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GROUPS WITH SUBNORMAL NORMALIZERS OF SUBNORMAL SUBGROUPS

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Dedicated to John Cossey on the occasion of his seventieth birthday.

Abstract

We consider the class of solvable groups in which all subnormal subgroups have subnormal normalizers, a class containing many well-known classes of solvable groups. Groups of this class have Fitting length three at most; some other information connected with the Fitting series is given.

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1. Introduction

Among the classes of finite solvable groups there has been much attention to classes of groups with particular properties of their subnormal subgroups: we name here groups with all of their subnormal subgroups normal (T-groups), permutable with all subgroups (PT-groups), permutable with all Sylow subgroups (PST-groups), normalized by the nilpotent residual. These classes form a hierarchy; a class not considered so far and containing all the classes mentioned before is the class of solvable groups all of whose subnormal subgroups have subnormal normalizers. As we will see, this class does not only comprise metanilpotent groups, but the extension is very restricted.

2. A hierarchy of subgroup classes

A subgroup U of G is called permutable if it is permutable with all subgroups of G. Maier and Schmid [5] have shown that permutable subgroups are hypercentrally embedded, that is, U/U_G is contained in the hypercenter of G/U_G. In particular, U is subnormal.

A subgroup U of G is called S-permutable if it is permutable with all Sylow subgroups of G. Kegel [4] has shown that these subgroups are subnormal, Schmid [7] proved that $G^{\mathcal{N}} \subseteq N(U)$ and $U^{\mathcal{N}} \subseteq U_G$.

As a consequence we have the following hierarchy of subgroup classes.

PROPOSITION 1. *The following statements for the subgroup U of G are pairwise nonequivalent; the later mentioned statement follows from the preceding one.*

(i) *U is permutable.*
(ii) *U is hypercentrally embedded.*
(iii) *U is S-permutable.*
(iv) *$G^{\mathcal{N}} \subseteq N(U)$.*
(v) *$N(U)$ is subnormal.*

PROOF. (i) ⇒ (ii) was shown by Maier and Schmid [5], (ii) ⇒ (iii) is obvious, (iii) ⇒ (iv) follows from Kegel [4] and Schmid [7], and (iv) ⇒ (v) is again obvious.

We show the nonequivalence by means of three examples.

(A) If D is a nonabelian dihedral 2-group, any noncentral subgroups of order two are not permutable. So (ii) ⇏ (i).
(B) Consider the wreath product $W = A_4 \wr C_2$. Let K be the Klein 4-group of one of the factors A_4. Then $K_W = 1 = Z(W)$ and K is S-permutable, so (iii) ⇏ (ii).
(C) Consider $V = \langle x, y, z, w\rangle$ with the following relations.

$$z^7 = w^7 = [z, w] = y^3 = z^5y^2zy = w^3y^2wy = 1,$$
$$x^4 = yx^3yx = zx^3wx = wx^3zx = 1.$$

Here $\langle z\rangle_V = 1$, $N(\langle z\rangle) = V^{\mathcal{N}}$ and $\langle z\rangle\langle x\rangle \neq \langle x\rangle\langle z\rangle$. So (iv) ⇏ (iii). Further, $\langle zw\rangle$ is subnormal and $N(\langle zw\rangle) = \langle z, w, x^2\rangle$ is subnormal but does not contain $V^{\mathcal{N}}$, showing that (v) ⇏ (iv). □

We turn now to groups *all* of whose subnormal subgroups satisfy one of the statements (i)–(iv) of Proposition 1.

THEOREM 2. *Let G be a solvable group such that* all *of its subnormal subgroups satisfy one of the statements (i)–(iv) of Proposition 1. Then $G^{\mathcal{N}} = M$ is a Dedekind group, and the following statements hold:*

(a) *statement (iv) is true for all subnormal subgroups of G if and only if the following conditions hold for all Sylow p-subgroups S:*
 (i) *there is a supplement C_S of $S \cap M$ in S such that $[C_S, S \cap M] = 1$;*
 (ii) *if $M' \neq 1$ and $p = 2$, then C_S is elementary abelian;*
(b) *statement (iii) is true for all subnormal subgroups of G if and only if all subgroups of M are G-invariant and M is a Hall subgroup;*
(c) *statement (ii) is true for all subnormal subgroups of G if and only if all subgroups of M are G-invariant and M is a Hall subgroup;*
(d) *statement (i) is true for all subnormal subgroups of G if and only if all Sylow subgroups are modular, all subgroups of M are G-invariant, and M is a Hall subgroup.*

