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A TOPOLOGISED MEASURE HOMOLOGY

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Abstract. A homology theory based on measures, first mentioned by Thurston, is naturally defined here as a functor into the category of locally convex topological vector spaces. It is proved that the first homology space is Hausdorff.

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1. Introduction. The study of problems involving measure of theoretic and topological properties may demand the construction of algebraic functors on categories that blend measure and topology. This is the case in topological ergodic theory where one encounters various groups of measure preserving homeomorphisms. This present work begins by recalling the definition of a homology functor based on measures and described by Thurston [**23**, pp. 6.6–6.7]. It was Prof. D. B. A. Epstein who taught me about this theory a long time ago; so I want to express my deepest gratitude to him.

The measure homology groups to be defined become locally convex linear spaces under the appropriate weak topology over a suitable category of topological spaces. Two examples, the circle and an Eilenberg–Mac Lane space of type $(\mathbb{Q}, 1)$, are discussed. The latter is used to prove that the first homology topological vector space is Hausdorff.

From the algebraic point of view, the measure homology groups satisfy the Eilenberg–Steenrod axioms for a wide class of topological spaces including metric spaces and are naturally equivalent to the standard singular homology groups with real coefficients for at least the large class of CW-complexes [**11**]. Zastrow [**27**] suggests 'slight changes to Milnor's and Thurston's original definition (giving differences for wild topological spaces only)' (see [**27**, Abstract and pp. 380–381]). In addition, the author gives an example of a (wild) space for which the first measure homology group differs from the first real singular homology group.

The homology theory considered here has been used mainly for one application in three-manifold theory: the fundamental cycle of a hyperbolic manifold can be more elegantly expressed as a uniform measure in this theory than in the classical way as a finite sum of singular simplices. Therefore, this homology theory can be conveniently used in the proof that the volume of a closed hyperbolic manifold is a topological invariant. A short description of this fact can be also found in Ratcliffe's book [**17**, Section 11.5]. This proof of the invariance of the hyperbolic volume is considerably more elegant and stronger than in Benedetti's and Petronio's book [**1**, Chapter C], which uses classical singular homology theory.

Löh [**14**] proves that the isomorphism between measure homology and singular homology is isometric with respect to the l^1-semi-norm on singular homology and the semi-norm on measure homology induced by the total variation of the measure. The isometry is proved via dualisations of both homology theories in terms of bounded cohomology and a strong duality principle via a Kronecker product.

The appropriate duality principle has already been observed by Gromov [**10**, end of Section 2.3]. In the same paper, Gromov also mentions and introduces measure homology in his arguments (see [**10**, Section 2.2]). Ratcliffe credits Gromov and Thurston for the construction of measure homology (see [**17**, p. 571]), while Zastrow credits Milnor and Thurston (see [**27**, p. 369]). Other papers that use measure homology theory are Kuessner [**13**], Munkholm [**16**] and Soma [**21**].

The idea to consider the invariants of algebraic topology not only as objects of algebra but also as objects that have as well a compatible topological structure is not new. Dugundji [**4**] considers a topologised fundamental group.

Let (X, x) be a topological pointed space. Recently, Biss [**2**], generalising [**4**], puts a topology on the fundamental group $\pi_1(X, x)$ in quite a natural way and asserts that X is semi-locally simply connected if and only if $\pi_1(X, x)$ is discrete [**2**, Theorem 5.1, p. 365].

As part of his programme, Biss enlarges the classical theory of covering spaces. Nevertheless, Fabel [**8**] shows that it is indispensable to add the assumption that X must be 'locally path connected' in order to make [**2**, Theorem 5.1] correct. In [**5**] and, more explicitly in [**6**], Fabel shows that the description given in [**2**] of the fundamental group of the so-called harmonic archipelago as well as [**2**, Theorem 8.1] is false.

Fabel [**7**, Remark 2.1, p. 188], points out that there is a gap in the proof of [**2**, Proposition 3.1] which asserts that the topologised fundamental group is a topological group. The gap is created because the product of quotient maps may fail to be a quotient map itself. In [**7**], the question of whether the topologised fundamental group, as defined by Biss, has a jointly continuous multiplication is left unsolved. Biss [**2**] is certainly a valuable contribution to the literature. Even in his work attacking [**2**], Fabel does not hesitate from using other parts than those criticised as sources for his own arguments. In this work, we shall consider certain quotients of topological vector spaces. In our setting, there arise no difficulties with the joint continuity of the vector space operations involved.

I want to thank the referee for putting the present work into context through many pertinent observations.

2. A Homology Theory Based on Measures. Let $\mathbb{R}^\infty$ be the vector space of all sequences $x = \{x_i\}_{i=1}^\infty$ of real numbers that vanish from some point on, that is, there exists a non-negative integer n (depending on x) such that $x_i = 0$ for all $i \geq n$.

Let $e_i\,(i = 1, 2, \ldots)$ be the vector whose ith coordinate is one and all other coordinates are zero, and let e_0 be the zero vector. Identify $\mathbb{R}^n$ $(n \geq 0)$ with the sub-space of $\mathbb{R}^\infty$ having all components after the nth coordinate equal to zero.

For $q \in \mathbb{N}$, define the *standard (geometric) simplex* $\triangle^q$ as the convex hull of the set $\{e_0, e_1, \ldots, e_q\}$ (note that $\triangle^1$ is just the unit interval I).

