EMBEDDINGS IN COSET MONOIDS

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Abstract

A submonoid $S$ of a monoid $M$ is said to be cofull if it contains the group of units of $M$. We extract from the work of Easdown, East and FitzGerald (2002) a sufficient condition for a monoid to embed as a cofull submonoid of the coset monoid of its group of units, and show further that this condition is necessary. This yields a simple description of the class of finite monoids which embed in the coset monoids of their group of units. We apply our results to give a simple proof of the result of McAlister [D. B. McAlister, ‘Embedding inverse semigroups in coset semigroups’, Semigroup Forum 20 (1980), 255–267] which states that the symmetric inverse semigroup on a finite set $X$ does not embed in the coset monoid of the symmetric group on $X$. We also explore examples, which are necessarily infinite, of embeddings whose images are not cofull.

Keywords and phrases: factorizable inverse monoid, coset monoid, symmetric inverse semigroup.

1. Cofull embeddings

In [5], Easdown et al. used groups of units and coset monoids to develop a theory of factorizable inverse monoids which is dual to the classical construction of fundamental inverse semigroups from semilattices. In the classical theory, an inverse subsemigroup $S$ of an inverse semigroup $M$ is full if it contains the semilattice $E_M$ of idempotents of $M$. If $M$ is a monoid, then one also has the alternative group-theoretic point of view, and may focus instead on the group $G_M$ of units of $M$, and say that a submonoid $S$ of $M$ is cofull if $S$ contains $G_M$. In [5] it was shown that cofull inverse submonoids of coset monoids (defined below) play a role which is dual to that played by full inverse subsemigroups of the Munn inverse semigroup [13].

An inverse monoid $M$ is factorizable if $M = E_MG_M = G_ME_M$. The study of factorizable inverse monoids was initiated in [2]; for related studies see [4–6, 8–10, 12, 16, 17] and references therein. Key examples of factorizable inverse...
monoids include finite symmetric inverse semigroups, as well as the so-called coset monoids which we now describe. Let $G$ be a group and denote by $S(G)$ the join semilattice of all subgroups of $G$. (The join $H \lor K$ of two subgroups $H, K \in S(G)$ is defined to be $\langle H \cup K \rangle$, the smallest subgroup of $G$ containing $HK$. ) Now let

$$C(G) = \{ Hg \mid H \in S(G), \ g \in G \},$$

be the set of all cosets of all subgroups of $G$. An associative product $*$ is defined on $C(G)$, for $H, K \in S(G)$ and $g, l \in G$, by

$$(Hg) \ast (Kl) = (H \lor gK g^{-1})gl,$$

the smallest coset of $G$ containing $HgKl$. This operation turns $C(G)$ into a factorizable inverse monoid, with identity $\{1\}$, known as the coset monoid of $G$. Coset monoids were introduced in [14, 15]; see also [8]. We now collect a number of elementary properties of coset monoids which, along with other properties, were stated in [11].

**Lemma 1.** Let $G$ be a group. Then:

(i) $E_{C(G)} = S(G)$;

(ii) $G_{C(G)} = \{ \{g\} \mid g \in G \} \cong G$; and

(iii) the subgroups of $C(G)$ are precisely the sections of $G$. (A section of $G$ is a quotient of a subgroup of $G$.)

For $g \in G$ write $\overline{g} = \{g\}$ and for $S \subseteq G$ write $\overline{S} = \{ \overline{g} \mid g \in S \}$. In particular, we have $G_{C(G)} = \overline{G}$ by Lemma 1(ii).

**Theorem 2** (McAlister [11]). Let $X$ be a set. Then:

(i) the symmetric inverse semigroup $I_X$ embeds in the coset monoid $C(G_Y)$ of the symmetric group $G_Y$ where $Y$ is any set with $|Y| = |X| + 1$; and

(ii) $I_X$ does not embed in $C(G_X)$ if $X$ is finite and nonempty.

It follows by Theorem 2(i) and the Wagner–Preston theorem that any inverse monoid (indeed any inverse semigroup) embeds in the coset monoid of some group. An interesting question that arises is ‘given an inverse monoid $M$, what minimal features (such as cardinality) are required of a group $G$ to ensure that an embedding $M \to C(G)$ exists’? By Lemma 1(iii), $G$ would at least have to contain a section isomorphic to $G_M$, so in particular the cardinality of $G$ would be bounded below by that of $G_M$. Thus, another natural question arises: ‘which inverse monoids $M$ embed in $C(G_M)$’? The goal of this article is to give necessary and sufficient conditions for an inverse monoid $M$ to embed as a cofull submonoid of $C(G_M)$. This allows us to give an answer to the second question, above, in the case that $M$ is finite.

