

Direct numerical simulations of Taylor–Couette flow for vanishingly small inner cylinders

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We investigate turbulent Taylor–Couette flow between two concentric cylinders, where the inner cylinder of radius r_i rotates while the outer one of radius r_o remains stationary. Using direct numerical simulations, we examine how varying the radius ratio $\eta = r_i/r_o$ from $\eta = 0.714$ down to 0.0244 affects the flow characteristics at low to moderate Reynolds numbers. Our results show significant changes in the flow structures and statistics in the limit of a vanishingly small inner radius. The turbulent kinetic energy, scaled with the friction velocity at the inner cylinder, does not exhibit a self-similar scaling; instead, it decreases with decreasing η . The turbulent kinetic energy budgets reveal that the locations of peak production and total dissipation are independent of η , whereas their amplitudes decrease as η increases. The pressure–velocity correlation near the inner cylinder is large for small η and its amplitude decreases with increasing η , while the turbulent transport term exhibits the opposite trend. Numerical simulations for $\eta \leq 0.5$ show that, for our specific set-up, a rather good collapse of the distribution of the normalised torque versus the Taylor number (Ta) is obtained when the latter is defined according to Chandrasekhar (*Hydrodynamic and Hydromagnetic Stability*, Oxford Univ. Press, 1961), with a tendency towards a $Ta^{1/3}$ regime at sufficiently large Ta .

Key words: turbulence simulation, turbulent boundary layers

1. Introduction

Fluid dynamics in cylindrical geometries underpins a broad spectrum of engineering and scientific applications. In configurations where the fluid is confined between two concentric cylinders, the flow may develop either in the axial or azimuthal direction. In the former case, referred to as the concentric annular flow, the wall curvature is transverse

to the main flow (Quadrio & Luchini 2002); in the latter, known as a curved channel, the curvature is longitudinal. Understanding the contrasting effects of curvature on axial versus azimuthal flows is essential for optimising designs across a range of industries, including oil and gas, and biomedical engineering. In a recent study, Orlandi, Soldati & Pirozzoli (2025) performed direct numerical simulation (DNS) of both configurations at very small and very large values of the inner radius r_i , using the gap width $\delta = r_o - r_i$ (where r_o is the outer radius) as the reference length. For the axial configuration at large r_i , the flow exhibits characteristics similar to those of a classical planar channel, with comparable mean wall friction on both boundaries. Conversely, for $r_i < 1$, the inner-wall friction coefficient ($C_{f,i}$) exceeds the outer-wall counterpart ($C_{f,o}$).

A notable feature of flow in curved channels is the occurrence of negative turbulent kinetic energy (TKE) production, which Eskinazi & Erian (1969) attributed to a local ‘energy reversal’ mechanism, wherein energy is transferred from the turbulent fluctuations back to the mean flow. This reversal is associated with the presence of large-scale coherent structures near the convex wall of the curved channel (Soldati, Orlandi & Pirozzoli 2025). In this configuration, Orlandi *et al.* (2025) observed a distinctive behaviour of the friction coefficients $C_{f,i}$ and $C_{f,o}$ as functions of the inner radius r_i , indicating the development of a highly complex flow. This complexity manifests as a region of negative TKE production near the inner wall, along with a pronounced enhancement in the contribution of the pressure–velocity correlation term in the TKE budget, which balances the total dissipation. Such peculiar behaviour in the TKE budget is observed only under conditions of extreme curvature, specifically for $r_i < 0.5$. These simulations are primarily of academic interest, as replicating such conditions in practical experiments would be highly challenging.