If $P_0, P_1, \ldots, P_q$ are points in some vector space $\mathbb{E}$, $(P_0, P_1, \ldots, P_q)$ will denote the restriction to $\triangle^q$ of the unique affine map $\mathbb{R}^q \to \mathbb{E}$ taking e_i into P_i $(i = 0, 1, \ldots, q)$.

Given a topological space X, a *singular q-simplex in X* is a (continuous) map $\sigma\colon \triangle^q \to X$. Let $S(\triangle^q, X)_\kappa$ denote the space of all singular q-simplices with the compact-open topology. Note that $S(\triangle^q, X)_\kappa$ is Hausdorff if and only if X is Hausdorff (see [**3**, p. 258, Subsection 1.3(a)]). In this situation, if $\mathcal{K} \subset S(\triangle^q, X)_\kappa$ is compact, then it is closed; hence, $\mathcal{K}$ and its complement are Borel subsets of $S(\triangle^q, X)_\kappa$.

Assume X to be Hausdorff and let C_qX denote the real vector space of finite signed Borel measures on $S(\triangle^q, X)_\kappa$ with compact support. More precisely, $\mu \in C_qX$ if and only if μ is a σ-additive, real-valued function defined on all Borel subsets of $S(\triangle^q, X)_\kappa$ such that there exists a $\mathcal{K} \subset S(\triangle^q, X)_\kappa$ compact with the property that the complement of $\mathcal{K}$ in $S(\triangle^q, X)$ has zero μ measure, that is, $\mu(B) = 0$ for each Borel subset B of $S(\triangle^q, X)_\kappa$ satisfying $B \cap \mathcal{K} = \emptyset$. This definition is in agreement with that given by Hansen [**11**, p. 205] (also see [**20**, Theorem 6.4, p. 126]). We refer the reader to Zastrow [**27**, p. 373, 1.5(ii) and 1.6; p. 381, Lemma 3.2] for a relevant discussion on the definition of *the support of a measure* as a uniquely determined set.

Define the *weak* topology on C_qX as the weakest topology such that, for each $\lambda : S(\triangle^q, X)_\kappa \to \mathbb{R}$ continuous, the functional $C_qX \to \mathbb{R}$, given by $\nu \mapsto \int \lambda\, d\nu$, is continuous. For $q > 0$ and $0 \le i \le q$, define the ith *face* of $\triangle^q$ as the affine map from $\triangle^{q-1}$ to $\triangle^q$

$$F_q^i(e_j) = \begin{cases} e_j, & j < i \\ e_{j+1}, & j \ge i \end{cases}$$

[F_q^i may also be denoted by $(e_0, \ldots, \widehat{e_i}, \ldots, e_q)$].

There is a continuous ith *face map* on the singular q-simplices, say $(F_q^i)^* : S(\triangle^q, X)_\kappa \to S(\triangle^{q-1}, X)_\kappa$, defined by $\sigma \mapsto \sigma \circ F_q^i$ (see [**3**, p. 259, 2.1(1)]), which induces a linear transformation $\partial_q^i : C_qX \to C_{q-1}X$ such that $\partial_q^i(\nu) = [(F_q^i)^*]_*\nu$ for each $\nu \in C_qX$, where $\partial_q^i(\nu)(B) = \nu([(F_q^i)^*]^{-1}(B))$ for each Borel subset B contained in $S(\triangle^{q-1}, X)$.

ASSERTION 2.1. *Let X be a Hausdorff space, and let q, s be natural numbers. Let $G : S(\triangle^q, X)_\kappa \to S(\triangle^s, X)_\kappa$ be a continuous function. Then, $G_* : C_qX \to C_sX$ is continuous. In particular, the partial boundary operators ∂_q^i, $q \in \mathbb{N}$, are continuous.*

Proof. If $\mathcal{K} \subset S(\triangle^q, X)$ is a compact set supporting ν, then $G(\mathcal{K})$ is compact and supports $G_*\nu$, so G_* is a well-defined linear transformation.

Let $\lambda : S(\triangle^s, X)_\kappa \to \mathbb{R}$ be any continuous function and define Λ such that $\Lambda(\mu) = \int \lambda\, d\mu$, for each $\mu \in C_sX$. By the definition of the weak topology on C_sX, it is enough to prove that $\Lambda \circ (G_*)$ is continuous. But $(\Lambda \circ (G_*))(\nu) = \Lambda(G_*\nu) = \int_{S(\triangle^s, X)} \lambda\, d(G_*\nu) = \int_{S(\triangle^q, X)} (\lambda \circ G)\, d\nu$. Since $\lambda \circ G$ is continuous, by the definition of the weak topology on C_qX, the assertion follows. □

Since $\partial_{q-1}^j \circ \partial_q^i = \partial_{q-1}^{i-1} \circ \partial_q^j$ for $j < i$, the *boundary operator*

$$\partial_q = \sum_{i=0}^{q} (-1)^i\, \partial_q^i$$

is such that $\partial_{q-1} \circ \partial_q = 0$.

REMARK 2.1. It is easy to see that, for each $q \in \mathbb{N}$, addition in C_qX is continuous; therefore, the boundary operator is continuous.