Let $M$ be an inverse monoid and write $E = E_M$ and $G = G_M$. For $e \in E$ let

$$G_e = \{ g \in G \mid eg = e \} = \{ g \in G \mid ge = e \}.$$

1 Unless stated otherwise, all embeddings are assumed to be semigroup embeddings.
It is easy to check that $G_e$ is a subgroup of $G$, and that $G_e \lor G_f \subseteq G_{ef}$ for each $e, f \in E$. Define a map

$$\psi_M : E \to \mathcal{S}(G) : e \mapsto G_e \quad \text{for each } e \in E.$$ 

Let $\mathcal{C}$ denote the class of factorizable inverse monoids $M$ for which $\psi_M$ is a semilattice embedding; that is, $\psi_M$ is injective, and $G_{ef} = G_e \lor G_f$ for all $e, f \in E$. Motivating examples of members of $\mathcal{C}$ include the merge and part braid monoid introduced in [5] both geometrically (in terms of braids for which certain strings associated with an equivalence class are allowed to interact) and algebraically (as a quotient of a semidirect product of a semilattice of equivalence relations by the braid group), and the largest factorizable inverse submonoid of the dual symmetric inverse monoid on a finite set [9]. (See [5, Section 4] for proofs and details.) The recently discovered class of dual reflection monoids [7] also belongs to $\mathcal{C}$. We aim to show that membership of $\mathcal{C}$ is a necessary and sufficient condition for a monoid $M$ to embed as a cofull submonoid of $\mathcal{C}(G_M)$. The following fact is well known and is implicit in [2].

**Lemma 3.** Suppose that $M$ is a factorizable inverse monoid. Then, for all $e, f \in E_M$ and $g, h \in G_M$,

$$eg = fh \iff e = f \quad \text{and} \quad gh^{-1} \in G_e.$$ 

**Lemma 4.** Any cofull submonoid of a factorizable inverse monoid is itself a factorizable inverse monoid.

**Proof.** Suppose that $N$ is a cofull submonoid of a factorizable inverse monoid $M$, and choose $m \in N$. Then $m = eg$ for some $e \in E_M$ and $g \in G_M$. Since $N$ is cofull, we have $g^{-1} \in N$ and so $m^{-1} = g^{-1}e = g^{-1}(eg)g^{-1} = g^{-1}mg^{-1} \in N$, showing that $N$ is inverse. We also have $e = mg^{-1} \in N$ so that $N$ is factorizable.

**Theorem 5.** A monoid $M$ embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if and only if $M \in \mathcal{C}$.

**Proof.** Write $E = E_M$ and $G = G_M$, and suppose first that $M \in \mathcal{C}$. Define $\theta : M \to \mathcal{C}(G)$ by $e \mapsto G_e$ for $e \in E$ and $g \in G$. Then $\theta$ is well defined by Lemma 3, and injective since $\psi_M$ is. That $\theta$ is a homomorphism follows from the rule for $\star$ in the coset monoid, and the facts that $G_{ef} = G_e \lor G_f$ and $g^{-1}G_eg = G_{g^{-1}eg}$ for all $g \in G$ and $e, f \in E$. That $M\theta$ is cofull follows from the fact that $G_1 = \{1\}$ which shows that $G\theta = \overline{G}$ (using the notation defined after Lemma 1).

To show the converse it suffices, since $\mathcal{C}$ is clearly closed under isomorphisms, to show that $N \in \mathcal{C}$ for every cofull submonoid $N$ of $\mathcal{C}(G_M)$. Since $N$ is a factorizable inverse monoid by Lemma 4, it remains only to show that the map

$$\psi_N : E_N \to \mathcal{S}(G_N)$$ 

is an embedding. Now $E_N = \mathcal{S}(G) \cap N$, and $G_N = \overline{G}$, since $N$ is cofull. Furthermore, if $H \in E_N$, then $H \psi_N = \overline{H}$. It follows that $\psi_N$ is an embedding since $\overline{H} \lor \overline{K} = \overline{H \lor K}$ for any subgroups $H, K \in \mathcal{S}(G)$. 

\[\square\]
Remark 6. The sufficiency of Theorem 5 can also be deduced from [5, Theorem 3.2] (although the reader should note that the injectivity of \( \psi_M \) is not part of the definition of the class \( \mathcal{C} \) used in [5]), or alternatively from [10, Theorems 3.1.5 and 4.2.3].