The Taylor–Couette (TC) flow is a more suitable candidate to study curvature effects through laboratory investigation. The flow develops as the result of the relative rotation of two infinitely long, concentric cylinders. Limiting to the case of stationary outer cylinder, which is traditionally the most studied, the flow is controlled by one geometric parameter, for instance the inner-to-outer radius ratio $\eta = r_i/r_o$, and a Reynolds number, for instance $Re = V_i\delta/\nu$, where $V_i = \omega_i r_i$ and ω_i is the angular velocity of the inner cylinder. TC flow has been extensively studied, both experimentally and numerically, with particular attention to the formation of complex azimuthal structures as the Reynolds number increases. Although it is impossible to mention all of the foundational works, key early contributions certainly include those of Taylor (1923) and Coles (1965), who visualised wave-like patterns on Taylor vortices that evolve into turbulent structures superimposed on coherent azimuthal modes. A comprehensive review of the literature up to that time was provided by Di Prima & Swinney (1981), which highlighted that the majority of studies address radius ratios $\eta > 0.5$. An important parameter in practical (either experimental and numerical) realisations of TC flow is the aspect ratio $\Gamma = L/\delta$, where L is the (necessarily finite) axial length of the cylinders. In numerical simulations with $\eta = 0.909$, Ostilla-Mónico *et al.* (2015) demonstrated that the torque and mean azimuthal velocity profiles are well captured with $\Gamma = 2$. Brauckmann & Eckhardt (2013a) employed $\Gamma = 2$ and $\eta = 0.71$ in numerical simulations of TC flow at Reynolds numbers up to 30 000, and observed a good collapse of the torque across several experimental and numerical datasets when plotted against the shear Reynolds number, $Re_S = 2Re/(1 + \eta)$, especially for $\eta \approx 0.7$.

In general, TC flows in the limit of very small inner radii have received considerably less attention than the more conventional cases with $\eta = O(1)$, with only a few notable exceptions that will be discussed later. Nevertheless, this regime is of practical relevance. Devices based on TC flow with small rotating inner cylinders can provide efficient mixing at low Reynolds numbers in small volumes of biological fluids, for instance in DNA amplification (Li *et al.* 2024). Further applications arise in the food and cosmetics

industries, where controlled mixing of small batches can significantly reduce costs during the research and development phase (Nemri, Charton & Climent 2016). In such contexts, achieving a nearly constant shear stress in the gap between the cylinders is desirable, yet difficult to realise in TC flows with order-one radius ratios.

In this paper, we present simulations of TC flow with a stationary outer cylinder, spanning the laminar, transitional and fully turbulent regimes, with a particular emphasis on configurations with very small inner radii, which are systematically compared with the reference cases with $\eta = O(1)$. One of our primary goals is to assess the most suitable definition of the Taylor number for collapsing the dimensionless torque across a wide range of η , including very small values that, to the best of our knowledge, have not been previously explored either numerically or experimentally. In addition, we investigate whether the friction velocity near the inner wall can serve as a basis for scaling the mean flow and turbulence statistics. It is important to note that in the present set-up, the effects of rotation and shear are inherently intertwined, unlike in previous studies. Therefore, comparison with previous studies in which the inner and outer cylinder were independently actuated should be taken with care (Grossmann, Lohse & Sun 2016).

2. Methodology

The incompressible Navier–Stokes equations in cylindrical coordinates are solved using the second-order finite-difference scheme developed by Verzicco & Orlandi (1996). This scheme employs $q_r = rv_r$ as a computational variable, which simplifies the treatment of the axis within the computational domain. Although not strictly necessary for the present configuration, this approach is retained for consistency with the numerical code used for DNS of pipe flow by Orlandi & Fatica (1997). Time integration of the governing equations is performed using a hybrid third-order low-storage Runge–Kutta algorithm, in which the diffusive terms are treated implicitly and the convective terms are advanced explicitly. The code has been optimised for execution on GPU clusters through a combination of CUDA Fortran and OpenACC directives, and makes use of CUFFT libraries for fast Fourier transforms (FFTs).