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Now, let $f : X \to Y$ be a continuous map between Hausdorff spaces. Then, for $q \in \mathbb{N}$, the function $f_q : S(\triangle^q, X)_\kappa \to S(\triangle^q, Y)_\kappa$, with $\sigma \mapsto f \circ \sigma$ is continuous (see [**3**, p. 259, 2.1(2)]). Hence, the rule $(f_q)_* : C_q X \to C_q Y$, given by $\nu \mapsto (f_q)_*(\nu)$, defines a continuous linear transformation. Furthermore, the following relations are satisfied:

(1) $(f_{(q-1)})_* \circ \partial_q = \partial_q \circ (f_q)_*$.

(2) $(g_q)_* \circ (f_q)_* = ((g \circ f)_q)_*$.

Summarising, we have defined a covariant functor C from the category of topological Hausdorff spaces to the category of (non-negative) chain complexes over $\mathbb{R}$, which assigns to each Hausdorff space X the complex $(C X, \partial) = \{(C_q X, \partial_q) \mid q \in \mathbb{N}\}$.

If A is a subspace of X, there is a *relative chain complex* $(C(X, A), \overline{\partial}) = \{(C_q(X, A), \overline{\partial}_q) \mid q \in \mathbb{N}\} = \{(C_q X / C_q A, \overline{\partial}_q) \mid q \in \mathbb{N}\}$, where for each $q \in \mathbb{N}$, $\overline{\partial}_q$ is the unique map which makes the following diagram commutative.

$$\begin{array}{ccccc} C_q A & \hookrightarrow & C_q X & \twoheadrightarrow & C_q X / C_q A \\ \downarrow{\scriptstyle \partial_q} & & \downarrow{\scriptstyle \partial_q} & & \downarrow{\scriptstyle \overline{\partial}_q} \\ C_{q-1} A & \hookrightarrow & C_{q-1} X & \twoheadrightarrow & C_{q-1} X / C_{q-1} A. \end{array}$$

If X is a Hausdorff space, there is a natural identification of the space of zero-singular simplices $S(\triangle^0, X)_\kappa$ with X by associating to each simplex $\sigma : \triangle^0 \to X$ the point $\sigma(0)$ in X. In this manner, $C_0 X$ becomes the linear space of all signed Borel measures on X with compact support and $\epsilon : C_0 X \to \mathbb{R}$, $\nu \mapsto \nu(X)$ is an augmentation; that is, an epimorphism such that the composite $C_1 X \overset{\partial_1}{\to} C_0 X \overset{\epsilon}{\to} \mathbb{R}$ is trivial. Indeed, if $\nu \in C_1 X$, then $\epsilon(\partial_1 \nu) = \epsilon(\partial_1^0(\nu)) - \epsilon(\partial_1^1(\nu)) = \nu(S(\triangle^1, X)) - \nu(S(\triangle^1, X)) = 0$.

Let X be a topological Hausdorff space, and let $(\triangle(X, \mathbb{R}), \partial) = \{(\triangle_q(X, \mathbb{R}), \partial_q) \mid q \in \mathbb{N}\}$ be the standard singular real chain complex associated with X (in particular, $\triangle_q(X, \mathbb{R})$ is the vector space of formal finite linear combinations of singular q-simplices with real coefficients).

The linear space $\triangle_q(X, \mathbb{R})$ can naturally be embedded in $C_q X$ by sending each simplex $\sigma : \triangle^q \to X$ to the atomic probability supported in $\{\sigma\}$, and then, extending linearly to all of $\triangle_q(X, \mathbb{R})$.

The inclusions $i_X : \triangle(X, \mathbb{R}) \hookrightarrow C X$, for all Hausdorff topological spaces X, certainly define a natural chain map preserving augmentation.

Denote by $H_q^s(X, \mathbb{R})$ the qth singular homology group with real coefficients of the space X, and by $H_q(X, \mathbb{R})$ the qth measure homology $\mathbb{R}$-vector space of X defined in this section (cf. [**22**, p. 158; p. 214, Example 3]).

Now, assume that X is a second countable, locally compact, Hausdorff space. Then, X is metrizable (see [**24**, pp. 130, 166]). In this setting, $S(\triangle^q, X)_\kappa$ is second countable as well, for $\triangle^q$ and X are second countable (see [**3**, p. 265, 5.2]). Furthermore, $S(\triangle^q, X)_\kappa$ is metrizable because $\triangle^q$ is compact and X metrizable (see [**3**, p. 270, 8.2(3)]).

ASSERTION 2.2. *Let X be a second countable, locally compact, Hausdorff space. Then, $C_q X$ is a locally convex Hausdorff topological vector space in the weak topology.*

Proof. Suppose $\mathbb{E}$ is a vector space and Y a separating vector space of linear functionals on $\mathbb{E}$. Then, the weak topology on $\mathbb{E}$ induced by Y makes $\mathbb{E}$ a locally convex Hausdorff space [**19**, Theorem 3.10, p. 62]. Therefore, to prove our assertion, it is enough to verify that the family $Y = \{\lambda : S(\triangle^q, X)_\kappa \to \mathbb{R} \mid \lambda \text{ is continuous }\}$ separates measures in $C_q X$.

