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ON CLOSED RADICAL ORBITS IN HOMOGENEOUS COMPLEX MANIFOLDS

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Suppose G is a complex Lie group having a finite number of connected components and H is a closed complex subgroup of G with H° solvable. Let R_G denote the radical of G. We show the existence of closed complex subgroups I and J of G containing H such that I/H is a connected solvmanifold with $I^\circ \supset R_G$, the space G/J has a Klein form S_G/A, where A is an algebraic subgroup of the semisimple complex Lie group $S_G := G/R_G$, and, unless $I = J$, the space J/I has Klein form $\widehat{S}/\widehat{\Gamma}$, where $\widehat{\Gamma}$ is a Zariski dense discrete subgroup of some connected positive dimensional semisimple complex Lie group $\widehat{S}$.

1. INTRODUCTION

Methods which allow one to study quotients of Lie groups by discrete subgroups by considering quotients of solvable and semisimple Lie groups separately are very helpful. One such method is presented in [1, Theorem 1] for G/Γ compact, where Γ is a discrete subgroup of a Lie group G. Also decompositions of compact real homogeneous spaces have been introduced by Gorbatsevich. He developed what he calls the natural and structure bundles, where these involve double coset spaces, for example, see the fifth chapter in the book [6] for their descriptions. In the case of a quotient of complex Lie groups one would like to have a decomposition which is compatible with the complex structure.

In order to get a decomposition we consider the natural map $\pi : G \to S_G := G/R_G$, where R_G denotes the radical of G. For arbitrary G/H difficulties arise because $\pi(H)$ need not be closed in S_G. To handle such problems we use a result of Zassenhaus–Auslander, extended in Theorem A in the Appendix of [11], which is applicable whenever H°, the connected component of the identity of H, is solvable. It follows that U°, the connected component of the identity of the closure of $\pi(H)$, is solvable. The normaliser N in S_G of the complexification $\widetilde{U}$ of U° is *an algebraic group* containing

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$\pi(H)$ which cannot be semisimple. If N is solvable, then $J := \pi^{-1}(N)$ is a closed complex subgroup of G containing H with J° solvable such that G/J has a Klein form S_G/A, where A is an algebraic subgroup of the semisimple complex Lie group S_G. If N is not solvable, one proceeds by recursion and, if at some stage there is a normaliser which is solvable, one is in a situation similar to the one described above. Otherwise, because the dimension of G/H is finite, there is a closed complex subgroup J_k of G containing H such that the orbits of the radical of J_k are closed in J_k/H. It is then easy to show that there exist closed complex subgroups I and J of G containing H such that I/H is a connected complex solvmanifold, G/J has a Klein form S_G/A with A an algebraic subgroup of S_G, and, unless $I = J$, the space J/I has a Klein form $\widehat{S}/\widehat{\Gamma}$, where $\widehat{\Gamma}$ is a Zariski dense discrete subgroup of a semisimple complex group $\widehat{S}$. This generalises a fibration for homogeneous complex manifolds with discrete isotropy which can be found in [4, Theorem 2]. But *explicit Klein forms* are not given in that result, as they are here.

Now any solvmanifold (respectively, quotient of algebraic groups) admits the structure of a vector bundle over a compact manifold, see [**2**] or [**10**] (respectively, [**9**] or [**8**]). Thus complicated topological problems concerning G/H with H° solvable reduce to questions about the $\widehat{S}/\widehat{\Gamma}$ corresponding to J/I. For example, this idea was applied in [**5**] to homogeneous complex manifolds with more than two ends in order to give an explicit decomposition of these spaces that shows "essentially where their ends live".

Throughout we will denote the connected component of the identity of a complex Lie group L by L° and its radical by R_L. Let $\pi_L : L \to L/R_L$ be the natural map.