Unlike some previous studies, since our focus is on flows with $\eta < 0.5$, all simulations are carried out over the full azimuthal extent. An extensive set of runs has been performed covering $0.024 \leq \eta \leq 0.6$ and $80 \leq Re \leq 5000$. At larger r_i , the configuration of Ostilla-Mónico *et al.* (2014) with $\eta = 0.714$ has been simulated at intermediate and relatively low Reynolds numbers to validate the present numerical method. The list of high-Reynolds-number simulations is reported in table 1, showing that the computations were carried out over long time intervals – particularly for the smallest η – and that the resolution in wall and Kolmogorov units is comparable to that typically employed in canonical wall-bounded flows (Pirozzoli *et al.* 2021).

3. Results

3.1. Torque

Before presenting the full set of simulated cases, it is useful to discuss possible definitions of the Taylor number and to examine whether any of them provides a consistent parametrisation of the torque for arbitrary η in the case of a stationary outer cylinder. For example, Di Prima & Swinney (1981) define the Taylor number as $Ta_d = 4(1 - \eta)/(1 + \eta)Re^2$, whereas Ostilla-Mónico *et al.* (2015) use $Ta_o = \sigma(1/\eta + 1)Re^2$, with the geometric factor $\sigma = (1 + \eta)^4/(4\eta)^2$. Brauckmann & Eckhardt (2013a) further propose $Ta_b = \sigma^2 Re_S^2$. According to all these definitions, the Taylor number typically increases

r_i	η	T_F	T_F^+	T_F^*	$(r_K \Delta\theta)^+$	Δr_K^+	Δz^+	Symbols
0.0250	0.0244	5670	48.0059	319.7453	0.70(0.26)	1.10(0.40)	6.00(2.20)	●
0.0500	0.0476	1500	18.9303	80.4087	0.67(0.29)	0.81(0.35)	4.92(2.09)	●
0.0750	0.0698	1500	23.6833	73.3913	0.87(0.38)	0.82(0.36)	4.40(1.91)	●
0.1111	0.1000	3560	70.5156	161.5576	1.66(0.75)	1.06(0.48)	3.93(1.77)	●
0.2500	0.2000	4000	121.4113	151.6326	2.85(1.33)	1.07(0.50)	3.17(1.48)	●
0.5000	0.3333	1751	68.1155	59.5551	3.20(1.52)	0.60(0.29)	1.87(0.89)	●
0.7500	0.4286	1250	61.8109	39.3366	4.21(2.08)	0.58(0.29)	1.66(0.82)	●
1.0000	0.5000	3138	161.6752	95.2543	5.43(2.65)	0.73(0.36)	1.62(0.79)	●
2.4965	0.7140	1250	96.2738	36.6976	2.66(1.53)	0.58(0.33)	1.04(0.60)	▲
2.4965	0.7140	1250	70.5837	31.1918	3.92(2.08)	0.82(0.44)	1.54(0.82)	▲
2.4965	0.7140	1226	35.1872	25.2333	9.68(5.37)	1.52(0.85)	3.84(2.13)	▲
2.4965	0.7140	1110	22.9943	22.2249	13.85(7.74)	1.82(1.02)	5.51(3.08)	▲

Table 1. Flow parameters: inner radius (r_i), corresponding inner-to-outer radius ratio (η); simulation time in code units (T_F), in wall units (T_F^+) and in Kolmogorov units (T_F^*); grid resolution in wall units and in Kolmogorov units (in parentheses), with Kolmogorov scale evaluated at the peak position of K (r_K).

with η up to $\eta = 0.5$, and then either decreases or approaches a plateau for $\eta > 0.5$. In contrast, Chandrasekhar (1961, p. 296, (169)) define the Taylor number (for a stationary outer cylinder) as $Ta_c = 4Re^2\eta^2/(1-\eta)^4/(1+\eta)^2$, which yields a quadratic dependence for small η and steeper growth as $\eta \rightarrow 1$.