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Proof. Assume that the most general statement (iv) is true; then M is a T-group. The elements of $M/F(M)$ induce power automorphisms in $F(M)$ and these are central automorphisms of $F(M)$, so $[G, M] \subseteq F(M)$. However, $[G, M] = M$ since M is the nilpotent residual of G. We deduce that M $(=F(M))$ is a Dedekind group. Now $|M'| \leq 2$ and $Z_\infty(G) \cap M = Z(G) \cap M = M'$ (see Robinson [6, Theorem 9.2.7, p. 264]). Let C be a Carter subgroup of G. Then $CM = G$ and $C \cap M = M'$. Consider a prime p which divides $|M|$ and $|G/M|$ and choose a Sylow p-subgroup S of G such that $S \cap C$ is the Sylow p-subgroup of C. Then $S = (S \cap C)(S \cap M)$. If R is the Hall p'-subgroup of M, then $(S \cap C)R$ is subnormal in G and $S \cap M \subseteq M \subseteq N_G((S \cap C)R)$ by statement (iv). Since $S \cap M$ is a normal subgroup of G, we obtain $[S \cap M, S \cap C] \subseteq S \cap M'$. The statement is true for all Sylow p-subgroups since Sylow subgroups and Carter subgroups are conjugacy classes, and we have shown the necessity of (i) for $p \neq 2$ or $M' = 1$. For the remaining case, assume the existence of $x \in S \cap C$ with $[x, S \cap M] = M' \neq 1$. Let $V = C_{S \cap M}(x)$. Then $[S \cap M, V] = 2$ and V is normalized by the Hall $2'$-subgroup K of C; which is impossible since $[K, S \cap M] = S \cap M$. This shows that $[S \cap M, S \cap C] = 1$. Now let $y \in S \cap C$ and $y^2 \neq 1$. Since $S \cap M$ is a nonabelian Dedekind group it contains a subgroup $\langle a, b\rangle \cong Q_8$. Let R be the Hall $2'$-subgroup of M. Then $\langle ya, R\rangle$ is a subnormal subgroup of G and $b \notin N(\langle ya, R\rangle)$, contradicting the condition that all normalizers of subnormal subgroups contain M. This shows that both (i) and (ii) are necessary in (a).

On the other hand, pick a Sylow p-subgroup S and C_S as mentioned in (a), and assume further that $C_S(S \cap M) = S$ with $[C_S, S \cap M] = 1$ for all Sylow p-subgroups. For $p = 2$ and $M' \neq 1$ assume further that $(C_S)^2 = 1$. Every subnormal subgroup T of G is generated by the subnormal hulls of the elements $x \in T$ of p-power order, and $N(T)$ contains the intersection of the normalizers of these subnormal hulls. Therefore, it suffices to show that if $x \in G$ is an element of p-power order and V is its subnormal hull in G, then $M \subseteq N(V)$. So pick a Sylow p-subgroup S with $x \in S$ and V as described, and denote the Hall p'-subgroup of M by R. Since $SR = SM$ is a normal subgroup of G, we have $VR = \langle x, R\rangle$ and $V = \langle x, [x, R]\rangle$. If $p \neq 2$ or $M' = 1$, then also $[x, M] = [x, R]$ and $M \subseteq N(V)$. If $M' \neq 1$ and $p = 2$, we inspect x more closely. We have $x = yz$ where $y \in C_S$ and $z \in M \cap S$, where $y^2 = 1$ and $[y, M \cap S] = 1$ by hypothesis. If $z^2 = 1$, then $[x, M] = [x, R]$ as before, and $M \subseteq N(V)$. If $z^2 \neq 1$, then $|M, C(z) \cap M| = |M, C(yz) \cap M| = 2$ and there is an element $w \in M \cap S$ such that $w^{-1}zw = z^{-1}$. Thus $w^{-1}(yz)w = (yz)^{-1}$ and $w \in N(\langle yz\rangle)$. So $M \subseteq N(V)$ in this case too, and (a) is proved.

The remaining statements are ordered such that the lower number is used for a stronger statement for (single) subnormal subgroups. Therefore we are able to use statement (a) and prove the possibly stronger property (b), (c), (d) of *all* subnormal subgroups.

Now suppose first that all subnormal subgroups G are S-permutable. Then G is known to be a PST-group (that is, the S-permutability property is transitive), all subgroups of $M = G^{\mathcal{N}}$ are normal in G, and M is a Hall subgroup of G (see [1]). On the

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other hand, provided that M satisfies these two conditions then, by [2, Theorem 2.4.4, p. 90], all subnormal subgroups of G are hypercentrally embedded. We have obtained that all subnormal subgroups of a group are S-permutable if and only if all of them are hypercentrally embedded (and if and only if G is a PST-group). This shows (b) and (c), so (b) and (c) are equivalent.

