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JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

Some Formulae in connection with the Parabolic Section of the Canonical Quadric.

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§ 1. I have ventured to bring the formulae of this paper before the Society, as I have been unable to find reference to them in any text-book or any original contribution to mathematical literature which I have come across. I confine my attention completely to the central surface, as the corresponding formulae for the paraboloids are very readily deduced by a similar process.

§ 2. Let the equation to the quadric be

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z}{\gamma} = 1 \qquad \ldots \qquad \ldots \qquad (1)$$

where $\alpha\,\beta\,\gamma$ are not all of like sign.

Let the equation to a plane parabolic section be

$$lx + my + nz = p \qquad \ldots \qquad \ldots \qquad (2)$$

As the parallel central plane section must also be parabolic and touch the asymptotic cone $\Sigma\frac{x^2}{\alpha} = 0$, we have the conditions

$$l^2 + m^2 + n^2 = 1 \qquad \ldots \qquad \ldots \qquad (\mathrm{I})$$

$$\alpha l^2 + \beta m^2 + \gamma n^2 = 0 \qquad \ldots \qquad \ldots \qquad (\mathrm{II})$$

Let $$\alpha^2 l^2 + \beta^2 m^2 + \gamma^2 n^2 = \mathrm{S} \qquad \ldots \qquad \ldots \qquad (\mathrm{III})$$

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Then S can not vanish for real values of $l\, m\, n$, and the following relations may be established:

$$S = \Sigma a^2 l^2 = \Sigma m^2 n^2 (\beta - \gamma)^2 = \sqrt{-\Sigma m^2 n^2 \beta\gamma(\beta - \gamma)^2} \quad \ldots \quad \ldots \quad \text{(A)}$$

$$= \beta m^2(\beta - a) + \gamma n^2(\gamma - a) = \text{etc.} = n^2(\gamma - a)(\gamma - \beta) - a\beta = \text{etc.}$$

$$\beta m^2(\beta - a)^2 + \gamma n^2(\gamma - a)^2 = S(\gamma + \beta - a) + a\beta\gamma, \text{ etc.} \quad \ldots \quad \ldots \quad \text{(B)}$$

These might be added to, but they are all that are made use of in what follows.

VERTEX.

§ 3. The coordinates of the vertex may be found by finding the equation of the plane in which the vertices of the parallel parabolic sections lie (just as for elliptic or hyperbolic sections), and solving for x, y, z with the aid of (1) and (2).

Now the axis of the parabola is parallel to what would be the diameter of the section as on the quadric, and therefore its direction cosines are $(\rho a l,\ \rho\beta m,\ \rho\gamma n)$, where ρ^2 is therefore equal to $1/S$.

If (x, y, z) be the vertex, the tangent to the parabola is perpendicular to $(\rho a l)$

i.e., $(\rho a l)$ is perpendicular to the direction given by the intersection of

$$lx + my + nz = p$$

and $$\frac{xx_1}{a} + \frac{yy_1}{\beta} + \frac{zz_1}{\gamma} = 1$$

and therefore the vertex satisfies the equation

$$\Sigma \frac{x}{al}(\beta - \gamma) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Solving (2) and (3) for y and z in terms of x, and remembering (A), we deduce

$$y/\beta m = x/al - p(a - \beta)/S \quad \ldots \quad \ldots \quad (4)$$

$$z/\gamma n = x/al - p(a - \gamma)/S \quad \ldots \quad \ldots \quad (5)$$

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Substitute these values in (1); we have a quadratic in x, which, however, must have one infinite root, and which, in virtue of (II) and (A) reduces to

$$2px/al = 1 - p^2\{\beta m^2(a-\beta)^2 + \gamma n^2(a-\gamma)\}^2/S^2$$

$$= 1 - \frac{p^2}{S}(\gamma + \beta - a) - a\beta\gamma p^2/S^2 \qquad \text{by (B).} \quad \ldots \quad (6)$$

Similarly

$$\frac{2py}{\beta m} = 1 - p^2(a + \gamma - \beta)/S - a\beta\gamma p^2/S^2 . \quad \ldots \quad (7)$$

$$\frac{2pz}{\gamma n} = 1 - p^2(a + \beta - \gamma)/S - a\beta\gamma p^2/S^2 . \quad \ldots \quad (8)$$

These are the coordinates of the vertex.

Parameter.

§ 4. Let V be the vertex, F the focus, FL the semi-latus rectum.

