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DETERMINATION OF GRASSMANN MANIFOLDS WHICH ARE BOUNDARIES

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ABSTRACT. Let $FG_{n,k}$ denote the Grassmann manifold of all k-dimensional (left) F-vector subspace of F^n for $F = \mathbb{R}$, the reals, $\mathbb{C}$, the complex numbers, or $\mathbb{H}$ the quaternions. The problem of determining which of the Grassmannians bound was addressed by the author in [4]. Partial results were obtained in [4] for the case $F = \mathbb{R}$, including a sufficient condition, due to A. Dold, on n and k for $\mathbb{R}\,G_{n,k}$ to bound. Here, we show that Dold's condition is also necessary, and obtain a new proof of sufficiency using the methods of this paper, which cover the complex and quaternionic cases as well.

1. **Introduction.** For a positive integer n, let $\nu(n)$ be the integer such that $2^{\nu(n)} | n$ and $2^{\nu(n)+1} \not| n$. Let $1 \leq k \leq n$. The purpose of this paper is to prove

THEOREM 1.1. *Let $F = \mathbb{R}, \mathbb{C}, or\, \mathbb{H}$. Then $FG_{n,k}$ bounds if and only if $\nu(n) > \nu(k)$.*

Let [M] denote the unoriented cobordism class of a smooth closed manifold M. From [5] we have

$$(1.2) \qquad [FG_{n,k}] = [\mathbb{R}\,G_{n,k}]^d \ldots$$

for $F = \mathbb{R}, \mathbb{C}, or \mathbb{H}$, where $d = \dim_{\mathbb{R}} F$. We will prove that $[\mathbb{C}\,G_{2n,2k}] = [\mathbb{H}\,G_{n,k}]^2$. Using these, and the known facts (see [4]) about $\mathbb{R}\,G_{n,k}$, we will prove Theorem 1.1.

We regard the left $\mathbb{H}$-vector space $\mathbb{H}^n$ as $\mathbb{C}^{2n}$ as follows: the n-tuple $(q_1, \ldots, q_n) \in \mathbb{H}^n$ of quaternions $q_p = a_{p0} + ia_{p1} + ja_{p2} + ka_{p3}$, $1 \leq p \leq n$, is identified with the $2n$-tuple $(a_{10} + ia_{11}, a_{12} + ia_{13}, \ldots, a_{n0} + ia_{nl}, a_{n2} + ia_{n3})$ in $\mathbb{C}^{2n}$. Then multiplication on the left by $j \in \mathbb{H}$ yields the conjugate linear automorphism $J: \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$ where

$$J(z_1, \ldots, z_{2n}) = (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n}, \bar{z}_{2n-1}).$$

Clearly, $J^2 = -\mathrm{Id}$. Also, a $\mathbb{C}$-subspace V of $\mathbb{C}^{2n} = \mathbb{H}^n$ is a left $\mathbb{H}$-subspace if and only if $J(V) = V$. Further, when $J(V) = V$, $\dim_{\mathbb{H}} V = \frac{1}{2}\dim_{\mathbb{C}} V$. From this identification of $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ we obtain an imbedding of $\mathbb{H}G_{n,k} = \{V \subset \mathbb{H}^n \mid \dim_{\mathbb{H}} V = k\}$ into $\mathbb{C}\,G_{2n,2k} = \{w \subset \mathbb{C}^{2n} \mid \dim_{\mathbb{C}} W = 2k\}$. We identify $\mathbb{H}G_{n,k}$ with its image in $\mathbb{C}\,G_{2n,2k}$ under the above imbedding.

We put the usual $\mathbb{R}$-valued inner product on $\mathbb{C}^{2n} = \mathbb{H}^n$ so that $\langle jv, jw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{H}^n$.

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2. **Proof of Theorem 1.1.** Let $\gamma^F_{m,\ell}$ denote the canonical F-vector bundle of F-rank ℓ over $FG_{m,\ell}$, and let $\beta^F_{m,\ell}$ denote its *orthogonal complement* bundle, whose fibre over $W \in FG_{m,\ell}$ is $W^\perp \subset F^m$. Then the tangent bundle $\tau^F_{m,\ell}$ is isomorphic to $\mathrm{Hom}_F(\gamma^F_{m,\ell}, \beta^F_{m,\ell})$. $\tau^F_{m,\ell}$ is a complex vector bundle when $F = \mathbb{C}$, and is a real vector bundle when $F = \mathbb{R}$ or $\mathbb{H}$ (see [2]).

For $x \in M$, let T_xM denote the tangent space at x to the smooth manifold M.

Because $J: \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$ is a conjugate linear automorphism, it induces a map Ψ: $\mathbb{C}G_{2n,2k} \longrightarrow \mathbb{C}G_{2n,2k}$ where $\Psi(W) = J(W)$ for all $W \in \mathbb{C}G_{2n,2k}$.