The key output in TC flow is the non-dimensional torque, defined as

$$Nu^\omega = \frac{J^\omega}{J_0^\omega} = \frac{r^3 \left(\langle v_r' v_\theta' \rangle - \frac{1}{Re} \frac{\partial \langle v_\theta \rangle / r}{\partial r} \right)}{J_0^\omega} = Nu_T + Nu_V, \quad J_0^\omega = \frac{1}{Re} \frac{2r_i r_o^2}{r_i + r_o}, \quad (3.1)$$

where Nu_T and Nu_V are the turbulent and viscous contributions, respectively, and J_0^ω denotes the value in laminar flow. In (3.1), the angle brackets $\langle \rangle$ denote averaging in time and over the azimuthal (θ) and axial (z) directions. Momentum conservation implies that Nu^ω must remain exactly constant across the radial direction; therefore, the statistical convergence of the numerical simulations can be assessed from deviations from this constancy. This requirement is particularly stringent for flows with very small η , since a long integration time is needed for the inner cylinder to transfer angular momentum to the entire flow field. This constraint helps explain why such configurations have seldom been explored in previous simulations or experiments. Moreover, it has been found that the grid resolution can significantly influence the accuracy of the computed torque, especially in cases with extremely small inner radii.

Eckhardt, Grossmann & Lohse (2007) provided a detailed derivation of the non-dimensional torque expression starting from the Navier–Stokes equations, concluding that the proper field to consider as the transported quantity is the angular velocity, $\omega = v_\theta/r$, rather than the azimuthal velocity or the angular momentum, $L = r v_\theta$. From a numerical standpoint, it is important to note that radial derivatives appear in the expression for Nu^ω . Their evaluation on strongly non-uniform grids – required to achieve higher resolution near the walls – may depend on the variable locations and the quantity being differentiated. For instance, we have found that more accurate results near the walls are obtained by expressing

$$\partial \langle \omega \rangle / \partial r = \partial \langle v_\theta \rangle / r \partial r = (\langle \omega_z \rangle - 2 \langle v_\theta \rangle / r) / r. \quad (3.2)$$

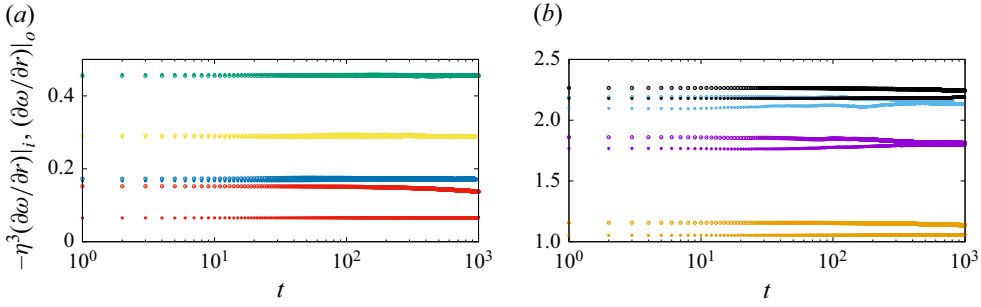


Figure 1. Time evolution (in the last thousand time units) of the terms in the identity (3.3), namely $\partial\omega/\partial r|_o$ (solid symbols) and $-\eta^3\partial\omega/\partial r|_i$ (open symbols). Data at $\eta < 0.2$ are shown in panel (a) and data at $\eta \geq 0.2$ are shown in panel (b). For all cases $Re = 5000$, with values of η provided in table 1.

It is important to recall that in the staggered numerical discretisation that we are using, the velocity components are defined at the centres of the cell faces; hence, the vorticity components are located at the centres of the edges (e.g. $\omega_z(i, j, k + 1/2)$). Therefore, derivatives as $\partial\langle\omega\rangle/\partial r$ are evaluated at the grid index j .

For the more general case in which the outer cylinder also rotates independently, Brauckmann & Eckhardt (2013b) decomposed the numerator of (3.1), into a large-scale component – representing the mean variations in the radial and axial directions – and the turbulent fluctuations. The evaluation of this decomposition is rather involved, as it requires axially shifting the instantaneous flow field to keep the Taylor vortices at a fixed height. The results reported by Brauckmann & Eckhardt (2013b, figure 4) show that, for the special case of a stationary outer cylinder as the present one, the two contributions are nearly identical.