Statement (d) follows by specialization of (c) and the fact that all subgroups of $G/G^{\mathcal{N}}$ must be permutable and that M is an abelian Hall subgroup with all subgroups normal in G by [2, Theorem 2.1.11, p. 60]. $\square$

3. NSS-groups

The widest class in the hierarchy requires more. For brevity we will call these groups NSS-groups, since normalizers of subnormal subgroups are subnormal. We will see by means of counterexamples that NSS-groups need not be metanilpotent (see Lemma 5). However, they will be shown later to be of Fitting length three at most.

Before we look into some special cases needed later on we notice the following: if G is an NSS-group and $U \subseteq V$ where U is a subnormal subgroup of G, then $N_V(U)$ is a subnormal subgroup of V. In particular, the class of NSS-groups is closed with respect to subnormal subgroups and quotient groups. We begin with three special cases.

Lemma 3. *Assume that the Fitting subgroup $F(G)$ of the NSS-group G is a p-group. Then $G/F(G)$ is a p'-group.*

Proof. Let $G^* = G/F(G)$. We must show that G^* is a p'-group. To derive a contradiction, we assume the existence of some element $xF(G) \in G^*$ of order p. Let x be an element of order a power of p. We denote the subnormal hull of x by X. Let A/B be a p-group and a chief factor of X, and let $U/B \subseteq A/B$ be a cyclic x- invariant subgroup of A/B. Now U is a subnormal subgroup of X and of G and $x \in N(U)$, so $X \subseteq N(U)$. This shows that $U = A$ and all p-chief factors of X are cyclic and $(X/B)/C_{X/B}(A/B)$ is an abelian p'-group. But $X = X^{\mathcal{N}}\langle x\rangle$ so that $X/X^{\mathcal{N}}$ is of order some power of p, so all p-chief factors of X are central. Hence $F(X)$ is a p-group, it is the hypercenter of X and X is nilpotent contrary to construction, so the element $xF(G)$ of order p does not exist. Lemma 3 is shown. $\square$

Corollary 4. *NSS-groups are of p-length one for all primes p.*

Proof. If L is an NSS-group and W is the maximal normal p'-subgroup of L, consider L/W. Here $F(L/W)$ is a p-group and $(L/W)/F(L/W)$ is, by Lemma 3, a p'-group. $\square$

First nonmetanilpotent NSS-groups can be derived from the following.

Lemma 5. *Assume that the Fitting subgroup $F(G)$ of the NSS-group G is a minimal normal subgroup of G and that $|F(G)| = p^2$ for some prime p. If G is not metanilpotent, then $p \notin \{2, 3\}$ and one of the following three cases arises:*

(a) $p \equiv 2 \pmod 3$ *and $G/F(G)$ is isomorphic to a subgroup of the central product of* $\mathrm{SL}(2, 3)$ *and C_n with $2n = p - 1$;*

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(b) $p \equiv 3 \pmod 4$ *and* $G/F(G)$ *is isomorphic to a subgroup of the direct product*

$$C_k \times \langle x, y \mid x^{2k} = y^4 = x^{-p}y^3xy = x^k y^2 = 1\rangle$$

where $2k = p - 1$*;*

(c) $p \equiv 3 \pmod 4$ *and* $G/F(G)$ *is isomorphic to a subgroup of the direct product*

$$C_k \times \langle x, y \mid x^{2k+2} = y^4 = x^{-p}y^3xy = x^{k+1}y^2 = 1\rangle$$

where $2k = p - 1$.

Proof. For $|F(G)| = p^2$ and $p \in \{2, 3\}$, G is either metanilpotent or of p-length 2 and not an NSS-group by Corollary 4. From now on $p \notin \{2, 3\}$.

A subgroup of $\mathrm{GL}(2, p)$ which is not nilpotent and not contained in $(\mathrm{GL}(2, p))^2$ but in $\mathrm{GL}(2, p)$ possesses a nonsubnormal subgroup of order two, contrary to the NSS-property. Therefore the subgroups mentioned in all cases have to be subgroups of $(\mathrm{GL}(2, p))^2$.

We consider first the maximal NSS-subgroups of $\mathrm{PSL}(2, p)$ that are p' groups: they are isomorphic to (a) A_4, (b) D_{2k}, or (c) D_{2k+2} where D_n is the dihedral group of order n and $2k = p - 1$.