2. The structure theorem

Theorem 1. *Suppose $X = G/H$ is a connected homogeneous complex manifold, where G is a complex Lie group which has a finite number of connected components and is not solvable and H is a closed complex subgroup of G with H° solvable. Then there exist closed complex subgroups I and J of G containing H such that I° is solvable, $I^\circ \supset R_G$, and I/H is connected, and J has a finite number of connected components with G/J having a Klein form S_G/A, where A is an algebraic subgroup of the semisimple complex Lie group $S_G := G/R_G$. Moreover, unless $I = J$, the space $J/I = \widehat{S}/\widehat{\Gamma}$, where $\widehat{\Gamma}$ is a Zariski dense discrete subgroup of a connected positive dimensional semisimple complex Lie group $\widehat{S}$.*

Proof: The proof proceeds by induction on $\dim_{\mathbb{C}} G/H$. The assertions are clear if $\dim_{\mathbb{C}} G/H = 1$. Suppose first that G° is semisimple. If H is an algebraic subgroup of G, then set $J = I := H$ and we are done. If H is not an algebraic subgroup of G, then let $J := \overline{H}^Z$ be the Zariski closure of H in G. If $J = G$, then set $I = H$. Since G admits a faithful representation into some linear group, it follows that the

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normaliser $N_G(H^\circ)$, as the intersection of the image of G with the normaliser of the image of H° in the linear group, is algebraic. The fact that $H \subset N_G(H^\circ) \subset G$, while H is Zariski dense in G implies $N_G(H^\circ) = G$. Because H° is solvable and G° is semisimple, this is impossible, unless $H^\circ = \{e\}$, that is, H is discrete. If $J \neq G$, then $\dim J/H < \dim G/H$. By induction there are subgroups $H \subset I \subset J_1 \subset J$ with I° solvable and $I^\circ \supset R_J$, $J_1/I = \widehat{S}/\widehat{\Gamma}$, where $\widehat{\Gamma}$ is a Zariski dense discrete subgroup of some semisimple complex Lie group $\widehat{S}$, and $J/J_1 = S_J/A$, where A is an algebraic subgroup of $S_J = J/R_J$. Since $\pi_J : J \to S_J$ is an algebraic map, $J_1 = \pi_J^{-1}(A)$ is an algebraic subgroup of G containing H and contained in J. Hence $J_1 = J$ and $G/H \to G/I \to G/J$ gives us the fibrations we seek.

In the rest of the proof assume G° is not semisimple, that is, $\dim R_G > 0$. We assume first $\pi_G(H)$ is *not closed* in $S_G := G/R_G$. Consider its closure $U := \overline{\pi_G(H)}$ in S_G. Since H° is solvable, U° is also solvable by the result of Zassenhaus–Auslander, as proved in Theorem A in the Appendix of [**11**]. In general, U is a *real* Lie group. Let $\widetilde{U}$ be the connected complex Lie subgroup of S_G whose Lie algebra $\widetilde{\mathfrak{u}}$ is the complexification in the Lie algebra of S_G of the Lie algebra $\mathfrak{u}$ of U°. Note that $\widetilde{U}$ need not be *closed*. Let $N_1 := N_{S_G}\left(\widetilde{U}\right)$ be the normaliser in S_G of $\widetilde{U}$. Now N_1 is the intersection of the image of S_G under a faithful representation into $GL(n,\mathbb{C})$ with the normaliser in $GL(n,\mathbb{C})$ of $\widetilde{U}$ and thus is an algebraic subgroup. By assumption $U^\circ \neq \{e\}$ and hence $\widetilde{U}$ has positive dimension. Thus $N_1 \neq S_G$, since this latter group is semisimple and so the positive dimensional connected solvable subgroup $\widetilde{U}$ cannot be normal in S_G. If $g \in U$, then $g\widetilde{U}g^{-1} \cap \widetilde{U}$ is a complex subgroup of S_G containing U°. By construction $\widetilde{U}$ is the smallest such subgroup, and thus $U \subset N_1$. Hence $\pi_G(H) \subset N_1$. Let $J_1 := \pi_G^{-1}(N_1) \subset G$. Note that $J_1 \supset R_G$ and that J_1 has a finite number of connected components, because N_1 does and the fibres of the map π_G are connected.