To highlight the pronounced differences in the flow near the inner and outer cylinders, Eckhardt *et al.* (2007) noted that, at steady state,

$$\left. \frac{\partial\langle\omega\rangle}{\partial r} \right|_o = \eta^3 \left. \frac{\partial\langle\omega\rangle}{\partial r} \right|_i. \quad (3.3)$$

The time evolution of the two sides of this relation illustrates the approach to steady state and demonstrates that, for small η , the simulations must be run for long durations before $\eta^3 \partial\langle\omega\rangle/\partial r|_i$ converges to $\partial\langle\omega\rangle/\partial r|_o$. The relevant values are reported in table 2. It is worth noting that, for very small η , the wall-normal gradients at the two boundaries differ significantly, with the difference decreasing rapidly as η increases. The identity (3.3) is indeed satisfied with good accuracy. Figure 1(a) further illustrates that, for small η , (3.3) is satisfied with fewer samples than for larger η (see figure 1b). Table 2 corresponds to $Re = 5000$, at which a fully turbulent regime is established.

A further verification can be made by evaluating the resolution as a fraction of the Kolmogorov scale, whose local value we estimate as

$$\ell_\eta = (\nu^3/\mathcal{D})^{1/4}, \quad (3.4)$$

where $\mathcal{D}$ is the total dissipation, which will be defined later. The grid spacing in Kolmogorov units is provided in table 1 (the values in parentheses), where ℓ_η is evaluated for each flow case at the peak location of the turbulent kinetic energy. Again, the table confirms that the simulations are properly resolved.

Table 2 also reports minimum and maximum values of $\varepsilon_{Nu} = 1 - Nu^\omega/\langle Nu^\omega \rangle$, which can be interpreted as deviations from perfect time convergence and satisfactory grid

η	$\varepsilon_{Nu _{max}}(\%)$	$\varepsilon_{Nu _{min}}(\%)$	$Re_{\tau i}(\times 10^{-2})$	$Re_{\tau o}(\times 10^{-2})$	$\ell_{\eta}(\times 10)$	$-\frac{\partial\langle\omega\rangle}{\partial r} _o$	$\frac{\partial\langle\omega\rangle}{\partial r} _i$
0.0244	10.4074	-3.7157	6.2063	0.1554	0.0355	0.1375	4535.6040
0.0476	8.3344	-1.3432	5.4274	0.2783	0.0373	0.1716	1574.6038
0.0698	7.1824	-1.0760	4.9919	0.3653	0.0409	0.2881	843.9775
0.1000	5.5963	-1.2863	4.5949	0.4098	0.0441	0.4553	457.5412
0.2000	3.1273	-1.4078	3.9513	0.6688	0.0528	1.1365	132.2882
0.3333	1.9784	-0.8915	3.1003	0.6169	0.0588	1.8109	48.5261
0.4286	2.3441	-0.9078	3.0464	0.7271	0.0636	2.1567	27.1189
0.5000	2.7414	-0.8664	2.7547	1.3816	0.0659	2.2446	17.4975
0.7140	0.3449	-0.3229	2.6339	1.7157	0.0681	1.9549	5.3442
0.7140	0.6385	-0.8664	3.3487	2.4266	0.0501	2.4518	6.5866
0.7140	3.6925	-1.1144	9.4562	6.2498	0.0192	5.1244	14.0889
0.7140	5.3911	-1.1176	13.5803	8.9847	0.0133	7.1182	19.3077

Table 2. Flow parameters: η , maximum and minimum per cent errors of $1 - Nu^{\omega}/\langle Nu^{\omega} \rangle$, friction Reynolds number $R_{\tau} = u_{\tau} Re$ at the inner and outer cylinders, Kolmogorov length scale evaluated with the total dissipation at location of maximum K^+ , derivatives of the angular velocity at the inner and outer cylinders.