For case (a), we consider the preimage of A_4 under the epimorphism of $\mathrm{SL}(2, p)$ on $\mathrm{PSL}(2, p)$. This is isomorphic to $\mathrm{SL}(2, 3)$. In $(\mathrm{GL}(2, p))^2$ we obtain the product of this group with the center C_n with $2n = p - 1$ with coinciding subgroup of order two. If $p - 1$ is divisible by 3, there is a cyclic subgroup of $F(G)$ which is invariant under an element of order three and its normalizer is not subnormal in G. So $p \equiv 2 \pmod 3$.

Cases (b) and (c) are treated in the same way: the preimages of the dihedral groups again have only one involution, and this is therefore central. If the subgroup mentioned in (b) is nonnilpotent, it possesses a nonsubnormal subgroup of order four which does not leave invariant a cyclic subgroup in $F(G)$ by the NSS property. Therefore the condition $p \equiv 3 \pmod 4$ is necessary, and so the central subgroup C_k is of odd order. □

We will have to consider (not necessarily subnormal) subgroups of NSS-groups later. For this we will need some statements in a more general setting.

Lemma 6. *Assume that the group G and its minimal normal subgroup M satisfy the following conditions.*

(a) $Z(F(G/C(M))) = Z(G/C(M))$ *and* $G/C(M)$ *is nonnilpotent.*

(b) *There is an element x of order a power of a prime q such that:*

   (b1) $G/C(M) = \langle xC(M), F(G/C(M))\rangle$*; and*

   (b2) $G/C(M)$ *is the subnormal hull of* $xC(M)$.

(c) $(G/C(M))^{\mathcal{N}}$ *is nilpotent.*

(d) *If U is a subnormal subgroup of $G^{\mathcal{N}}$ then $N(U)$ is subnormal in G.*

Then:

(I) $F(G/C(M))$ is an extraspecial group of order 2^{2m+1};

(II) $q = 2^m + 1$ is a prime and $x^q \in C(M)$;

(III) $\exp(M)$ is a nonsquare modulo q.

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Proof. For brevity we put $S^* = SC(M)/C(M)$ for all subgroups $S \subseteq G$. Consider a nonabelian Sylow r-subgroup R^* of $(G^{\mathcal{N}})^*$. By (b1) and (b2) we know that $r \neq q$, and from (a) we deduce that $Z(R^*) \subseteq Z(G^*)$. Thus $Z(R^*) \subseteq Z(G^*)$ is cyclic by Schur's lemma.

To derive a contradiction, we assume first that R^* possesses a noncyclic elementary abelian characteristic subgroup C^* and choose a Hall r'-subgroup H^* of G^*. The r-module C^* may be considered an H^*-module; it splits into a direct product $U^* \times Z(G^*) \cap R^*$ such that U^* is H^*-invariant. Therefore, $H^* \subset N(U^*)$ and $R^* \not\subseteq N(U^*)$, so $N(U^*)$ is not subnormal but U^* is. Note that we are using (c) and (d) of the hypothesis here.

We obtain:

(i) abelian characteristic subgroups of R^* are cyclic.

By a theorem of Hall (see, for instance, Huppert [3, Satz III.13.10]), we obtain that R^* is a central product of an extraspecial group and a cyclic, dihedral, generalized quaternion or semidihedral group.

We want to prove first that $r \neq 2$ is impossible. For odd r, we have that R^* is a central product of an extraspecial group E^* and a cyclic group. Let $\langle x \rangle C(M)/C(M) = Q^*$. Then $[Q^*, E^*] \neq [(Q^*)^p, E^*] = 1$ and we may assume that $Q^* \subseteq N(E^*)$. Let F^* be a minimal Q^*-invariant subgroup of E^* that is not centralized by R^*. If F^* is abelian, it is elementary abelian and $F^* \cap Z(R^*) = 1$. Now $[Q^*, E^*] \not\subseteq N(F^*)$ but $Q^* \subseteq N(F^*)$, in contradiction to (d). The same happens if $Z(F^*) \neq F^* \cap Z(R^*)$. We deduce that F^* must be an extraspecial subgroup of E^*. There is a number k such that $|F^* : T^*| = s^k$ for all maximal abelian subgroups T^* of F^* and $|F^*| = s^{2k+1}$. By minimality of F^*, q divides $s^k + 1$ and is also different from 2 and smaller than s^k since s is an odd prime. By minimality of M we have also $[M, Z(E^*)] = M$. Consider a minimal Q^*-invariant subgroup L of M. By construction, $Q^* \subseteq N(L)$ and therefore $F^* \subseteq N(L)$ since F^* is contained in the subnormal hull of Q^* by (d). We consider the rank $d(L)$ of L. Since F^* is a normal subgroup of $N(L)$, we obtain that s^k divides $d(L)$. On the other hand, since L is minimal Q^*-invariant and $|Q^*| = q^m$ for some m, we have $d(L) = q^t w$ where w divides $q - 1$ and $t \leq m$. Clearly $w < q - 1$. Since s and q are different primes, we obtain by comparison that s^k divides w, so $s^k \leq w < p$. However, Q^* operates on $F^*/(F^*)'$ without fixed points and q divides $s^{2k} - 1$, so $q < s^k$, a contradiction. This shows that $r = 2$, that is,

(ii) $F(G/C(M))/Z(F(G/C(M)))$ is a 2-group.