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Let μ_0 and μ_1 be distinct measures in C_qX. Let $\mathcal{K}$ be a compact set in $S(\triangle^q, X)_\kappa$ supporting μ_0 and μ_1. Define $\nu = \mu_1 - \mu_0$ and let $\nu|_{\mathcal{K}}$ be the restriction of ν to $\mathcal{K}$. Since $\mathcal{K}$ is a locally compact, metrizable Hausdorff space, the uniqueness assertion of the Riesz Representation Theorem [**18**, Theorem 8, p. 310; p. 302, second paragraph] implies that there is a continuous function $f : \mathcal{K} \to \mathbb{R}$ such that $\int_{\mathcal{K}} f\, d(\nu|_{\mathcal{K}}) \neq 0$. Since every metric space is normal, it is then possible to apply Tietze's Extension Theorem [**24**, p. 103] to get a continuous extension $\lambda : S(\triangle^q, X)_\kappa \to \mathbb{R}$ of $f : \mathcal{K} \to \mathbb{R}$. Now, μ_0 and μ_1 are separated by λ, for $\int_{S(\triangle^q,X)} \lambda\, d\mu_0 = \int_{\mathcal{K}} f\, d(\mu_0|_{\mathcal{K}}) \neq \int_{\mathcal{K}} f\, d(\mu_1|_{\mathcal{K}}) = \int_{S(\triangle^q,X)} \lambda\, d\mu_1$. □

We will not try to generalise Assertion 2.2 and stress; instead, the fact that although we have defined a covariant functor C in the topological category of Hausdorff spaces, it is its restriction to the category of second countable, locally compact, Hausdorff (hence metrizable) spaces that takes values into the category of chain complexes of locally convex Hausdorff topological vector spaces over $\mathbb{R}$ and continuous linear maps. The composite of C, when restricted, and the homology functor H (see [**22**, p. 158]) is a covariant functor to the category of graded topological (locally convex) vector spaces over $\mathbb{R}$ and continuous linear transformations. The following remarks clarify the situation further.

REMARKS 2.2. By a topological vector space we mean a (real) vector space V with a topology τ such that the vector space operations are continuous with respect to τ. Since translations are homeomorphisms of V, then points in V are closed if and only if $\bar{0} \in V$ is closed. Furthermore, the linear space V is Hausdorff if and only if $\bar{0} \in V$ is closed (see [**19**, pp. 7–10]). If $H \subset V$ is a linear subspace and V/H denotes the quotient space, then the natural projection $\rho : V \to V/H$ is open when V/H is given the quotient topology; indeed, let A be any subset in V, then $\rho^{-1}(\rho(A)) = \bigcup_{h\in H} (h + A) = H + A$. Now, if A is open in V, it follows that $\rho^{-1}(\rho(A))$ is open in V for it is the union of open sets. Therefore, by the definition of the quotient topology, $\rho(A)$ is open in V/H. Consequences of the fact that ρ is open are as follows (cf. [**19**, Theorem 1.41, p. 29]):

(1) V/H is a topological vector space.
(2) If $\mathcal{B}$ is a local base for τ, then the collection of all sets $\rho(A)$ with $A \in \mathcal{B}$ is a local base for the quotient topology.
(3) If V is locally convex, so is V/H.

Finally, observe that subspaces of locally convex spaces are locally convex and that a quotient space V/H is Hausdorff if and only if H is closed in V.

According to Hansen [**11**], for a wide class of topological spaces that contains all metrizable spaces, the homology functor $H \circ C$ just constructed defines a homology theory with compact supports in the sense of Eilenberg and Steenrod, and that for any CW-complex X, the inclusion $i_X : \triangle(X, \mathbb{R}) \hookrightarrow CX$ induces an isomorphism $H(i_X) : H(\triangle(X, \mathbb{R}), \partial) \cong H(CX, \partial)$.

Following Milnor, let $\mathfrak{W}_\circ$ be the category of all spaces that have the homotopy type of countable CW-complexes as objects and arbitrary continuous maps as morphisms. A space X belongs to $\mathfrak{W}_\circ$ if and only if it has the homotopy type of a countable, locally finite simplicial complex [**15**, Theorem 1(c)]. Therefore, since a simplicial complex is a CW-complex on its underlying polyhedron, a space X belongs to $\mathfrak{W}_\circ$ if and only if it has the homotopy type of a countable simplicial complex or a countable, locally finite CW-complex. In particular, every $X \in \mathfrak{W}_\circ$ has a countable number of connected components and is homotopically equivalent to a separable, locally compact metric space (see [**26**, p. 223, paragraph 5]). Note that separability and second countability are equivalent properties for metric spaces [**24**, Theorem 16.11, p. 112]. Not every second countable, locally compact space is in $\mathfrak{W}_\circ$; being totally disconnected and uncountable, the Cantor set is an example.

With these provisions, now it is possible to define $H \circ C$ as a functor from $\mathfrak{W}_\circ$ to the category of graded topological, locally convex, vector spaces over $\mathbb{R}$. Indeed, for X in $\mathfrak{W}_\circ$, $(H \circ C)(X)$ is well defined as a graded vector space (and naturally isomorphic to the singular homology of X). Now let $\xi : X \simeq Y$ be a homotopy equivalence between X and a second countable, locally compact Hausdorff space Y. By Assertion 2.2 and Remarks 2.2, $(H \circ C)(Y)$ is a graded topological, locally convex, vector space.

The isomorphism that this equivalence induces on homology defines a (locally convex) topological structure on $(H \circ C)(X)$ independent of Y and ξ.

3. Examples. The first of our examples is just the unit circle. The computations performed on the 'infinite telescope' described afterwards will allow us to show that the first homology space is separated.