As a corollary, we have the following characterization for finite monoids.

**Theorem 7.** A finite monoid \( M \) embeds in \( \mathcal{C}(G_M) \) if and only if \( M \in \mathcal{C} \).

**Proof.** Write \( G = G_M \) and let \( \Psi : M \rightarrow \mathcal{C}(G) \) be an embedding. Now \( G\Psi \) is a section of \( G \) by Lemma 1(iii) and, since \( G\Psi \cong G \) is finite, we must have \( G\Psi = G/\{1\} = G_{\mathcal{C}(G)} \). Thus \( \Psi \) is cofull, and we are done by Theorem 5. \( \square \)

We now apply Theorem 7 to provide a short proof of Theorem 2(ii).

**Corollary 8.** Let \( X \) be a finite nonempty set. Then \( \mathcal{I}_X \) does not embed in \( \mathcal{C}(\mathcal{G}_X) \).

**Proof.** Put \( G = \mathcal{G}_X = G_{\mathcal{I}_X} \) and, for \( A \subseteq X \), denote by \( \text{id}_A \) the identity map on \( A \) so that \( E_{\mathcal{I}_X} = \{\text{id}_A | A \subseteq X\} \). For each \( A \subseteq X \), we have \( G_{\text{id}_A} = \{\pi \in G | a\pi = a \text{ (for all } a \in A)\} \). We will denote this subgroup, the pointwise stabilizer of \( A \), by \( \text{Stab}(A) \). Now if \( x \in X \), then \( \text{Stab}(X) = \text{Stab}(X\setminus\{x\}) = \{\text{id}_X\} \) so that \( \Psi_{\mathcal{I}_X} \) is not injective, and we are done by Theorem 7. \( \square \)

Remark 9. If \( X \) is any (possibly infinite) set with \( |X| \geq 2 \), then the map \( \Psi_{\mathcal{I}_X} \) is not even a homomorphism since if \( x, y \in X \) with \( x \neq y \) then, writing \( A = X\setminus\{x\} \) and \( B = X\setminus\{y\} \),

\[
G_{\text{id}_A} \lor G_{\text{id}_B} = \text{Stab}(A) \lor \text{Stab}(B) = \text{Stab}(A) = \text{Stab}(B) = \{\text{id}_X\},
\]

while the transposition which interchanges \( x \) and \( y \) is in

\[
\text{Stab}(X\setminus\{x, y\}) = \text{Stab}(A \cap B) = G_{\text{id}_{A\cap B}} = G_{\text{id}_A \circ \text{id}_B}.
\]

2. Other embeddings

In this final section we consider examples of factorizable inverse monoids \( M \) which embed in \( \mathcal{C}(G_M) \) but do not belong to \( \mathcal{C} \). These monoids are necessarily infinite and, of course, the embeddings are not cofull.

**Example 1.** Let \( X \) be an infinite set. Then the symmetric inverse semigroup \( \mathcal{I}_X \) does not belong to \( \mathcal{C} \) as noted in Remark 9. On the other hand, \( \mathcal{I}_X \) does embed in the coset monoid of the symmetric group \( \mathcal{G}_X = G_{\mathcal{I}_X} \) by Theorem 2(i).

Now \( \mathcal{I}_X \) (indeed \( E_{\mathcal{I}_X} \)) is uncountable for any infinite set \( X \). Our second example is a countable factorizable inverse monoid \( M \) for which \( |E_M| = 3 \) and \( \text{rank}(G_M) = 1 \). (For a group \( G \), \( \text{rank}(G) \) denotes the minimal cardinality of a set which generates \( G \) as a group.)

**Example 2.** Let \( G = \langle x \rangle \) be the infinite cyclic group generated by \( x \), and let \( G^y \) be the semigroup obtained by adjoining a zero \( y \) to \( G \). Let \( M = (G^y)\mathcal{C} \) be the semigroup
obtained by adjoining a new zero $z$ to $G^y$. It is easy to check that $M$ is a factorizable inverse monoid with $G_M = G$ and $E_M = \{1, y, z\}$. We also have $G_y = G_z = G$ so that $M \notin \mathcal{E}$. Now define

$$
\Psi : M \rightarrow \mathcal{C}(G) : \begin{cases} 
  x \mapsto \langle x^2 \rangle, \\
y \mapsto \langle x^2 \rangle, \\
z \mapsto G.
\end{cases}
$$

Then one may easily check that $\Psi$ is an embedding.

