm{GM}^*(\omega_2)$ *holds. Then, every tree of height* ω_2 *without cofinal branches is specialisable via a proper and* $\aleph_2$*-preserving forcing with finite conditions. Moreover, the forcing has the* ω_1*-approximation property.*

Proof. By Lemma 2.5, we may also assume that T is a Hausdorff tree. By Corollaries 4.27 and 4.40, $\mathbb{P}_T$ preserves $\aleph_1$ and $\aleph_2$, respectively. Let $G \subseteq \mathbb{P}_T$ be V-generic filter, and set

$$f_G = \bigcup \{f_p : p \in G\}.$$

By Lemma 4.43, $f_G : T \to \omega_1$ is a total function on T. It is clear that f_G is a specialising function on T. ⊣

Since PFA implies $\mathrm{GM}^*(\omega_2)$ by Proposition 2.15, we obtain the following corollary.

Corollary 5.2. *Assume* PFA. *Suppose T is a tree of height* ω_2 *without cofinal branches. Then there is a proper and* $\aleph_2$*-preserving forcing with the* ω_1*-approximation property such that T is special in generic extensions by* $\mathbb{P}_T$.

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