(1) *An Eilenberg–Mac Lane space of type* $(\mathbb{Z},1)$.

Let T^1 denote the topological group $\mathbb{R}/\mathbb{Z}$, which is isomorphic to the unit circle $S^1 = \{z \in \mathbb{C} | \, |z| = 1\}$. Denote by Π the canonical projection from $\mathbb{R}$ onto the quotient group T^1. We want to construct an explicit isomorphism of the first homology vector space $H_1(T^1, \mathbb{R})$ onto $\mathbb{R}$. For this purpose, let $\lambda : S(\triangle^1, T^1)_\kappa \to \mathbb{R}$ be the continuous map defined by $\lambda(\sigma) = \overline{\sigma}(1) - \overline{\sigma}(0)$ for all $\sigma : \triangle^1 \to T^1$ continuous, where $\overline{\sigma} : \triangle^1 \to \mathbb{R}$ is any lifting of σ. Alternatively, $\lambda(\sigma) = [\overline{\sigma - \sigma(0)}](1)$, where $\overline{\sigma - \sigma(0)} : \triangle^1 \to \mathbb{R}$ is the lifting of $\sigma - \sigma(0)$ such that $[\overline{\sigma - \sigma(0)}](0) = 0$.

Now, we define two linear functionals $D : C_1 T^1 \to \mathbb{R}$ such that $D(\nu) = \int \lambda d\nu$ for all $\nu \in C_1 T^1$ and $d : \triangle_1(T^1, \mathbb{R}) \to \mathbb{R}$ determined by the formula $d(\sigma) = \lambda(\sigma)$ for all $\sigma \in S(\triangle^1, T^1)$. Recall that $\triangle_1(T^1, \mathbb{R})$ is the free $\mathbb{R}$-linear space over $S(\triangle^1, T^1)$, and note that if σ is a closed curve in T^1, then $d(\sigma)$ is just the degree of σ.

ASSERTION 3.1. *Both d and D are zero on boundaries.*

Proof. Let $\sigma \in S(\triangle^2, T^1)$ and let $\overline{\sigma} : \triangle^2 \to \mathbb{R}$ be any lifting of σ. Then $\partial_2\sigma = \Pi(\overline{\sigma} \circ F_2^0 - \overline{\sigma} \circ F_2^1 + \overline{\sigma} \circ F_2^2)$ and $d\partial_2\sigma = \overline{\sigma} \circ F_2^0(1) - \overline{\sigma} \circ F_2^0(0) - \overline{\sigma} \circ F_2^1(1) + \overline{\sigma} \circ F_2^1(0) + \overline{\sigma} \circ F_2^2(1) - \overline{\sigma} \circ F_2^2(0) = \overline{\sigma}(e_2) - \overline{\sigma}(e_1) - \overline{\sigma}(e_2) + \overline{\sigma}(e_0) + \overline{\sigma}(e_1) - \overline{\sigma}(e_0) = 0 = (\lambda \circ (F_2^0)^* - \lambda \circ (F_2^1)^* + \lambda \circ (F_2^2)^*)\sigma$.

Now, let $\nu \in C_2 T^1$, then $D\partial_2\nu = \int_{S(\triangle^1, X)} \lambda d\partial_2\nu = \int_{S(\triangle^2, X)} (\lambda \circ (F_2^0)^* - \lambda \circ (F_2^1)^* + \lambda \circ (F_2^2)^*) d\nu$. But $\lambda \circ (F_2^0)^* - \lambda \circ (F_2^1)^* + \lambda \circ (F_2^2)^* \equiv 0$, as the previous computation has shown. □

ASSERTION 3.2. *Let* $i : \triangle_1(T^1, \mathbb{R}) \to C_1 T^1$ *be the canonical injection. Then,* $D \circ i = d$.

Proof. Let $\sigma : \triangle^1 \to T^1$ be a generator in $\triangle_1(T^1, \mathbb{R})$. Then, $D \circ i(\sigma) = \int \lambda di(\sigma) = \lambda(\sigma) = d(\sigma)$. □

Because of the above assertions, the maps d, D and i induce the following commutative diagram in homology

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$$\begin{array}{ccc} H_1^s(T^1,\mathbb{R}) & \xrightarrow{i_*} & H_1(T^1,\mathbb{R}) \\ & \overline{d}\searrow \quad \mathbb{R} \quad \swarrow \overline{D}. & \end{array}$$

But i_* is an isomorphism and $\overline{d}$ is an isomorphism too, for it has been already noticed that d is the degree function for closed curves. Hence, $\overline{D}$ is an isomorphism as well.

(2) *An Eilenberg–Mac Lane space of type (*$\mathbb{Q}$*, 1).*

Let j be a positive integer and let T_j^1 denote the topological group $\mathbb{R}/(1/j!)\,\mathbb{Z}$, where $j! = 1\times 2\times 3\times \cdots \times j$ and $(1/j!)\,\mathbb{Z}$ denotes the discrete group of all integral multiples of $1/j!$.

If $0 < j \leq \ell$, then there is a natural group homomorphism $T_j^1 \to T_\ell^1$ such that $x + 1/j!\,\mathbb{Z} \mapsto x + 1/\ell!\,\mathbb{Z}$. Observe that, in particular, the kernel of $T_j^1 \to T_{j+1}^1$ is cyclic of order $j+1$.