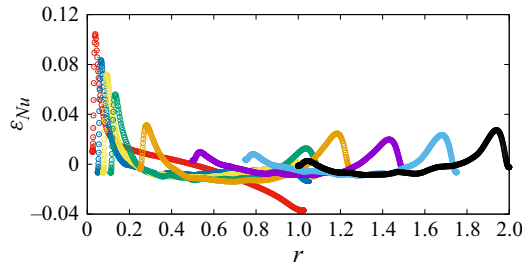


Figure 2. Profiles of local torque unbalance, $\varepsilon_{Nu} = 1 - Nu^{\omega}/\langle Nu^{\omega} \rangle$, obtained by averaging the last thousand and profiles saved every one non-dimensional time unit, for values of η indicated in table 1 ($Re = 5000$ for all cases).

resolution. For $\eta = O(1)$, the errors are similar to those reported by Ostilla-Mónico *et al.* (2013). Slightly higher errors are found at low η , which could be partly due to discretisation errors in the evaluation of the radial derivative of $\langle \omega \rangle$ under such conditions. As reported in table 2, the time convergence error strongly depends on η . To identify the regions within the gap which contribute most to the error, in figure 2, we show profiles of ε_{Nu} . It is evident that only in the cases with extremely small η does a peak appear near the inner cylinder, which coincides with the region where Nu_V sharply decreases. As previously mentioned, in this region, the evaluation of radial derivatives on a non-uniform grid is sensitive to the numerical scheme used. This peak decreases as η increases and the resulting distributions become similar to those reported by Ostilla-Mónico *et al.* (2013), although their results refer to a lower Reynolds number $Re = 1600$ compared with the present case with $Re = 5000$.

As stated in § 1, most previous numerical and experimental studies were conducted for $\eta > 0.5$, for which good collapse of the $Nu^{\omega} - Ta$ relationship is obtained regardless of the definition of Ta . As a preliminary step, we have computed two of the test cases carried out by Ostilla-Mónico *et al.* (2015). The first case correspond to $Re = 80$, $\eta = 0.5$, $\Gamma = 2$, with a $32 \times 64 \times 64$ grid and a rotational symmetry of order four, for which we find $Nu^{\omega} = 1.136$, in good agreement with their value $Nu^{\omega} = 1.138$. The second case corresponds to $Re = 1120$, $\eta = 5/7$, $\Gamma = 2\pi$, which as in Ostilla-Mónico *et al.* (2015),

