FINITE DIMENSIONAL APPROXIMATIONS TO SOME FLOWS ON THE PROJECTIVE LIMIT SPACE OF SPHERES II

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Dedicated to Professor K. NOSHIRO on his sixtieth birthday

§1. **Introduction.** In the previous paper [6], we have considered flows on the measure space $(\Omega, \mathscr{B}, P)$, which was the projective limit of a certain subspace $(\Omega_n, \mathscr{B}_n, P_n)$ of the measure space $(S_n, \mathscr{B}(S_n), P_n)$, where S_n is the $(n-1)$-sphere with radius $\sqrt{n}$ and P_n is the uniform probability distribution over S_n. In particular, we have discussed how to construct a *canonical flow* $\{T_t\}$ by a consistent system of flows $\{T_t^{(n)}\}_{n=2,4,\ldots}$, where each $\{T_t^{(n)}\}$ is derived from a one-parameter subgroup of rotations of the sphere S_n. In [6, Theorem 2.2 and Proposition 2.2], we have proved that every canonical flow $\{T_t\}$ is characterized by the sequence $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ which is obtained by eigenvalues of finite dimensional rotations $T_t^{(n)}$ by which the flow $\{T_t\}$ is formed. Also, in [6], we have pointed out that the sequence Λ, called *the spectral set* of $\{T_t\}$, is a part of the spectrum of the flow $\{T_t\}$. The purpose of this paper is to determine not only the spectral type but also the ergodic property of canonical flows.

In Section 2, it will turn out that any canonical flow has a discrete spectrum which forms a subgroup of the additive group of real numbers which is generated by its spectral set. In Sections 3 and 4, we shall consider the decomposition of a canonical flow into its ergodic parts. Although the ergodic parts of the basic spaces, in general, are determined depending on the flows under consideration, we can, in our case, find the universal decomposition ζ of the basic space Ω such that

(i) *ζ is the invariant partion for any canonical flow $\{T_t\}$,*

(ii) *if $\{T_t\}$ has linearly independent spectral set, then ζ gives the decomposition of $\{T_t\}$ into its ergodic components every one of which is isomorphic to the flow on*

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the infinite dimensional torus.

In [8], *G.—C.* Rota has presented several questions in ergodic theory. For the canonical flows, above partion ζ enables us to give a partial answer to his problem (2) which asks the structure of the invariant σ-subalgebras of a given flow [cf. Theorem 3].

§ **2. Spectrum of canonical flows.** Before discussing our problems on flows, we shall summarize some results obtained in [6] as preliminaries. In order to avoid complicated discriptions, we will consider only the flow $\{T_t\}$ which is formed by the consistent system of flows $\{T_t^{(2n)}\}_{n=1,2,\dots}$. The other type of canonical flows can be treated in a simlar way.

Let $(S_n, \mathscr{B}(S_n), P_n)$*) be the probability space defined in Section 1. Let $x^{(n)}=(x_1^{(n)},\dots,x_n^{(n)})$ be a point of S_n and let Ω_n denote an open subset of S_n defined by

$$\Omega_n=\{x^{(n)}\ ;\ x^{(n)}\in S_n,\ x_1^{(n)2}+x_2^{(n)2}>0\}.$$

Then, we have a subspace $(\Omega_n, \mathscr{B}_n, P_n)$ of the measure space S_n where $\mathscr{B}_n=\mathscr{B}(S_n)\cap\Omega_n$. For any $m<n$, we shall define a point transformation $f_{m,n}$ from Ω_n onto Ω_m as follows;

$$x_k^{(m)}=\frac{\sqrt{m}}{[x_1^{(n)2}+\cdots+x_m^{(n)2}]^{1/2}}\,x_k^{(n)},\quad 1\le k\le m. \tag{2.1}$$

Then, the system $[(\Omega_n, \mathscr{B}_n, P_n)\,;f_{m,n}]$ and its subsystem $[(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})\,;f_{2m,2n}]$ determine the projective limit spaces $(\Omega, \mathscr{B}, P)$ and $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{P})$ respectively. Let $(S_\infty, \mathscr{B}_\infty)$ be the direct product measurable spaces of $(S_n, \mathscr{B}(S_n))_{n=2,4,\dots}$;

$$S_\infty=\prod_{n\geq 1} S_{2n}\ \text{(weak product)},\quad \mathscr{B}_\infty=\mathscr{B}(S_\infty).$$

Then, by putting $P_\infty(A)=\tilde{P}(A\cap\tilde{\Omega})$ for $A\in\mathscr{B}_\infty$, we have an extension $(S_\infty, \mathscr{B}_\infty, P_\infty)$ of $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{P})$.

