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On monoids related to braid groups

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OUTLINE. In the first half of the thesis, we introduce the class of 'chainable monoids' and describe general techniques for solving the word and division problems and for obtaining normal forms for monoids in this class. Their growth functions are shown to be rational, and calculable. We also show that a monoid with a certain central 'fundamental element' embeds in a group or related monoid. Motivating examples of monoids in this class are positive braid monoids and positive singular braid monoids. Much of the rest of the thesis is devoted to applying the techniques described to singular Artin monoids, and we obtain further results regarding conjugacy and parabolic submonoids. In the final chapter we apply the techniques of chainable monoids to braid groups of complex reflection groups.

A *chainable monoid* is a monoid which may be defined by a finite presentation with four properties; three of which can be determined immediately from the length of the relations and their starting letters, and the fourth, the *reduction property*, which is more involved to verify. It has been shown that positive Artin monoids, including braid monoids, have these properties. Other examples of monoids in this class include free monoids, free commutative monoids, positive singular braid monoids (see later), a large class of one relation monoids, trace monoids and divisibility monoids. The class of chainable monoids is closed under taking direct and free products.

Chainable monoids are always left cancellative. The method of *chains* enables the construction of a calculable partial map on pairs of words, which returns the left quotient of the first word by the second, whenever it exists, thus giving a solution to the division problem, and hence word problem, in a chainable monoid. Again using chains, we construct another partial map on pairs of words which is shown to return their least common left multiple, whenever it exists. We show that in a chainable monoid, greatest common left divisors always exist, and least common left multiples exist whenever common left multiples exist. [The author is grateful to Dietrich Kuske for recently pointing out that chainable monoids may be algebraically characterised as left cancellative monoids that

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are finitely generated by their atoms, possess greatest common divisors for all pairs of elements, and for which maximal divisor chains of an element always have the same size.] Consequently we may define a unique normal form recursively on word length: the normal form of a word W is the least common multiple L_W of the generators which left divide W, concatenated with the normal form of the quotient of W by L_W (which is strictly shorter). These techniques generalise the approaches found in [**6**] and [**3**].

In Chapter 2 we show that chainable monoids have rational growth—in fact the growth function is shown to be of the form $1/(1 - p(z))$ where p is a polynomial with integer coefficients and zero constant term. The function is computable, indeed simple to compute, and examples are given for some 'favourite' chainable monoids.

We next describe a technique for embedding a monoid with a *fundamental element* into a larger monoid, or group. Suppose the monoid M is generated by $X \cup Y$, and suppose further that there exists an element Ψ which is central, cancellable, and a common multiple of X for which the left quotient by any x in X equals the right quotient. Then M is shown to embed in a monoid G generated by $X \cup X^{-1} \cup Y$—in particular, if Y is empty (that is, if X generates M) then M embeds in a group. Furthermore, any element of G may be decomposed as the product of a power of Ψ with an element from M. In this way, any unique normal form for M may be extended to give a unique normal form for G.

The techniques developed thus far may now be applied to the motivating example—*singular braid monoids*. Singular braid monoids were introduced in the context of Vassiliev invariants of knots, and presentations were given in Baez [**1**] and Birman [**2**]. Analogously to the way in which Artin groups may be considered as generalisations of braid groups, we generalise the presentation of Baez and Birman, and introduce singular Artin monoids of arbitrary type. These are defined by a presentation associated to a labelled graph, in a similar way to Artin groups. Positive singular Artin monoids are defined by presentations involving only 'positive generators'—no inverses allowed.

In Chapter 5 we show (via a technically involved proof) that positive singular Artin monoids have the reduction property, and subsequently are chainable. This may be found in [**5**]. Thus we have solutions to the division and word problems, can calculate common multiples, quotients and the unique normal form described above. Furthermore these monoids are left cancellative, and by the symmetry of the defining relations, right cancellative as well. We show that there is a fundamental element in the sense defined earlier, with respect to which a positive singular Artin monoid of finite type embeds in the corresponding singular Artin monoid, giving a unique normal form in the larger monoid as well. We describe a (calculable) partial map on pairs of words in the singular Artin monoid which returns the quotient when it exists, and thus obtain a solution to the division problem in singular Artin monoids of finite type.

There is more than one way to define conjugacy in a monoid, and in Chapter 6 of the thesis we consider two of these relations in singular Artin monoids. There is a Markov type theorem for singular braids [**7**], which relates conjugacy of singular braids and equivalence of the singular knots which are obtained by closing the singular braids; the question of determining conjugacy is thus immediately relevant. We describe algorithms for determining when elements of positive singular Artin monoids are conjugate, and when elements of the singular Artin monoids of finite type are conjugate, using the fundamental element decomposition described earlier. Thus we have a solution to the conjugacy problem in all these monoids.

We consider other algorithmic questions related to conjugacy in singular Artin monoids of finite type—for example, we give algorithms for determining when parabolic submonoids are conjugate, for determining membership of the centraliser of a generator, and for determining when an element z has the property that $az = zb$ for generators a and b (these are the (a, b)-*bands* of [**8**]).

In the final chapter, we describe some results regarding braid groups of complex reflection groups. Presentations for these groups were given in [**4**], where the question was posed as to whether the corresponding positive braid monoids embed in the groups. By showing that all but one of the exceptional types is chainable and possesses a fundamental element, we prove that they do embed in the corresponding groups for these cases; moreover the machinery developed in the first part of the thesis gives solutions to the word and division problems as well as a unique normal form. The remaining exceptional type G_{31} is shown not to embed. For the infinite families, we show that monoids of type $B(e, e, r)$ fail to be cancellative, and hence cannot embed; finally type $B(de, e, r)$ is shown not to satisfy the reduction property, although it remains unclear as to whether the monoid embeds in the corresponding group.

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