Let ℓ be a positive integer and define $K_\ell(\mathbb{Q}, 1)$ as the quotient space obtained from the disjoint union $I\times T_1^1 \sqcup I\times T_2^1 \sqcup \cdots \sqcup I\times T_\ell^1 \sqcup \{0\}\times T_{\ell+1}^1$ by identifying the point $(1,x)_j = (1, x+1/j!\,\mathbb{Z}) \in I\times T_j^1$ with $(0,x)_{j+1} = (0, x+1/(j+1)!\,\mathbb{Z}) \in I\times T_{j+1}^1$ (i.e., $(1,x)_j \sim (0,x)_{j+1}$). If $(t,x)_j \in I\times T_j^1$, then $[t,x]_j$ will denote the corresponding point in $K_\ell(\mathbb{Q}, 1)$.

Observe that $K_\ell(\mathbb{Q}, 1)$ has a 'horizontal' axis (each cylinder $I\times T_j^1$ has one), so we can deform it in itself by 'pushing to the right' along this axis. Therefore, $\{0\}\times T_{\ell+1}^1$ is a (strong) deformation retract of $K_\ell(\mathbb{Q}, 1)$.

An explicit retraction by deformation for the Möbius strip $K_1(\mathbb{Q}, 1)$ may be given by the formulas $([t,x]_1, s) \mapsto [t(1-s)+s, x]_1$ and $([0,x]_2, s) \mapsto [0,x]_2$. In particular, the function $\zeta_\ell : K_\ell(\mathbb{Q}, 1) \to T_{\ell+1}^1$ such that $[t,x]_j \mapsto x + 1/(\ell+1)!\,\mathbb{Z}$ is a homotopy equivalence, showing that the fundamental group of $K_\ell(\mathbb{Q}, 1)$ is infinite cyclic, and that the circle $\{0\}\times T_{\ell+1}^1$ (when traversed in the positive direction) defines a generator of the group.

Note that $K_\ell(\mathbb{Q}, 1)$ is naturally embedded in $K_{\ell+1}(\mathbb{Q}, 1)$. Finally, define the space

$$K(\mathbb{Q}, 1) = \left\{ \bigsqcup_{j=1}^{\infty} I\times T_j^1 \right\} \Big/ \; ((1,x)_j \sim (0,x)_{j+1}).$$

Let us mention that the topology of $K(\mathbb{Q}, 1)$ as a quotient space coincides with the topology coinduced by the family $\{K_\ell(\mathbb{Q}, 1) \mid \ell = 1, 2, \ldots\}$. The commutative diagram

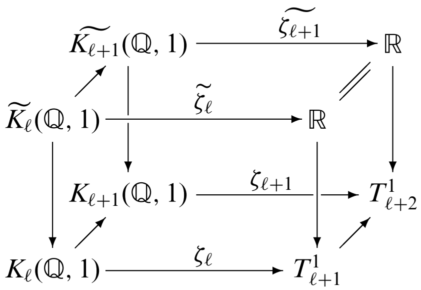
$$\begin{array}{ccccccc} T_2^1 & \to & T_3^1 & \to \cdots \to & T_{\ell+1}^1 & \to \cdots \\ \uparrow \zeta_1 & & \uparrow \zeta_2 & & \uparrow \zeta_\ell & \\ K_1(\mathbb{Q}, 1) & \hookrightarrow & K_2(\mathbb{Q}, 1) & \hookrightarrow \cdots \hookrightarrow & K_\ell(\mathbb{Q}, 1) & \hookrightarrow \cdots \end{array}$$

induces the following commutative diagram:

$$\begin{array}{ccccccc} 1/2!\,\mathbb{Z} & \hookrightarrow & 1/3!\,\mathbb{Z} & \hookrightarrow \cdots \hookrightarrow & 1/(\ell+1)!\,\mathbb{Z} & \hookrightarrow \cdots \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & \\ \pi(K_1(\mathbb{Q}, 1)) & \hookrightarrow & \pi(K_2(\mathbb{Q}, 1)) & \hookrightarrow \cdots \hookrightarrow & \pi(K_\ell(\mathbb{Q}, 1)) & \hookrightarrow \cdots \end{array}$$

It easily follows that there is a group isomorphism from $\pi(K(\mathbb{Q}, 1))$ onto the additive group $\mathbb{Q}$ of rational numbers. Now, $\pi(K(\mathbb{Q}, 1))$ being a rank one, torsion-free divisible group, its field structure depends only on a choice of multiplicative unit, so let $(\{0\} \times T_1^1) \subset K_1(\mathbb{Q}, 1)$ (positively traversed) be our preferred unit. The fact that the higher homotopy groups of a circle vanish, imply that the space we have defined has trivial higher homotopy groups (for any continuous $\alpha : S^n \to K(\mathbb{Q}, 1)$ has its image in some $K_\ell(\mathbb{Q}, 1)$).