Let $\{T_t^{(2n)}\}$ be a canonical flow on S_{2n} such that

$$T_t^{(2n)}=\begin{bmatrix} A_1(t) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & A_n(t)\end{bmatrix},\quad A_k(t)=\begin{bmatrix}\cos\lambda_k t & -\sin\lambda_k t\\ \sin\lambda_k t & \cos\lambda_k t\end{bmatrix},\ 1\le k\le n, \tag{2.2}$$

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*) We denote by $\mathscr{B}(X)$ the topological Borel field of the topological space X.

where $\lambda_k \neq 0$ are all reals. Since $\{T_t^{(2n)}\ ;\ n\geq 1\}$ is consistent, we can form a measurable flow $\{T_t\}$ on S_∞ with the spectral set $\Lambda=\{\lambda_k\ ;\ k\geq 1\}$ as follows

(2.3) $T_t x=(T_t^{(2n)}x^{(2n)}\ ;\ n\geq 1),\ x=(x^{(2)}, x^{(4)}, \ldots)\in S_\infty$ (cf. [6, Section 2]).

Now, we shall consider the spectral type of the above flow. Denote by $\{U_t\}$ the unitary group which is determined by the flow $\{T_t\}$: $U_t f(x)=f(T_t x)$. We define random variables $\varphi_k^{(2n)}$ and $\psi_k^{(2n)}$ by

$$(2.4)\qquad \begin{cases} \varphi_k^{(2n)}(x)=x_k^{(2n)}, & 1\leq k\leq 2n,\\ \psi_k^{(2n)}(x)=\varphi_{2k-1}^{(2n)}(x)+i\varphi_{2k}^{(2n)}(x), & 1\leq k\leq n,\end{cases}$$

where $x=(x^{(2)}, x^{(4)}, \ldots)\in S_\infty$ and $x^{(2n)}=(x_1^{(2n)}, \ldots, x_{2n}^{(2n)})\in S_{2n}$. Then, in view of (2.2), (2.3) and (2.4), it is easy to show that every $\psi_k^{(2n)}$ $(2n\geq k)$, and $\eta_k=\lim\limits_{n\to\infty}\psi_k^{(2n)}$ $(k\geq 1)$ (Gaussiam random variable with $\mathbf{E}(\eta_k)=0$ and $\mathbf{E}|\eta_k|^2=2$, [3]) are eigenfunctions of $\{U_t\}$ all of which belong to the same eigenvalue λ_k. This proves that λ_k has infinite multiplicity, since $\{\psi_k^{(2n)};2n\geq k, n\geq 1\}$ is linearly independent.

Let $G(\Lambda)$ be the additive subgroup of R^1 which is generated by $\Lambda=\{\lambda_1, \lambda_2, \ldots\}$ and let $\mathbf{H}_\lambda(\lambda\in G(\Lambda))$ denote the eigenspace which belongs to λ. As is well known, any function in $L^2(S_{2n})$ can be developed into a series of homogeneous polynomials of $x_k^{(2n)}$'s $(1\leq k\leq 2n)$, so that, by considering $L^2(S_{2n})$ as a subspace of $L^2(S_\infty)$, it is not hard to show that

$$\bigcup_{n\geq 1} L^2(S_{2n})\subset \sum_{\lambda\in G(\Lambda)}\oplus\mathbf{H}_\lambda \qquad \text{(direct sum)}$$

since U_t is multiplicative. On the other hand, $\tilde{\mathscr{B}}=\bigcup\limits_{n\geq 1} f_{2n}^{-1}(\mathscr{B}_n)$**) implies that $\bigcup\limits_{n\geq 1} L^2(S_{2n})$ is dense in $L^2(S_\infty)$, therefore we have

$$(2.5)\qquad L^2(S_\infty)=\sum_{\lambda\in G(\Lambda)}\oplus\mathbf{H}_\lambda \qquad \text{(direct sum)}.$$

Thus we have the following theorem.