Our nonabelian Sylow subgroup R^* of $F(G/C(M))$ is a 2-group and is a central product where one factor is an extraspecial group E^* and the other, K^*, say, is cyclic, dihedral, semidihedral, or generalized quaternion. If K^* is noncyclic, it is of order 16 at least. We will reduce the possibilities first to the cyclic case. If K^* is nonabelian, then $\exp(K^*) > \exp(E^*)$ and the subgroup $L^* \subseteq R^*$ generated by the elements of order (exactly) $\exp(K^*)$ is of index two in R^*. Now $[Q^*, R^*] \subseteq L^*$ by (b2) and because L^* is characteristic in R^*. So $L^* = R^*$, $K^* = Z(R^*)$ and R^* is a central product of E^* and the cyclic subgroup $Z(R^*)$. Now $[Q^*, Z(R^*)] = 1$ and $[Q^*, R^*] = R^*$ by (b2), so $R^* = E^*$. If $|E^*| = 2^{2m+1}$, we obtain that $d(M)$ is a multiple of 2^m since E^* is a normal subgroup of G^*. On the other hand, since x is of some order q^{r+1}, $d(M)$ must be a divisor of $q^r(q-1)$. In particular, $q-1$ must be a multiple of 2^m. Since xM induces a nontrivial automorphism on E^*, the only possibility is $q = 1 + 2^m$ and $E^*/Z(E^*)$ is x-irreducible. Also $d(M)$ is divisible by $q-1$ and $\exp(M)$ must be a nonsquare modulo q.

We have shown that:

(iii) $F(G/C(M))$ is an extraspecial 2-group of order 2^{2m+1};

(iv) $q = 2^m + 1$;

(v) $\exp(M)$ is a nonsquare modulo q.

Thus (I) and (III) and the first part of (II) are shown.

Let $x^q \notin C(M)$ and consider $z \in \langle x \rangle$ such that $z^q C(M) \in F(G^*)$ and $zC(M) \notin F(G^*)$. We have $1 = [x^q C(M), E^*] = [zC(M), E^*]$ and $\langle E^*, zC(M) \rangle = W^*$ is a normal subgroup of G^*. Minimal W^*-invariant subgroups of M must be of rank 2^{2m} and so also $d(M)$ must be divisible by 2^{2m}. But this is impossible: $d(M) = q^r(q-1)$. This contradiction shows that:

(vi) $x^q \in C(M)$.

Now (II) is proved by (iv) and (vi), and the proof is complete. □

Lemma 7. *Assume that the group G and its minimal normal subgroup M satisfy the following conditions.*

(a) $Z(F(G/C(M))) \neq Z(G/C(M))$.

(b) *There is an element x and a prime q such that:*

(b1) $G = \langle x, G^{\mathcal{N}}, C(M) \rangle$*;*

(b2) $|x|$ *is a power of* q.

(c) $G^{\mathcal{N}} C(M)/C(M)$ *is nilpotent.*

(d) *If U is a subnormal subgroup of $G^{\mathcal{N}}$ then $N(U)$ is subnormal in G.*

Then:

(I) $x^q \notin C(M)$;

(II) $x^q C(M) \in Z(G/C(M))$;

(III) $(G^{\mathcal{N}})' \subseteq C(M)$.

Proof. M is a p-subgroup for some prime p. If $G^{\mathcal{N}} \cap M = 1$, then $G^{\mathcal{N}} \subseteq C(M)$ and $G = \langle x, C(M) \rangle$ by (b1), and $F(G/C(M)) = G/C(M)$, contradicting (a). Thus:

(i) $M \subseteq G^{\mathcal{N}}$.

Assume the existence of a proper x-invariant subgroup L of M. Then $x \in N(L)$ and $N(L)$ is a subnormal subgroup of G by (d). By construction, $N(L) = G$ contradicting the minimality of M. We have derived that:

(ii) M is a minimal x-invariant subgroup.