By recursion we assume $G \supset J_i$, where $J_i \supset H$ and $J_i \supset R_G$. We also assume that N_i and thus J_i are not solvable; see below. If the orbits of $R_i := R_{J_i}$ in J_i/H are not closed, then we construct a closed complex subgroup J_{i+1} of G in the following way. Let $\pi_i : J_i \to S_i := J_i/R_i$ denote the natural map. Note that $\dim R_i > \dim R_G > 0$, where the first inequality follows from the fact that N_i cannot be a semisimple Lie group, because it contains the positive dimensional normal solvable complex subgroup $\widetilde{U}_{i-1}$. The group $\widetilde{U}_i$ is the complexification in S_i° of the identity component (which, by the Zassenhaus–Auslander result, is again solvable) of the closure of $\pi_i(H)$. The normaliser N_{i+1} of $\widetilde{U}_i$ in S_i is a proper algebraic subgroup of S_i containing $\pi_i(H)$ and we set $J_{i+1} := \pi_i^{-1}(N_{i+1})$. Note that $J_{i+1} \supset R_i \supset R_G$. If the subgroup N_{i+1}° is solvable, then we set $I = J := J_{i+1}$ and this is the group we seek. For, in this case it is clear I° is solvable. The fact that G/J is the quotient of S_G by an algebraic subgroup is shown below in the last paragraph of the proof. The Klein form of J/I is obvious in this case.

Otherwise, since $\dim_{\mathbb{C}} G/H$ is finite, the process described above terminates after finitely many steps yielding a chain of closed complex subgroups J_i, for $1 \leqslant i \leqslant k$, of G with

$$H \subset J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset G,$$

such that the R_k–orbits are closed in J_k/H and thus in G/H too. Each J_i has a finite number of connected components, since it is the inverse image of an algebraic subgroup by a map with connected fibres. In addition, $J_i/J_{i+1} = S_i/N_{i+1}$, where $S_i := J_i/R_i$ and $N_{i+1} \subset S_i$ is an algebraic subgroup, given by the above procedure. Because H is a subgroup of J_k and R_k is normal in J_k, it follows that $H \cdot R_k$ is a closed subgroup of J_k. Incidentally, note that if $\pi_G(H)$ is closed in S_G to begin with, then we set $J_k := G$ and the above properties are trivially verified. The projection $H_k := \pi_k(H)$ is closed in $S_k := J_k/R_k$ and H_k° is solvable. Since S_k° is semisimple, it follows from the first paragraph of the proof that there exist a closed complex subgroup I_S with I_S° solvable and an algebraic subgroup A_S with $H_k \subset I_S \subset A_S \subset S_k$ such that $A_S/I_S = \widehat{S}/\widehat{\Gamma}$ is the quotient of a semisimple complex Lie group $\widehat{S}$ by a Zariski dense discrete subgroup $\widehat{\Gamma}$, unless $I_S = A_S$. Now set $I := \pi_k^{-1}(I_S)$ and $J := \pi_k^{-1}(A_S)$. Because $\pi_k(H) = H_k \subset I_S \subset A_S$, it follows that $H \subset I \subset J$. Since $I^\circ = \pi_k^{-1}(I_S^\circ)$ is the inverse image of a solvable group by a homomorphism with solvable kernel, I° is solvable. Also

$$J/I = A_S/I_S = \widehat{S}/\widehat{\Gamma}, \tag{1}$$

and we see from (1) that J/I has the required Klein form.