THEOREM 1. *If $\{T_t\}$ is a canonical flow on S_∞ with the spectral set $\Lambda=\{\lambda_k\,;\,k\geq 1\}$ then its spectrum forms a subgroup of R^1 which is generated by Λ.*

§ 3. Decomposition of the basic space $\tilde{\Omega}$. Let Q_n and Q_n^+ be the subsets of $(n-1)$-sphere defined by

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**) f_{2n} is the projection from $\tilde{\Omega}$ to Ω_{2n}.

$$Q_n = \{c^{(n)} = (c_1^{(n)}, \ldots, c_n^{(n)}) \ ; \ \sum_{i=1}^{n} c_i^{(n)2} = 2n, \ c_i^{(n)} \geq 0\}$$

and

$$Q_n^+ = \{c^{(n)} \ ; \ c^{(n)} \in Q_n, \ c_1^{(n)} > 0, \ldots, c_n^{(n)} > 0\}$$

respectively. We shall define a mapping q_n from S_{2n} onto Q_n as follows;

$$q_n(x^{(2n)}) = c^{(n)},$$

where

$$x^{(2n)} = (x_1^{(2n)}, \ldots, x_{2n}^{(2n)}) \in S_{2n}, \ c^{(n)} = (c_1^{(n)}, \ldots, c_n^{(n)}) \in Q_n, \text{ and}$$

(3. 1) $$c_i^{(n)} = [x_{2i-1}^{(2n)2} + x_{2i}^{(2n)2}]^{1/2}, \quad 1 \leq i \leq n.$$

We denote by ζ_n the partion of S_{2n} which is obtained from the mapping q_n, i.e. $\zeta_n = \{q_n^{-1}(c^{(n)}) \ ; \ c^{(n)} \in Q_n\}$. If $c^{(n)} \in Q_n^+$, then every point $x^{(2n)} \in q_n^{-1}(c^{(n)})$ can be expressed uniquely in the following form

(3. 2) $$\begin{cases} x_{2k-1}^{(2n)} = c_k^{(n)} \cos \varphi_k^{(n)}, \\ x_{2k}^{(2n)} = c_k^{(n)} \sin \varphi_k^{(n)}, \quad 1 \leq k \leq n. \end{cases}$$

Thus we may write $x^{(2n)} = (c^{(n)}, \varphi^{(n)})$, where $\varphi^{(n)} = (\varphi_1^{(n)}, \ldots, \varphi_n^{(n)})$ is a point of n-dimensional torus $\mathbf{T}^n = [0, 2\pi)^n$. Let P_{ζ_n} be the factor measure over the factor space S_{2n}/ζ_n. Then, by definition, P_{ζ_n} can be considered as a measure on the space Q_n which is expressed in the following form;

(3. 3) $$dP_{\zeta_n}(c^{(n)}) = 2^{n-1}\Gamma(n)\Big[\prod_{k=1}^{n-1} \cos \vartheta_k \sin^{2k-1} \vartheta_k\Big] d\vartheta_1 \cdots d\vartheta_{n-1},$$

where ϑ_k's are the polar coordinates of the point $c^{(n)} \in Q_n$, that is,

$$\begin{cases} c_1^{(n)} = \sqrt{2n} \prod_{i=1}^{n-1} \sin \vartheta_i \\ c_k^{(n)} = \sqrt{2n} \cos \vartheta_{k-1} \prod_{i=k}^{n-1} \sin \vartheta_k, \qquad 2 \leq k \leq n-1, \\ c_n^{(n)} = \sqrt{2n} \cos \vartheta_{n-1}, \end{cases}$$

$$0 \leq \vartheta_1 < 2\pi, \ 0 \leq \vartheta_2, \ldots, \vartheta_{n-1} \leq \pi.$$

On the other hand, we shall consider a measure on $\mathscr{B}(\mathbf{T}^n)$ defined by

(3. 4) $$d\mu_n(\varphi^{(n)}) = \left(\frac{1}{2\pi}\right)^n d\varphi_1^{(n)} \cdots d\varphi_n^{(n)}.$$

Then, the measure $dP_{2n}(x^{(2n)})$ can be decomposed in the following way;

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(3.5) $$dP_{2n}(x^{(2n)}) = P(d\varphi^{(n)}\ ;\ c^{(n)})dP_{\zeta_n}(c^{(n)}),\quad x^{(2n)} = (c^{(n)},\ \varphi^{(n)}),\quad c^{(n)} \in Q_n^+,$$

where the family of measures $[P(d\varphi^{(n)}\ ;\ c^{(n)}),\ dP_{\zeta_n}(c^{(n)})]$ in (3.5) is the canonical system of measures corresponding to the partition ζ_n [7]. It is easy to show that the measure $P(d\varphi^{(n)}\ ;\ c^{(n)})$ is independent of the parameter $c^{(n)}$;