In particular, if $p = q$ we would have $|M| = p = q$ and again $F(G/C(M)) = G/C(M)$. Thus:

(iii) M is not a q-subgroup.

Again we denote $XC(M)/C(M)$ by X^*, for all $X \subseteq G$. Put also $T^* = (G^{\mathcal{N}})^*$. Since M is a minimal x-invariant subgroup by (ii), the rank $r(M)$ of M is of the form aq^s

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where a is a divisor of $q-1$, namely the minimal integer such that q divides $p^a - 1$. We will show that $r(M)$ must be divisible by q and consequently $x^q \notin C(M)$.

By (a), $Z(F(G^*)) \neq Z(G^*)$. Assume first that $Z(F(G^*))$ is not cyclic and consider M under the operation of $Z(F(G^*))$. By Schur's lemma we know that $Z(V/C(N))$ is cyclic whenever N is a minimal normal subgroup of V. So in our case M is a direct product of conjugate $Z(F(G^*))$-invariant subgroups, where the number of factors is $|G^* : C(Z(F(G^*)))|$, a power $q^n \neq 1$. So:

(iv) $r(M)$ is divisible by q if $Z(F(G^*))$ is noncyclic.

The second possibility is that $Z(F(G^*))$ is cyclic. Let M be a direct product of minimal $Z(F(G^*))$-invariant subgroups L_j; their rank $r(L_j)$ is the same since $Z(F(G^*))$ is a normal subgroup of G^*. The element $xC(M)$ induces a nontrivial automorphism on $Z(F(G^*))$. If $xC(M)$ permutes the factors L_j by conjugation such that no factor is fixed, then the number of factors and $r(M)$ are divisible by q. If one factor L_j is normalized by $xC(M)$, then the conjugation of $xC(M)$ induced in $Z(F(G^*))/(C(L_j))^*$ is a field automorphism of the field of order $|L_j|$. This happens only if $r(L_j)$ is divisible by q (and $L_j = M$). We summarize:

(v) $r(M)$ is divisible by q if $Z(F(G^*))$ is cyclic.

In particular, $r(M) \geq q$ and we see that:

(vi) $x^q \notin C(M)$.

This shows (I).

Assume now that q is odd. We know that M is a minimal normal p-subgroup of $\langle x, M\rangle$ so $r(M) = q^t a$, with a as before, is divisible by q and $t > 0$. Now M splits into a direct product of q minimal x^q-invariant subgroups L_j; they are all of rank $r(L_j) = q^{t-1}a$ and they are all x^q-operator isomorphic because M is minimal x-invariant. Therefore M may be considered as a vector space over the field F of order $p^{q^{t-1}a}$ such that x^p operates as field multiplication, so x^p is contained in the intersection of all normalizers of F-subspaces of M, and this is contained in $Z(\mathrm{GL}(q, F))$ and subnormal in $G/C(M)$ by (d). So $x^q C(M) \in Z(G^*)$ if q is odd. The same argument is correct for $q = 2$ provided that $r(M) > 2$. For $r(M) = 2$ all possibilities are given in Lemma 5, and we obtain in general:

(vii) $x^q C(M) \in Z(G/C(M))$.

We have proved (II).

It remains to show the commutativity of $(G^{\mathcal{N}})^*$. We distinguish three cases: $Z(F(G^*))$ is noncyclic; $Z(F(G^*))$ is cyclic and $|Z(F(G^*))|$ does not divide $p^{t-1} - 1$; and $Z(F(G^*))$ is cyclic and $|Z(F(G^*))|$ divides $p^{t-1} - 1$.

If $Z(F(G^*))$ is noncyclic, then M splits into a direct product of q factors L_j which are $Z(F(G^*))$-invariant. For any two different factors L_j the centralizers $(C(L_J))^* \cap Z(F(G^*))$ are different and normal in $F(G^*)$. It follows that the factors L_j are $F(G^*)$-invariant and also $(C(L_j))^* \cap F(G^*)$ are normal subgroups of $F(G^*)$. Since the factors L_j are minimal x^p-invariant and $\langle x^p\rangle^* \subseteq Z(F(G^*))$, we obtain that $F(G^*)/F(G^*) \cap (C(L_j))^*$ is abelian, and since the intersection of all $C(L_j)$ is $C(M)$, we have that $F(G^*)$ is abelian.

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If $Z(F(G)$ is cyclic and $|Z(F(G^*))|$ does not divide $p^{t-1}-1$, then M is minimal $Z(F(G^*))$ invariant and $F(G^*) = Z(F(G^*))$ is abelian.