If (λ) be the direction of the tangent at the vertex, it will also be that of FL

Now (λ) is perpendicular to (l) of plane and to (al) of the axis

$$\left.\begin{aligned} \therefore \quad \lambda l + \mu m + \nu n &= 0 \\ \text{and} \quad \lambda a l + \mu\beta m + \nu\gamma n &= 0 \end{aligned}\right\} \quad \ldots \quad (9)$$

$$\text{and} \;\therefore\; \lambda : \mu : \nu = mn(\beta - \gamma) : nl(\gamma - a),\; lm(a - \beta)$$

$$\text{and} \;\therefore\; \lambda = \frac{mn(\beta-\gamma)}{\sqrt{\Sigma m^2n^2(\beta-\gamma)^2}} = \rho mn(\beta - \gamma), \text{ etc.} \quad \ldots \quad (10)$$

Let 4π be the latus rectum then the coordinates of F are, if V be (xyz),

$$x + \pi\rho a l, \quad y + \pi\rho\beta m, \quad z + \pi\rho\gamma n$$

$$\text{and} \qquad \text{L is } (x + \pi\rho a l + 2\pi\rho mn\overline{\beta - \gamma}).$$

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But L is on the quadric, and V is on the quadric

$$\therefore \quad \Sigma\frac{1}{a}\{x+\pi\rho a l+2\pi\rho m n(\beta-\gamma)\}^2=1$$

$$\therefore \quad 2\pi\rho\left\{\Sigma l x+4\Sigma\frac{m n x}{a}(\beta \quad \gamma)\right\}$$

$$+\pi^2\rho^2\left\{\Sigma a l^2+4 l m n\Sigma(\beta-\gamma)+\Sigma\frac{m^2n^2(\beta-\gamma)^2}{a}\right\}=0$$

which by (2) and (3) reduces to

$$2\pi\rho p+\frac{4\pi^2\rho^2}{a\beta\gamma}\Sigma m^2n^2\beta\gamma(\beta-\gamma)^2=0$$

and $\therefore$

$$\begin{aligned} 4\pi &= -2a\beta\gamma p/(-S^2)\rho \\ &= 2a\beta\gamma p\rho^3 \quad \text{by (A).} \qquad \ldots \qquad (11) \end{aligned}$$

So that for a system of parallel parabolic sections the parameter is not constant, as I had at first supposed, but varies directly as the distance of the section from the origin.

For the central section itself $p=0$, *i.e.* the parameter vanishes. The section then reduces to a pair of parallel lines for the hyperboloid of one sheet, which are coincident asymptotes in the two-sheeted surface. In the former there are, therefore, an infinite number of pairs of parallel generators, and, from the present analysis, one might infer that the parameter of a pair of parallel lines, considered as the limiting case of a parabola, is zero.

Focus.

§ 5. The coordinates of the focus are given by

$$\xi=x+\pi\rho a l, \text{ etc.}$$

They are therefore given by

$$\left.\begin{aligned} \frac{2p\xi}{a l} &= 1-\frac{p^2}{S}(\gamma+\beta-a) \\ \frac{2p\eta}{\beta m} &= 1-\frac{p^2}{S}(\gamma+a-\beta) \\ \frac{2p\zeta}{\gamma n} &= 1-\frac{p^2}{S}(a+\beta-\gamma). \end{aligned}\right\} \qquad (12)$$

If we consider l, m, and n as connected by the equations (I) and (II) these contain implicitly equations (2) and (3).

For a system of parallel sections the locus of the foci is a plane curve, and the orthogonal projection on a reference plane as obtained by eliminating p between any two of the equations (12) is found to be a hyperbola whose centre is at the origin. Hence the locus of the foci of a series of parallel parabolic sections $\Sigma lx = p$ is a hyperbola whose centre is at the origin, and which lies in the plane

$$\Sigma \frac{x}{al}(\beta - \gamma) = 0.$$

Moreover $\Sigma lx = 0$ and $\Sigma \frac{x}{al}(\beta - \gamma) = 0$ are conjugate planes, and hence the foci for any system of parallel plane sections parallel to $\Sigma lx = 0$ is a conic lying in a plane conjugate to $\Sigma lx = 0$.

§ 6. The equation $\Sigma \frac{x}{al}(\beta - \gamma) = 0$ might be considered as given along with the equation $\Sigma al^2 = 0$; then we know that in it lies a hyperbola which is the locus of the foci of a system of parabolic sections. To find such a system, all we have to do is to take the diameter conjugate to the given plane, and draw the planes passing through it and touching the asymptotic cone. There are two such planes, real or coincident, since one is $\Sigma lx = 0$. They are not coincident since that would require that the diameter in question $\left(\frac{\beta - \gamma}{l}\right)$ should be a generator of the cone and $\therefore \Sigma m^2 n^2 \beta\gamma(\beta - \gamma)^2 - 0$, *i.e.*, $S = 0$ which is impossible.