(3.6) $$P(d\varphi^{(n)}\ ;\ c^{(n)}) = d\mu_n(\varphi^{(n)}) \text{ for almost all } c^{(n)}(P_{\zeta_n}).$$

Let $x^{(2n)} = (c^{(n)},\ \varphi^{(n)})$ and $x^{(2m)} = (c^{(m)},\ \varphi^{(m)})$ be points in Ω_{2n} and Ω_{2m} respectively. If $x^{(2m)} = f_{2m,2n}(x^{(2n)})$, then, in view of (2.1), (3.1) and (3.2), the relations

(3.7) $$c_k^{(m)} = \frac{\sqrt{2m}}{[c_1^{(n)2} + \cdots + c_m^{(n)2}]^{1/2}}\, c_k^{(n)},\quad 1 \leq k \leq m,$$

and

(3.8) $$\varphi_k^{(m)} = \varphi_k^{(n)},\qquad 1 \leq k \leq m$$

hold. This means that, via the projection $f_{2m,2n}$ from Ω_{2n} to Ω_{2m}, we have two projections such as

(3.7)′ $$\tilde{f}_{m,n}\ :\ Q_n^+ \to Q_m^+$$

and

(3.8)′ $$\pi_{m,n}\ :\ \mathbf{T}^n \to \mathbf{T}^m$$

defined by (3.7) and (3.8) respectively. Moreover, on account of (3.3) and (3.4), it is not hard to show that the systems of measure spaces with indicated projection mappings

$$[(Q_n^+,\ \mathscr{B}(Q_n^+),\ P_{\zeta_n})\ ;\ \tilde{f}_{m,n}]$$

and

$$[(\mathbf{T}^n,\ \mathscr{B}(\mathbf{T}^n),\ \mu_n)\ ;\ \pi_{m,n}]$$

determine projective limit spaces

(3.9) $$(Q^+,\ \mathscr{B}(Q^+),\ P^+)$$

and

(3.10) $$(\mathbf{T},\ \mathscr{B}(\mathbf{T}),\ \mu)$$

respectively.

Next, we shall introduce a partition ζ of the space S_∞ based on the sequence of the partitions $\{\zeta_n\}$. Let $x=(x^{(2n)}\ ;\ n\geq 1)$ and $y=(y^{(2n)}\ ;\ n\geq 1)$ be points in S_∞. Then we say that they are equivalent if $x^{(2n)}$ and $y^{(2n)}$ belong to the same equivalence class with respect to the partition ζ_n for every n. Thus we have the partition ζ of S_∞ and it is clear that an equivalence class D containning a point $x(\in S_\infty)$ is of the form

$$D=\bigcap_{n\geq 1} p_{2n}^{-1}(D^{(2n)}), \tag{3.11}$$

where p_{2n} is the projection from S_∞ to S_{2n} and $D^{(2n)}\in\zeta_n$ with $p_{2n}(x)\in D^{(2n)}=q_n^{-1}(c^{(n)})$. Hereafter, we shall write the set D in (3.11) as

$$D(c)=(D(c^{(1)}),\ D(c^{(2)}),\ \ldots),\ c=(c^{(n)}\ ;\ n\geq 1) \tag{3.12}$$

or simply

$$D=(D^{(2)},\ D^{(4)},\ \ldots).$$

Here note that, if $c^{(n)}$ is a point such that $c_1^{(n)}>0$, then $D(c^{(n)})$ is contained in Ω_{2n}, so that the set $D(c)$ in (3.12) is contained in $\tilde{\Omega}$.

Now, let P_ζ be the factor measure of P_∞ with respect to the partition ζ. We wish to obtain the decomposition of P_∞ similar to (3.5). To get the measures which play the same role as $P(d\varphi^{(n)}\ ;\ c^{(n)})$ in (3.5), we proceed as follows.