LEMMA 2.1. *Let $\alpha: \tau^{\mathbb{C}}_{2n,2k} \longrightarrow \tau^{\mathbb{C}}_{2n,2k}$ denote the map $\alpha(h) = ih$ for all $h \in T_X\mathbb{C}G_{2n,2k} = \mathrm{Hom}_{\mathbb{C}}(X, X^\perp)$, for $X \in \mathbb{C}G_{2n,2k}$. Then the following diagram anti-commutes for any $X \in \mathbb{C}G_{2n,2k}$:*

$$\begin{array}{ccc} T_X\mathbb{C}G_{2n,2k} & \xrightarrow{\alpha} & T_X\mathbb{C}G_{2n,2k} \\ \downarrow T\Psi & & \downarrow T\Psi \\ T_{\Psi(X)}\mathbb{C}G_{2n,2k} & \xrightarrow{\alpha} & T_{\Psi(X)}\mathbb{C}G_{2n,2k} \end{array}$$

That is, $T\Psi \circ \alpha = -\alpha \circ T\Psi$.

PROOF. Recall, from [2], that the identification of $\tau^F_{m,\ell}$ with $\mathrm{Hom}_F(\gamma^F_{m,\ell}, \beta^F_{m,\ell})$ is obtained as follows. Given an F-linear map $h: X \longrightarrow X^\perp$ with $X \in FG_{m,\ell}$, let $\sigma(h)(t) =$ "graph of th" $= \{x + th(x) \mid x \in X\}$ for $t \in \mathbb{R}$. Then $\sigma(h)$ is a smooth curve in $FG_{m,\ell}$ with $\sigma(h)(0) = X$. The h corresponds to the tangent vector $\frac{d}{dt}\sigma(h)(t)\mid_{t=0}$ at X to $FG_{m,\ell}$.

Now let $X \in \mathbb{C}G_{2n,2k}$, and let $h: X \longrightarrow X^\perp$ be $\mathbb{C}$-linear, that is, $h \in T_X\mathbb{C}G_{2n,2k}$. Then $T\Psi(h): \Psi(X) = J(X) \longrightarrow \big(\Psi(X)\big)^\perp = \big(J(X)\big)^\perp = J(X^\perp)$ is the tangent vector $\frac{d}{dt}\big(\Psi \circ \sigma(h)(t)\big)\mid_{t=0}$.

Consider

$$\begin{aligned} \Psi \circ \sigma(h)(t) &= J\big(\sigma(h)(t)\big) \\ &= \big\{jv \mid v \in \sigma(h)(t)\big\} \\ &= \big\{jx + tjh(x) \mid x \in X\big\} \\ &= \big\{jx + tjhj^{-1}(jx) \mid x \in X\big\} \\ &= \big\{y + t(-jhj)(y) \mid y \in J(X)\big\} \\ &= \sigma(-jhj)(t). \end{aligned}$$

Therefore $\frac{d}{dt}\big(\Psi \circ \sigma(h)(t)\big)\mid_{t=0}$ is the tangent vector $-jhj: J(X) \longrightarrow J(X^\perp)$. Hence $T\Psi(h) = -jhj$. Therefore $(T\Psi \circ \alpha)(h) = T\Psi(ih) = -jihj = ijhj = \alpha\big(T\Psi(-h)\big) = -(\alpha \circ T\Psi)(h)$, as required. ■

THEOREM 2.2. *For $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ one has*

$$[FG_{n,k}]^4 = [FG_{2n,2k}].$$

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PROOF. Note that $\Psi: \mathbb{C}G_{2n,2k} \longrightarrow \mathbb{C}G_{2n,2k}$ is a smooth involution as $\Psi^2(W) = \Psi\big(J(W)\big) = J^2(W) = (-\mathrm{Id})(W) = W$, for any $W \in \mathbb{C}G_{2n,2k}$. Clearly $\mathrm{Fix}(\Psi) = \{V \in \mathbb{C}G_{2n,2k} \mid J(V) = V\} = \mathbb{H}G_{n,k}$. From Lemma 2.1 above and by Theorem 22.4 of [1], we see that

$$[\mathbb{C}G_{2n,2k}] = [\mathrm{Fix}(\Psi)]^2 = [\mathbb{H}G_{n,k}]^2$$

Using (1.2) we get

$$[\mathbb{R}G_{2n,2k}]^2 = [\mathbb{C}G_{2n,2k}] = [\mathbb{H}G_{n,k}]^2 = [\mathbb{R}G_{n,k}]^8$$

Since the unoriented cobordism ring is a polynomial ring over $\mathbb{Z}/2$, it follows that the theorem holds for $F = \mathbb{R}$. The general case now follows from (1.2). ∎