Since $J \supset R_G$, it is clear $G/J = S_G/\pi_G(J)$, where $\pi_G(J)$ is a closed complex subgroup of S_G. We would like to show that $\pi_G(J)$ is an algebraic subgroup of S_G, that is, that one can write G/J as a quotient of S_G by an algebraic subgroup. We have already defined the natural map $\pi_i : J_i \to S_i := J_i/R_i$ and we also consider the natural map

$$\widehat{\pi}_i : N_i \to N_i/R_{N_i}.$$

Note that $R_{N_i} = \pi_{i-1}(R_i)$ and thus $S_i = \widehat{\pi}_i(N_i)$. For any $1 \leqslant i \leqslant k-1$ we have the following diagram

$$\begin{array}{ccccc} J_{i+1} & := \pi_i^{-1}(N_{i+1}) & \subset J_i \\ \pi_{i+1} \downarrow & & \downarrow \pi_i \\ S_{i+1} & \stackrel{\widehat{\pi}_{i+1}}{\longleftarrow} \; N_{i+1} & \subset S_i \end{array}$$

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and thus

$$\pi_{i+1} = \widehat{\pi}_{i+1} \circ \pi_i|_{J_{i+1}}. \tag{2}$$

There is a similar diagram for $i = 0$, if one replaces J_0 by G, S_0 by S_G and π_0 by π_G. Then

$$\pi_1 = \widehat{\pi}_1 \circ \pi_G|_{J_1}. \tag{3}$$

We now define a group

$$A := \widehat{\pi}_1^{-1} \circ \cdots \circ \widehat{\pi}_{k-1}^{-1} \circ \widehat{\pi}_k^{-1}(A_S) \subset S_G.$$

Note that A is an algebraic group, because A_S is algebraic and each of the maps $\widehat{\pi}_i$ is an algebraic map. Now by using (2) and (3) we see that

$$\begin{aligned} J &= \pi_k^{-1}(A_S) \\ &= \pi_{k-1}^{-1} \circ \widehat{\pi}_k^{-1}(A_S) \\ &= \cdots \\ &= \pi_G^{-1} \circ \widehat{\pi}_1^{-1} \circ \widehat{\pi}_2^{-1} \circ \cdots \circ \widehat{\pi}_k^{-1}(A_S) \\ &= \pi_G^{-1}(A). \end{aligned}$$

Hence $\pi_G(J) = A$ and thus $G/J = S_G/A$. If the process ended because N_k° is solvable, then we set

$$A := \widehat{\pi}_1^{-1} \circ \cdots \circ \widehat{\pi}_{k-1}^{-1} \circ \widehat{\pi}_k^{-1}(N_k) \subset S_G.$$

In this case A is algebraic for the same reasons as before and A° is also solvable. Similar to above, it follows that $G/J = S_G/A$ and the proof of the theorem is complete. □

3. The structure of arbitrary G/H

In conclusion we would like to point out how the structure of any homogeneous complex manifold G/H depends on six components. First let $N := N_G(H^\circ)$ be the normaliser in G of the connected component H° of the identity of H. Then N is a closed complex subgroup of G containing H and one has the fibration $G/H \to G/N$ which is commonly called the *normaliser fibration* of G/H. By a Theorem of Chevalley [3] the commutator subgroup G' of G has closed orbits in G/N and one has the fibration $G/N \to G/NG'$. Its base G/NG' is a Stein Abelian Lie group, see [7], and its fibre $NG'/N = G'/N \cap G'$ can be written as a quotient of algebraic groups. Since the radical $R_{G'}$ of the commutator subgroup G' has closed orbits in $G'/G' \cap N$, one can consider the fibration

$$G'/G' \cap N \overset{R_{G'}/R_{G'} \cap N}{\longrightarrow} S/S \cap NR_{G'},$$

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where S denotes a maximal semisimple subgroup of G. As well, the fibre N/H of the normaliser fibration can be handled by the theorem above, because $N/H = (N/H^\circ)/(H/H^\circ)$ is the quotient of the complex Lie group N/H° by the discrete subgroup H/H°. Thus one can study G/NG', $R_{G'}/R_{G'} \cap N$ and $S/S \cap NR_{G'}$ along with the three components of N/H given by our structure theorem. Variants of these ideas can be found in many different works in the literature, for example, see [4]. There it sufficed to know that if G is a mixed complex Lie group (neither solvable nor semisimple) and Γ is a discrete subgroup of G, then there exists a proper closed complex subgroup J of G containing both Γ and the radical R_G and one then has the fibration $G/\Gamma \to G/J$ to work with.

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