Define a mapping h_n from $D(c^{(n)})=q_n^{-1}(c^{(n)})$ to $\mathbf{T}^n$ by

$$h_n(x^{(2n)})=\varphi^{(n)},\ x^{(2n)}=(c^{(n)},\ \varphi^{(n)}) \text{ and } c^{(n)}\in Q_n^+. \tag{3.13}$$

Then h_n is a homeomorphism between the compact sets $D(c^{(n)})$ and $\mathbf{T}^n$. Furthermore, it is an isomorphic mapping from the measure space $(D(c^{(n)}),\ P(d\varphi^{(n)}\ ;\ c^{(n)}))$ to $(\mathbf{T}^n,\ d\mu_n)$. Summing up the above arguments we obtain the following diagram;

$$\begin{array}{ccccccc}
\cdots\longleftarrow D(c^{(m)}) & \xleftarrow{f_{2m,2n}} & D(c^{(n)}) & \xleftarrow{f_{2n}} & D(c) & & \\
h_m\big\downarrow\big\uparrow h_m^{-1} & & h_n\big\downarrow\big\uparrow h_n^{-1} & & h\big\downarrow\big\uparrow h^{-1} & & m<n, \\
\cdots\longleftarrow \mathbf{T}^m & \xleftarrow{\pi_{m,n}} & \mathbf{T}^n & \xleftarrow{\pi_n} & \mathbf{T} & &
\end{array} \tag{3.13}$$

in which π_n is the projection from $\mathbf{T}$ to $\mathbf{T}^n$ and $h\ :\ D(c)\rightarrow\mathbf{T}$ is the homeomorphic mapping which is defined by an obvious method.

On account of the mapping h, we can define a probability measure on the measurable space $(D(c),\ \mathscr{B}(D(c)))$ similar to $P(d\varphi^{(n)}\ ;\ c^{(n)})$ by

$$P(d\varphi\ ;\ c)=d\mu(\varphi) \tag{3.14}$$

where $\varphi=(\varphi^{(n)}\ ;\ n\geq 1)\in \mathbf{T}$, $c=(c^{(n)}\ :\ n\geq 1)\in Q^{+}$. Since, by definition, $P_{\zeta}(Q^{+})=1$, so that the measure $P(d\varphi\ :\ c)$ is defined for almost all c (considered as a point of S_{∞}/ζ) (mod P_{ζ}). Now, denote the point $x=(x^{(2)}, x^{(4)}, \ldots)$ by $x=(c, \varphi)$ if $c=(c^{(n)}\ ;\ n\geq 1)$ and $\varphi=(\varphi^{(n)}\ ;\ n\geq 1)$ with $x^{(2n)}=(c^{(n)}, \varphi^{(n)})$. Then

$$d\widetilde{P}(x)=P(d\varphi\ ;\ c)\,dP_{\zeta}(c),\quad x=(c,\ \varphi)\in Q^{+}\times \mathbf{T} \tag{3.15}$$

is the desired decomposition of the measure $\widetilde{P}$, that is, the family of measures $[P(d\varphi\ ;\ c),\ dP_{\zeta}(c)]$ is the canonical system of measures corresponding to the partition ζ of S_{∞}. Thus we have the following theorem.

THEOREM 2. *Both the spaces* $(\widetilde{\Omega}, \widetilde{\mathscr{B}}, \widetilde{P})$ *and* $(S_{\infty}, \mathscr{B}_{\infty}, P_{\infty})$ *are isomorphic (mod* 0) *to the space* $(Q^{+}\times \mathbf{T},\ \mathscr{B}(Q^{+}\times \mathbf{T}),\ P_{\zeta}\times\mu)$ *by the mapping* $x\rightarrow(c, \varphi)$ *with the relation* (3.15).

§ **4. Decomposition of canonical flows.** Let $\{T_t\}$ be the canonical flow defined by (2.3). Then, observing the decompositions ζ_n and ζ of S_n and S_{∞} respectively, we can prove the following

(a) *If* $c^{(n)}\in Q_n^{+}$, *then* $D(c^{(n)})=q_n^{-1}(c^{(n)})$ *is invariant under the flow* $\{T_t^{(2n)}\}$, *and the restriction* $\{T_t(c^{(n)})\}$ *of the flow* $\{T_t^{(2n)}\}$ *to* $D(c^{(n)})$ *is isomorphic to a flow* $\{S_t^{(n)}\}$ *determined by*

$$S_t^{(n)}\varphi^{(n)}=(\varphi_1^{(n)}+\lambda_1 t,\ \ldots,\ \varphi_n^{(n)}+\lambda_n t),\quad \varphi^{(n)}\in \mathbf{T}^{n}. \tag{4.1}$$

This flow $\{S_t^{(n)}\}$ *turns out to be* $h_n\cdot T_t(c^{(n)})\cdot h_n^{-1}$.