We now turn to the proof of Theorem 1.1. First note that, using (1.2), we need only consider the case $F = \mathbb{R}$. Let $\nu = \min\{\nu(n), \nu(k)\}$. Let $m = n \cdot 2^{-\nu}$, and $\ell = k \cdot 2^{-\nu}$. By repeated use of Theorem 2.2 we get

$$[\mathbb{R}G_{n,k}] = [\mathbb{R}G_{m,\ell}]^{4^\nu}.$$

If $\nu(n) > \nu(k)$, then m is even and ℓ is odd. In this case $\mathbb{R}G_{m,\ell}$ admits a fixed point free involution. To obtain such an involution one uses the $\mathbb{R}$-linear automorphism of $\mathbb{R}^m$ which is "multiplication by i" when $\mathbb{R}^m$ is regarded as $\mathbb{C}^{m/2}$. Hence $\mathbb{R}G_{n,k}$ also bounds. (See also [4].)

If $\nu(n) \leq \nu(k)$, then m is odd. In this case $\mathbb{R}G_{m,\ell}$ does not bound [4]. This completes the proof. ∎

REMARK. The analogue of Theorem 2.2 for the more general F-flag manifolds is true, and can be proved similarly.

3. **Decomposability of Grassmannians.** It is a well-known result [3] that the projective space $\mathbb{R}P^n = \mathbb{R}G_{n+1,1} \cong \mathbb{R}G_{n+1,n}$ is indecomposable if and only if n is even. Thus $[\mathbb{R}P^n]$ can be chosen as the n-dimensional generator for the cobordism ring for n even. From Theorems 1.1 and 2.2 it follows that any $\mathbb{R}G_{n,k}$ with n even is decomposable. Indeed $\mathbb{R}G_{n,k}$ is either a boundary or is cobordant to a cartesian power of $\mathbb{R}G_{m,\ell}$ for suitable integers m, ℓ with m odd. We prove below that

THEOREM 3.1. *For $2 \leq k \leq n-2$, $\mathbb{R}G_{n,k}$ is decomposable.*

PROOF. Let $p = \dim \mathbb{R}G_{n,k} = k(n-k)$. Denote by $S_p = S_p(\sigma_1, \ldots, \sigma_p)$ the power sum $\sum_{1 \leq j \leq q} y_j^p$ expressed as a polynomial in the elementary symmetric polynomials σ_j's in q "unknowns" $y_1, \ldots, y_q, q \geq p$. Let $w_j = w_j(\gamma_{n,k}) \in H^j(\mathbb{R}G_{n,k}; \mathbb{Z}/2)$, $1 \leq j \leq k$. Write $w(\gamma_{n,k}) = 1 + w_1 + \cdots + w_k = \prod_{1 \leq j \leq k}(1 + e_j)$. Then

$$w(\mathbb{R}G_{n,k}) = \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{-2} \prod_{1 \leq j \leq k} (1 + e_j)^n,$$

where the e_i's satisfy—among others—the following relation: $e_i^n = 0$ (see [6]).

Choose r such that $n \leq 2^r$. Then $(1 + e_i + e_j)^{2^r} = 1$. Therefore

$$(1 + e_i + e_j)^{-2} = (1 + e_i + e_j)^{2^r - 2}.$$

Hence $w(\mathbb{R}\, G_{n,k}) = \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{2^r - 2} . \prod_{1 \leq i \leq k} (1 + e_i)^n$. Thus we can (and do) regard the j^{th} Stiefel-Whitney class $w(j)$ of $\mathbb{R}\, G_{n,k}$ as j^{th} elementary symmetric polynomial in $e_i + e_j$, $1 \leq i < j \leq k$, each with multiplicity $2^r - 2$, and e_i, $1 \leq i \leq k$, each with multiplicity n. Thus

$$\begin{aligned} S_p(\mathbb{R}\, G_{n,k}) :&= S_p\big(w(1), \ldots, w(p)\big) \\ &= \text{sum of the } p^{\text{th}} \text{ powers of } (e_i + e_j)\text{'s}, \ 1 \leq i < j \leq k, \\ &\qquad \text{and of } e_i\text{'s counted with appropriate multiplicities.} \\ &= \sum_{1 \leq i < j \leq k} (2^r - 2)(e_i + e_j)^p + \sum_{1 \leq i \leq k} n e_i^p \\ &= \sum_{1 \leq i \leq k} n e_i^p. \end{aligned}$$

Now $2 \leq k \leq n - 2$ implies that $p = k(n - k) \geq n$. Since $e_i^n = 0$ for all i, we get $S_p(\mathbb{R}\, G_{n,k}) = 0$. The theorem now follows from this by a well-known result of R. Thom. See, for example [3]. ∎

NOTE. The decomposability of $[FG_{n,k}]$ for $F = \mathbb{C}$ or $\mathbb{H}$ follows from (1.2).

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