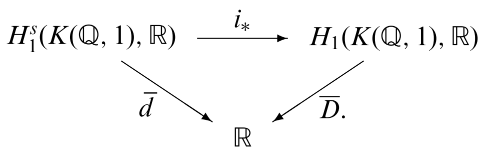
Now we want to exhibit an isomorphism of the homology vector space $H_1(K(\mathbb{Q}, 1), \mathbb{R})$ onto $\mathbb{R}$. For this purpose, we first observe that the inclusion $K_\ell(\mathbb{Q}, 1) \hookrightarrow K_{\ell+1}(\mathbb{Q}, 1)$ can be lifted to a map from the universal covering $\widetilde{K_\ell}(\mathbb{Q}, 1)$ of $K_\ell(\mathbb{Q}, 1)$ into the universal covering $\widetilde{K_{\ell+1}}(\mathbb{Q}, 1)$ of $K_{\ell+1}(\mathbb{Q}, 1)$, and that the continuous function $\zeta_\ell : K_\ell(\mathbb{Q}, 1) \to T_{\ell+1}^1$ can be lifted to a map $\widetilde{\zeta_\ell} : \widetilde{K_\ell}(\mathbb{Q}, 1) \to \mathbb{R}$ in such a way that the following diagram commutes:



Therefore, there is a well-defined continuous map $\lambda : S(\triangle^1, K(\mathbb{Q}, 1))_\kappa \to \mathbb{R}$ defined by the formula $\lambda(\sigma) = \widetilde{\zeta_\ell} \circ \overline{\sigma}(1) - \widetilde{\zeta_\ell} \circ \overline{\sigma}(0)$, where ℓ is any integer such that $\sigma(\triangle^1)$ is contained in $K_\ell(\mathbb{Q}, 1)$ and $\overline{\sigma} : \triangle^1 \to \widetilde{K_\ell}(\mathbb{Q}, 1)$ is any lifting of σ.

Define two linear functionals, $D : C_1\, K(\mathbb{Q}, 1) \to \mathbb{R}$ given by the integral expression $D(\nu) = \int \lambda\, d\nu$ for all $\nu \in C_1\, K(\mathbb{Q}, 1)$ and $d : \triangle_1(K(\mathbb{Q}, 1), \mathbb{R}) \to \mathbb{R}$ determined by the formula $d(\sigma) = \lambda(\sigma)$ for all $\sigma \in S(\triangle^1, K(\mathbb{Q}, 1))$.

Using the previous example it is not difficult to see that both D and d are zero on boundaries and that the following induced diagram in homology commutes,



In the same way as in Example (1), it follows that $\overline{D}$ is an isomorphism.

4. The first homology space is separated. Let X be a second countable, locally compact, Hausdorff space and let q be a non-negative integer. By Asertion 2.2, the space $C_q\, X$ of q-chains has been given a locally convex Hausdorff topology.

Let $Z_q\, X$ and $B_q\, X$ denote the topological vector sub-space of q-cycles and the sub-space of q-boundaries, respectively.

Certainly, $H_q(X, \mathbb{R}) = Z_q\, X \,/\, B_q\, X$ is endowed with the quotient topology. As in any quotient of topological vector spaces, $H_q(X, \mathbb{R})$ is Hausdorff if and only if $B_q\, X$ is closed in $Z_q\, X$. Observe that $Z_q\, X$ is the kernel of a boundary operator and therefore is closed in $C_q\, X$ (see Remark 2.1).

Question: *Is $H_q(X, \mathbb{R})$ a Hausdorff space in general?*

For example, if we let X be the space $K(\mathbb{Q}, 1)$ of the above section, then the continuous commutative diagram

$$\begin{array}{ccc} Z_1 X & & \\ \downarrow & \searrow^{D} & \\ H_1(X, \mathbb{R}) & \xrightarrow[\overline{D}]{\cong} & \mathbb{R} \end{array}$$

shows that the space $B_1 X$ is equal to the kernel of D; hence, in this case, $H_1(X, \mathbb{R})$ is Hausdorff.

Furthermore, using the fact that any linear bijection from a finite-dimensional Hausdorff topological vector space over $\mathbb{R}$ onto some euclidean space is also a topological isomorphism ([**3**, Theorem 2.1, p. 413]), we can conclude that $\overline{D}$ is a homeomorphism as well.

When $q = 1$, this particular example implies that the answer to the above question is affirmative for topological spaces homotopically equivalent to countable CW-complexes. This is the content of the next theorem.

THEOREM 4.1. *Let X be a space of the same homotopy type as a countable CW-complex. Then, $H_1(X, \mathbb{R})$ is a Hausdorff space.*

Proof. (cf. [**9**, pp. 68–69]) Assume, without loss of generality, that X is a countable, locally compact, CW-complex. It is enough to prove that for each $z \in H_1(X, \mathbb{R})$ with z different than zero, there exists a continuous linear functional $\Lambda : H_1(X, \mathbb{R}) \to \mathbb{R}$ such that $\Lambda(z) \neq 0$.

Let $H_1^s(X, \mathbb{Q})$ be the first singular homology group of X with rational coefficients and denote by $H_s^1(X, \mathbb{Q})$ the first singular cohomology group of X over $\mathbb{Q}$. Observe that the zeroth singular homology group $H_0^s(X, \mathbb{Q})$ is a free $\mathbb{Q}$-module on as many generators as there are path components of X; hence, the torsion product $\mathrm{Tor}_1(H_0^s(X, \mathbb{Q}), B)$ (also denoted by $H_0^s(X, \mathbb{Q}) \star B$) and the module of extensions $\mathrm{Ext}(H_0^s(X, \mathbb{Q}), B)$ are equal to zero for any $\mathbb{Q}$-module B (see [**22**, p. 220, Example 3; p. 241, last paragraph]).