(b) *For every* t, $\{S_t^{(n)}\ ;\ n\geq 1\}$ *is consistent with* $\{\pi_{m,n}\}$, *that is,*

$$S_t^{(m)}\cdot\pi_{m,n}=\pi_{m,n}\cdot S_t^{(n)}\ \text{ on }\ \mathbf{T}^{n}, \tag{4.2}$$

so that they determine a flow $\{S_t\}$ *on* $\mathbf{T}$ [6].

(c) *If* $c=(c^{(n)}\ ;\ n\geq 1)\in Q^{+}$, *then* $D(c)$ *(see* (3.12)) *is invariant under the flow* $\{T_t\}$ *and its restriction* $\{T_t(c)\}$ *to* $D(c)$ *is isomorohic to the flow* $\{S_t\}$ *in such a way as* $S_t=h\cdot T_t(c)\cdot h^{-1}$.

In order to state the theorem, we prepare two lemmas.

LEMMA 1. *Let* $\{T_t\}$ *be a measurable flow on a probability space* $(X, \mathscr{F}, m)$ *and let* $\mathscr{F}_0$ *be any field which generates* $\mathscr{F}$. *Then* $\{T_t\}$ *is ergodic if and only if any* $\{T_t\}$*-invariant* A *in* $\mathscr{F}_0$ *is trivial, i.e.* $m(A)=0$ *or* 1.

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LEMMA 2. *The flow* $\{S_t^{(n)}\}$ *given by* (4. 1) *is ergodic if and only if* $\{\lambda_1, \ldots, \lambda_n\}$ *is linearly independent* ([4]).

Now, let $(\mathbf{T}^\infty, \mathscr{B}(\mathbf{T}^\infty), \mu_\infty)$ be the infinite direct product measure space of one dimensional tori $\mathbf{T}^1 = [0, 2\pi)$ with uniform distribution $d\mu_1(\varphi) = \frac{1}{2\pi} d\varphi$. Then we have the following theorem.

THEOREM 3. *Let* $\{T_t\}$ *be the canonical flow on* S_∞ *with spectral set* $\Lambda = \{\lambda_k ; k \geq 1\}$. *Then,*

(i) ζ *is the* $\{T_t\}$*-invariant partion of* S_∞ *and the flow* $\{T_t(c)\}$ $(c \in Q^+)$ *on the space* $(D(c), \mathscr{B}(D(c), P(d\varphi ; c))$ *is isomorphic to the flow* $\{\tilde{S}_t\}$ *on the space* $(\mathbf{T}^\infty, \mathscr{B}(\mathbf{T}^\infty), \mu_\infty)$:

$$\tilde{S}_t : \tilde{\varphi} = (\varphi_n : n \geq 1) \rightarrow \tilde{S}_t \tilde{\varphi} = (\varphi_n + \lambda_n t : n \geq 1) \tag{4. 3}$$

(ii) $\{T_t(c)\}$ *is ergodic if* $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ *is linearly independent.*

Proof. 1°. Let $\varphi = (\varphi^{(n)} ; n \geq 1)$ be a point of $\mathbf{T}$. Then (3. 8) implies that

$$\varphi_k^{(k)} = \varphi_k^{(k+1)} = \cdots, \quad k = 1, 2, \ldots,$$

so that we can define a mapping H from $\mathbf{T}$ to $\mathbf{T}^\infty$ by

$$H\varphi = \tilde{\varphi} \tag{4. 4}$$

where $\tilde{\varphi} = (\varphi_n ; n \geq 1)$ and $\varphi_n = \varphi_n^{(n)}$. By definitions of S_t, $\tilde{S}_t$ and H, it is abvious that H is the isomorphism between the measure spaces $\mathbf{T}$ and $\mathbf{T}^\infty$ and

$$\tilde{S}_t = H \cdot S_t \cdot H^{-1} \tag{4. 5}$$

holds. Therefore, (c) implies (i).

2°. To prove (ii), it is enough to show that the flow $\{S_t\}$ is ergodic. Let A be an $\{S_t\}$-invariant subset which is in $\mathscr{F}_0 = \bigcup_{m \geqq 1} \pi_m^{-1}(\mathscr{B}(\mathbf{T}^m))$, for example, $A \in \pi_n^{-1}(\mathscr{B}(\mathbf{T}^n))$. Then, by definition of S_t, we have $S_t A = S_t^{(n)} A$ for all t, that is, A is $\{S_t^{(n)}\}$-invariant. Therefore, lemma 2 implies that A is trivial, so that lemma 1 shows that $\{S_t\}$ is ergodic since $\mathscr{F}_0$ generates $\mathscr{B}(\mathbf{T})$. Thus the theorem is proved.

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