There are then two systems having their foci lying in the same plane conjugate to both (these three planes are not a conjugate system).

§ 7. The plane $\Sigma \frac{x}{al}(\beta - \gamma) = 0$ being conditioned by $\Sigma al^2 = 0$ must therefore envelope a cone of the fourth class, whose equation is

$$\Sigma\{\beta\gamma(\beta - \gamma)^2 x^2\}^{\frac{1}{3}} = 0 \qquad \ldots \qquad \ldots \qquad (13)$$

and which is therefore of the sixth degree.

I omit the discussion of this surface, but the following may be noted :—

All the foci of the parabolic sections of a quadric will lie on a surface whose equation may be found by eliminating l, m, n, p from

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any five independent equations involving them. This surface would seem to be of a fairly high degree, but, whatever its degree, every plane section whose plane is at the same time tangent to the cone (13) must be a degenerate curve, part or whole of whose real intersection consists of two concentric hyperbolas, with centre at the origin.

Surface of Revolution.

§ 8. For the surface of revolution, the formulae are much simpler. Suppose that $\beta=\alpha$, then $1/\rho^2 = S = -\alpha\gamma$, and $\therefore$ S is constant.

Equations (I) and (II) become

$$l^2+m^2+n^2=1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(I)}'$$

$$\alpha(l^2+m^2)+\gamma n^2=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(II)}'$$

whence

$$\left.\begin{aligned} n^2 &= -\alpha/(\gamma-\alpha) \\ l^2+m^2 &= \gamma/(\gamma-\alpha) \end{aligned}\right\} \quad \ldots \quad (14)$$

so that n^2 is constant, as is otherwise obvious.

The parameter

$$4\pi = 2\alpha\beta\gamma p\rho^3 \qquad \text{becomes}$$

$$4\pi = 2\alpha\beta\gamma p \frac{1}{\sqrt{-\alpha^3\gamma^3}} = \frac{2\alpha p}{\sqrt{-\alpha\gamma}} = 2p\sqrt{-\frac{\alpha}{\gamma}}, \quad \ldots \quad (15)$$

and therefore varies directly as the distance of the section from the origin.

The coordinates of the focus are given by

$$\frac{2p\xi}{\alpha l} = 1 - \frac{p^2}{-\alpha\gamma}\gamma = 1 + \frac{p^2}{\alpha} \quad \ldots \quad \ldots \quad (16)$$

$$\frac{2p\eta}{\alpha m} = \quad \ldots \quad = 1 + \frac{p^2}{\alpha} \quad \ldots \quad \ldots \quad (17)$$

$$\frac{2p\zeta}{\gamma n} = \quad 1 + \frac{p^2}{\alpha\gamma}(2\alpha-\gamma) \quad \ldots \quad \ldots \quad (18)$$

so that, (n being constant), when p is constant so also is ζ.

§ 9. Equations (16) and (17) give $l\eta - m\xi = 0$, and hence the cone of equation (13) shrinks up into the z-axis for the surface of revolution.

From the same equations we deduce

$$\frac{\xi}{l} = \frac{n}{m} = \sqrt{\frac{\xi^2 + \eta^2}{l^2 + m^2}} = cr, \text{ say, where}$$

$$r^2 = \xi^2 + \eta \quad \text{and} \quad c^2 = 1/(l^2 + m^2) = (\gamma - a)/\gamma.$$

Hence, substituting in (16) and (18), we deduce

$$p^2 - 2pcr + a = 0 \quad \dots \quad \dots \quad (19)$$

$$p^2(2a - \gamma) - 2p\frac{a\zeta}{n} + \gamma a = 0 \quad \dots \quad \dots \quad (20)$$

On eliminating p from these two equations, we obtain the locus of the foci of all the parabolic sections.

The eliminant in question is

$$\left(\gamma a - a \cdot \overline{2a - \gamma}\right)^2 = \left(-2cra\gamma + \frac{2a^2\zeta}{n}\right)\left(-2\frac{a\zeta}{n} + 2cr \cdot \overline{2a - \gamma}\right) \quad (21)$$

and this, on replacing c and n by their equivalents in terms of a and γ, reduces to

$$a(\gamma - a) - a\zeta^2 - r^2\gamma + 2ar^2 = 2a^2\zeta r/(1 - a\gamma). \quad \dots \quad (22)$$

Take the square on both sides, and put $x^2 + y^2$ for r^2, z for ζ, when we obtain the locus of the foci of all parabolic sections,—a surface of revolution of the fourth degree, and such that all sections passing through the z-axis split up into degenerate curves consisting of two concentric hyperbolas.