Assume, finally, that $Z(F(G^*))$ is cyclic and $|Z(F(G^*))|$ divides $p^{t-1}-1$. We have to show that M is a direct product of $F(G^*)$-invariant direct factors L_j. This happens if and only if the factors L_j are operator isomorphic with respect to $Z(F(G^*))$. Let $Z(F(G^*)) = \langle x^p\rangle^* \times \langle y\rangle^*$ so that y is an element of order prime to q. We know that $x^{-1}yxC(M) \neq yC(M)$ by (a). If k is minimal such that $|yC(M)|$ divides p^k-1, then $k \leq t-1$. Conjugation by y will permute the factors L_j; if they are operator isomorphic in $G^{\mathcal{N}}$, then x can be chosen such that $x^{-1}yxC(M) = y^{p^{k-1}}C(M)$. On the other hand, $x^{-1}x^pxC(M) = x^pC(M) \neq x^{p^k}C(M)$ and $x^{-1}(x^py)xC(M) = x^py^{p^{k-1}}C(M) \neq (x^py)^{p^{k-1}}C(M)$ have, as operators of different direct factors L_j, different minimal polynomials and therefore the factors L_j are not operator isomorphic. We deduce again that all L_j are $F(G^*)$-invariant and $F(G^*)/F(G^*) \cap (C(L_j))^*$ is abelian as in the first case, and $F(G^*)$ is abelian. Thus (III) follows and the proof is complete. □

Remark. All three cases mentioned in the last paragraph of the proof of Lemma 7 do appear. We indicate examples by mentioning possible triples $(|M|, F(G^*), |xC(M)|)$.

(A) $F(G^*)$ noncyclic: $(7^3, K_4, 9)$ with K_4 the Klein 4-group.

(B) $F(G^*)$ cyclic, $|F(G^*)|$ is not a divisor of $q^{t-1}-1 : (7^3, C_{19}, 9)$ (form the semidirect product of E_{7^3} by C_9 and extend by an element z of order three inducing by conjugation the automorphism of order three. Choose a subgroup of index three, not the semidirect product).

(C) $F(G^*)$ cyclic, $|F(G^*)|$ is a divisor of $q^{t-1}-1 : (7^9, C_{19}, 27)$.

For our main result we need two further auxiliary statements.

Lemma 8. *If G is an NSS-group and* $\exp(G/F(G))$ *is squarefree, then $G/F(G)$ is nilpotent.*

Proof. Consider a p-chief factor U/V of G for some prime p. Then U/V is a minimal normal subgroup of G/V and $F(G/V) \subseteq C(U/V)$. Since $F(G)V/V \subseteq F(G/V)$ we obtain:

(i) $(G/V)/C(U/V)$ is of squarefree exponent.

Assume first that $H = (G/V)/C(U/V)$ is metanilpotent. Consider an element $x \in H \backslash F(H)$ of prime order and call X the subnormal hull of $\langle x\rangle$ in H. We have $X \cong W/C(U/V)$ for some subnormal subgroup $W \subseteq G$. Now let U_1/V_1 with $V \subseteq V_1 \subset U_1 \subseteq U$ be a chief factor of W. Now $\{W/V_1, U_1/V_1\}$ satisfy the hypotheses of $\{G, M\}$ in Lemma 6 or 7. In both cases the quotient group $G/C(M)$ is not of squarefree exponent, so $W/C_{W/V_1}(U_1/V_1)$ cannot be nonnilpotent and $x \in F(H)$ contrary to construction. It follows also that $W/C_{W/V}(U/V)$ is nilpotent. In particular, $H = F(H)$ if H is metanilpotent. In the general case, if we have for H the Fitting series

$$1 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq H$$

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such that $A_i/A_{i-1} = F(H/A_{i-1})$ for all $i > 0$, we argue in the same way that $A_1 = A_2$ $(=H)$. So we obtain that $H = (G/V)/C(U/V)$ is nilpotent for every chief factor U/V of G. Lemma 8 is established. □

Lemma 9. *Assume that the solvable group G and the minimal normal subgroup M of G satisfy the following conditions.*

(a) *There is an element $x \in G$ and a prime p such that x is of order a power of p and $G = \langle G^{\mathcal{N}}, x\rangle$.*
(b) *$G/C(M)$ is not nilpotent.*
(c) *If U is a subnormal subgroup of $G^{\mathcal{N}}$ then $N(U)$ is subnormal in G.*

Then $[x^p, G] \subseteq M$.