Our final example is also countable, but in this case we have $|E_M| = \text{rank}(G_M) = 2$.

**Example 3.** Let $G = \langle x, y \rangle$ be the free group freely generated by $\{x, y\}$. Define a homomorphism

$$
\varphi : G \rightarrow G : x \mapsto x^2, \quad y \mapsto y^2,
$$

and put $H = \langle x^2, y^2 \rangle$, the image of $\varphi$. Let $B = G/N$ where $N$ is the normal closure in $G$ of $\{xyxy^{-1}x^{-1}y^{-1}\}$. So $B$ has the presentation $\langle x, y \mid xyx = yxy \rangle$ and is isomorphic to the braid group on three strings; see [1]. It is well known that $N x^2$ and $N y^2$ generate a free subgroup of $B$ of rank two; see for example [3]. It follows that $N \cap H = \{1\}$.

Now let $E = \{0, 1\}$, which we consider as a semilattice under multiplication, and put $M = E \times G$. So $M$ is a factorizable inverse monoid with $E_M = (E, 1) \cong E$ and $G_M = (1, G) \cong G$. We see that $M \notin \mathcal{E}$ since $G_{(1, 1)} = G_{(0, 1)} = \{(1, 1)\}$. Now define

$$
\Psi : M \rightarrow \mathcal{C}(G) : \begin{cases} 
  (1, g) \mapsto [g \varphi] & \text{for all } g \in G, \\
  (0, g) \mapsto N(g \varphi) & \text{for all } g \in G.
\end{cases}
$$

Then $\Psi$ is a homomorphism since $N$ is normal in $G$ and $\varphi$ is a homomorphism. To show that $\Psi$ is injective, suppose that $e_1, e_2 \in E$ and $g_1, g_2 \in G$ are such that $(e_1, g_1)\Psi = (e_2, g_2)\Psi$. Then we clearly must have $e_1 = e_2$. Suppose first that $e_1 = e_2 = 1$. Then

$$
[g_1 \varphi] = (e_1, g_1)\Psi = (e_2, g_2)\Psi = [g_2 \varphi].
$$

It then follows that $g_1 = g_2$, since $\varphi$ is injective, and so $(e_1, g_1) = (e_2, g_2)$. Finally, suppose that $e_1 = e_2 = 0$. Then

$$
N(g_1 \varphi) = (e_1, g_1)\Psi = (e_2, g_2)\Psi = N(g_2 \varphi),
$$

from which it follows that $(g_1 g_2^{-1})\varphi = (g_1 \varphi)(g_2 \varphi)^{-1} \in N$. But then $(g_1 g_2^{-1})\varphi = 1$ since $N \cap H = \{1\}$, and so $g_1 g_2^{-1} = 1$ since $\varphi$ is injective, whence $g_1 = g_2$ and $(e_1, g_1) = (e_2, g_2)$. This completes the proof that $\Psi$ is injective.

While the monoids $M$ considered in Examples 2 and 3 had different values of $|E_M|$ and $\text{rank}(G_M)$, they shared the property that $|E_M| + \text{rank}(G_M) = 4$. This number turns out to be minimal among all such examples, as the next proposition demonstrates.
PROPOSITION 10. Suppose that $M \not\in C$ is a factorizable inverse monoid and that there exists an embedding $\Psi : M \rightarrow C(G_M)$. Then $|E_M| + \text{rank}(G_M) \geq 4$.

PROOF. Write $E = E_M$ and $G = G_M$ and suppose that $|E| + \text{rank}(G) \leq 3$. Since $M \not\in C$, we have $|E| \geq 2$ and, since $\Psi$ is injective, we have $\text{rank}(G) \geq 1$. It then follows that $|E| = 2$ and $\text{rank}(G) = 1$. Write $E = \{1, e\}$ where 1 is the identity of $M$. Since $M \not\in C$ we must have $G_1 = \{1\}$. Since $M$ is infinite, $G$ must be an infinite cyclic group generated by an element $x$ (say) and, since $\Psi$ is an embedding, we have $x^{ij} = e$ for some $i, j \in \mathbb{Z} \setminus \{0\}$. But then $x^{ij} \neq e$ since $G_e = \{1\}$, yet $(x^{ij})^e = (e^i) \ast (x^j)^e = (x^j)^i = (x^j)x^i = x = e^i$, contradicting the injectivity of $\Psi$. This completes the proof. 

References


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