Let $[X, K(\mathbb{Q}, 1)]$ denote the set of homotopy classes of continuous functions from X into $K(\mathbb{Q}, 1)$. $\mathbb{Q}$ being Abelian, $K(\mathbb{Q}, 1)$ has an H-structure making $[X, K(\mathbb{Q}, 1)]$ an Abelian group. Moreover, $[X, K(\mathbb{Q}, 1)]$ is isomorphic to $H_s^1(X, \mathbb{Q})$ (see [**25**, Theorem 7.13, p. 249; Theorem 7.14, p. 250]). By the Universal-Coefficient Theorem for Cohomology [**22**, Theorem 3, p. 243], $H_s^1(X, \mathbb{Q})$ is isomorphic to $\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q})$, where $\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q})$ denotes the dual space of $H_1^s(X, \mathbb{Q})$ consisting of the $\mathbb{Q}$-linear homomorphisms of $H_1^s(X, \mathbb{Q})$ into $\mathbb{Q}$. The composite of these two isomorphisms gives a natural bijection

(1) $[X, K(\mathbb{Q}, 1)] \cong \mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q})$

which assigns to each class $[f]$ of a continuous map $f : X \to K(\mathbb{Q}, 1)$ the homomorphism $H_1^s(f, \mathbb{Q}) : H_1^s(X, \mathbb{Q}) \to \mathbb{Q} = H_1^s(K(\mathbb{Q}, 1), \mathbb{Q})$. Observe that for $\mathbb{Q}$-linear spaces, linear transformations coincide with the homomorphisms of Abelian groups. The set of reals $\mathbb{R}$ has the structure of a $\mathbb{Q}$-vector space, so, if A is a $\mathbb{Q}$-linear space, then $\mathrm{Hom}(A, \mathbb{R})$ is well defined; but now, real multiplication induces on $\mathrm{Hom}(A, \mathbb{R})$ the structure of a vector space over $\mathbb{R}$. In this way, the isomorphism

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(2) $\mathrm{Hom}(\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q}), \mathbb{R}) \cong \mathrm{Hom}([X, K(\mathbb{Q}, 1)], \mathbb{R})$

induced by the $\mathbb{Q}$-isomorphism in (1) is also an isomorphism of real vector spaces.

The singular homology group $H_1^s(X, \mathbb{Q})$ is naturally embedded in its double dual by the function $i : H_1^s(X, \mathbb{Q}) \hookrightarrow \mathrm{Hom}(\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q}), \mathbb{Q})$, given by $z \mapsto (\zeta \mapsto \zeta(z))$.

If $\phi : A \to B$ is a $\mathbb{Q}$-linear transformation between $\mathbb{Q}$-vector spaces, then $A \otimes \mathbb{R}$ and $B \otimes \mathbb{R}$ become vector spaces over $\mathbb{R}$ and $\phi \otimes Id : A \otimes \mathbb{R} \to B \otimes \mathbb{R}$ becomes an $\mathbb{R}$-linear transformation (note that, in this proof, $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{Q}}$ and $\otimes = \otimes_{\mathbb{Q}}$ always). If ϕ is injective, then $\phi \otimes Id$ is also injective (see [**12**, p. 111, Proposition 7.4]). In particular, the above homomorphism i gives an $\mathbb{R}$-embedding $i \otimes Id$: $H_1^s(X, \mathbb{Q}) \otimes \mathbb{R} \hookrightarrow \mathrm{Hom}(\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q}), \mathbb{Q}) \otimes \mathbb{R}$. By composing $i \otimes Id$ with the natural inclusion $\mathrm{Hom}(\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q}), \mathbb{Q}) \otimes \mathbb{R} \hookrightarrow \mathrm{Hom}(\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q}), \mathbb{R})$ and using (2), we can construct a commutative diagram

$$\begin{array}{ccccc} H_1(X, \mathbb{R}) \cong & H_1^s(X, \mathbb{R}) & \hookrightarrow & \mathrm{Hom}([X, K(\mathbb{Q}, 1)], \mathbb{R}) \\ & \updownarrow & & \updownarrow \\ & H_1^s(X, \mathbb{Q}) \otimes \mathbb{R} & \hookrightarrow & \mathrm{Hom}(\mathrm{Hom}(H_1^s(X, \mathbb{Q}), \mathbb{Q}), \mathbb{R}). \end{array}$$

The Universal-Coefficient Theorem for Homology [**22**, p. 222, Theorem 8] shows that the left vertical arrow in the diagram is an isomorphism.

The diagram now implies that, for every $z \in H_1(X, \mathbb{R})$ with $z \neq 0$, there is a continuous $f : X \to K(\mathbb{Q}, 1)$ such that $H_1(f, \mathbb{R})z \neq 0$. This concludes the proof, for $H_1(f, \mathbb{R})$ is a continuous functional when $\mathbb{R}$ is endowed with its standard topology. □

REMARKS 4.1. The obstruction to prove that all the homology spaces are Hausdorff certainly lies on adequately generalising the correspondence in (1). Surjectivity would be sufficient. It can also be seen that $H_1^s(X, \mathbb{R}) \to \mathrm{Hom}([X, T^1], \mathbb{R})$ is bijective when the first homology group is finitely generated, but not otherwise. A counter example is obtained by taking $X = K(\mathbb{Q}, 1)$ where $[X, T^1]$ is trivial.

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