Proof. Consider a chief factor K/L of G such that $C(M) \subseteq L \subset K \subseteq G^{\mathcal{N}}$. If K/L is a p-group, there is a cyclic subgroup Z/L such that $[x, Z] \subseteq L$. Since $Z \subseteq G^{\mathcal{N}}$ we have that $N(Z)$ subnormal in G. Since $x \in N(Z)$ we obtain that Z is normal in G and $Z = K$. Further, G induces by conjugation a group of order dividing $p - 1$ in K/L, while $G/G^{\mathcal{N}}$ is a p-group. We see by an obvious induction argument that $G^{\mathcal{N}}$ must be an extension of a p'-group by a p-group, and since $G/G^{\mathcal{N}}$ is a p-group we obtain:

(i) $G^{\mathcal{N}}C(M)/C(M)$ is a p'-group.

Assume that the proper subgroup T of M is x-invariant. Then $x \in N(T)$ and $N(T) = G$ contrary to minimality of M. So:

(ii) M is minimal x-invariant.

Now let $F/C(M) = F(G/C(M)) \cap G^{\mathcal{N}}C(M)/C(M)$. Then $\langle x, [x, F]\rangle$ also satisfies the hypotheses of Lemma 9. We apply Lemma 6 if $Z([x, F]C(M)/C(M)) \subseteq Z(\langle x, [x, F]\rangle)$ and obtain $x^p \in C(M)$. In the other alternative we apply Lemma 7 and obtain $[x^p, [x, F]] \subseteq C(M)$ and $[x^p, F] = [x^p, [x^p, F]] \subseteq [x^p, [x, F]] \subseteq C(M)$. Remember that $F/C(M) = F(G^{\mathcal{N}}C(M)/C(M))$ by the Fitting property. So $C_{F/C(M)}(F/C(M)) = Z(F/C(M))$ and $C_{G/C(M)}(F/C(M)) = \langle x^pC(M), Z(F/C(M))\rangle$ since $[x, F] \not\subseteq C(M)$. Now $(F/C(M))C_{G/C(M)}(F/C(M))$ is a nilpotent normal subgroup of $G/C(M)$—it is in fact $F(G/C(M))$—and $\langle x^pM\rangle$ is a characteristic subgroup of this group. So also:

(iii) $[x^p, G^{\mathcal{N}}] \subseteq M$,

since $x^pC(M)$ and $G^{\mathcal{N}}C(M)/C(M)$ are of relatively prime order. Now Lemma 9 follows since $G = \langle x, G^{\mathcal{N}}\rangle$. □

Theorem 10. *For the NSS-group G the following statements are true:*

(a) *G is of p-length 1 for all primes p;*
(b) *if $F_1 = F(G)$ and $F_2/F_1 = F(G/F_1)$, then G/F_2 is nilpotent of squarefree exponent;*
(c) *denoting $G^{\mathcal{N}}F(G)/F(G)$ by Q, then Q' and $Q/Z(Q)$ are of exponent two;*
(d) *$G^{\mathcal{N}\mathcal{N}}$ is nilpotent.*

Proof. Statement (a) is shown in Corollary 4.

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For (b) take an element $x \in G \backslash F_2$ of order some power of a prime p. Let X/F_1 be the subnormal hull of xF_1 in G/F_1. Certainly X is an NSS-group. Consider a normal subgroup Y of X which is maximal with respect to not containing $F(X)$. Then $F(X)Y/Y = F(X/Y)$ is a minimal normal subgroup of X/Y and we have $x^pY \in Z(X/Y)$ or equivalently $[x^p, X] \subseteq YF_1$ by Lemma 9. The intersection of all normal subgroups Y as defined above is F_1 itself, so:

(i) $[x^p, X] \subseteq F_1$.

In particular, $\langle x^p \rangle F_1$ is a subnormal subgroup of the subnormal subgroup X. Also:

(ii) $x^p \in F_2$.

This shows that G/F_2 is of squarefree exponent. By Lemma 8 we deduce that:

(iii) G/F_2 is nilpotent.

Denote by M a chief factor U/V of G with $U \subseteq F(G)$. We apply Lemmas 6 and 7 to obtain that if $R = G^{\mathcal{N}}C(M)/C(M)$, then R^2 and $R/Z(R)$ are of exponent two in both cases, $Z(R) = Z(G/C(M))$ or not. This shows (c).

Finally, G is of Fitting length 3 by (b), so $G^{\mathcal{N}\mathcal{N}}$ is nilpotent, and (d) is true. □

Reminder. For example, the following nonsolvable groups are obviously NSS-groups.

(I) Extensions of any direct product of simple nonabelian groups by nilpotent groups.

(II) Extensions of any elementary abelian group of order p^2 by SL(2, 5) , where p is a prime such that $p + 1$ is divisible by 60.

The restriction to solvable groups in this paper is therefore essential.

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