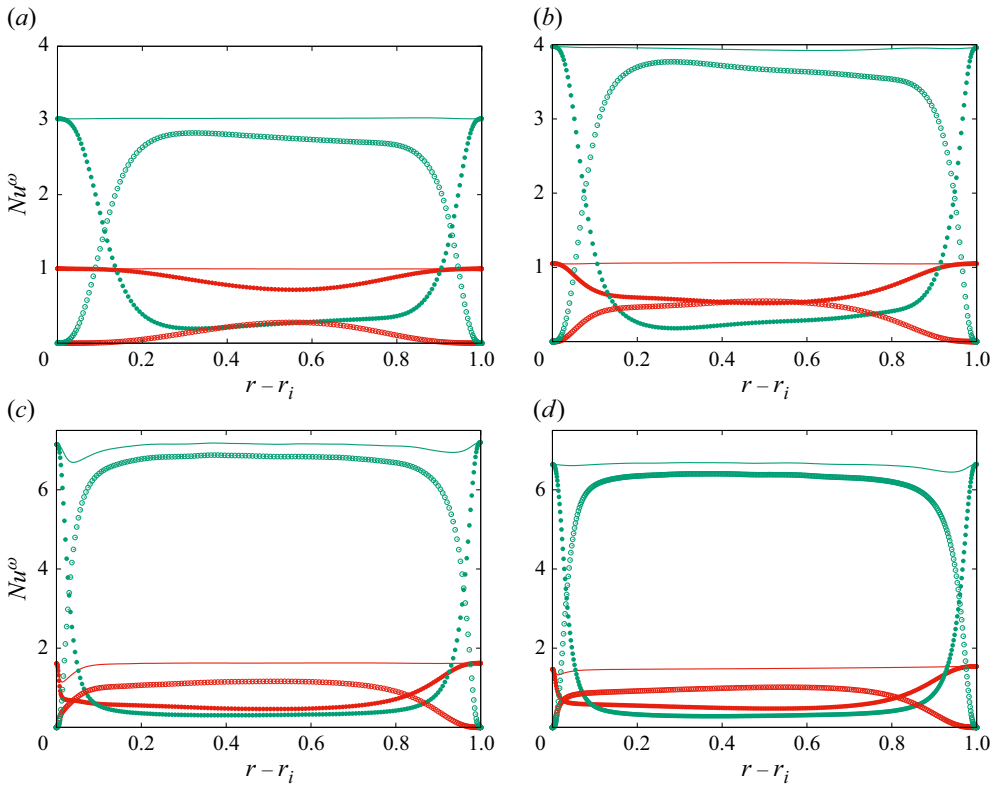


Figure 3. Distributions of non-dimensional total torque Nu^ω (lines), and its turbulent (Nu_T , open circles) and viscous (Nu_V , solid circles) components, versus distance from the inner wall for $\eta = 0.0244$ (red) and $\eta = 0.5$ (green): (a) $Re = 800$ for $\eta = 0.0244$ ($192 \times 192 \times 128$ grid) and for $\eta = 0.5$ ($384 \times 192 \times 128$ grid); (b) $Re = 1600$ for $\eta = 0.0244$ ($192 \times 192 \times 128$ grid) and for $\eta = 0.5$ ($384 \times 192 \times 128$ grid); (c) $Re = 5000$ with same resolution as for $Re = 1600$; (d) $Re = 5000$ for $\eta = 0.0244$ ($256 \times 256 \times 256$ grid) and for $\eta = 0.5$ ($384 \times 384 \times 256$ grid).

we have computed with an unresolved and a fully resolved mesh, with rotational symmetry of order 4. Ostilla-Mónico *et al.* (2015) found that, depending on the initial conditions, four or three pairs of Taylor vortices may form, with associated slight differences in Nu^ω . In the unresolved simulation, we find $Nu^\omega = 4.809$, as compared with $Nu^\omega = 4.835$. In the well-resolved simulation, we find instead $Nu^\omega = 4.688$, as compared with $Nu^\omega = 4.478$, with a 4.7 % difference. This difference may be due to the way that Nu^ω is evaluated. For instance, we estimate Nu^ω by averaging the total torque along the radial direction.

Representative distributions of Nu^ω are reported in figure 3, as obtained in a domain with $\Gamma = 2$, without any assumed rotational symmetry. At low Reynolds number ($Re = 800$), figure 3(a) shows that for $\eta = 0.0244$, Nu_T is smaller than Nu_V , indicating that the flow is nearly laminar. In contrast, for $\eta = 0.5$, the flow remains turbulent and Nu^ω decreases with decreasing Re . At $Re = 1600$, the simulation performed on a $192 \times 192 \times 128$ grid (figure 3b) yields a nearly constant Nu^ω profile. It is also noteworthy that, for $\eta = 0.5$, the turbulent contribution Nu_T decreases with distance from the inner wall near the centre of the gap, with a corresponding increase in the viscous contribution Nu_V , thereby maintaining a constant total torque. The difference between Nu_T and Nu_V diminishes as η and Re decrease. For instance, at $\eta = 0.0244$ in figure 3(b), the two contributions are approximately in balance around the gap centre. Figure 3(c) and 3(d) illustrate the

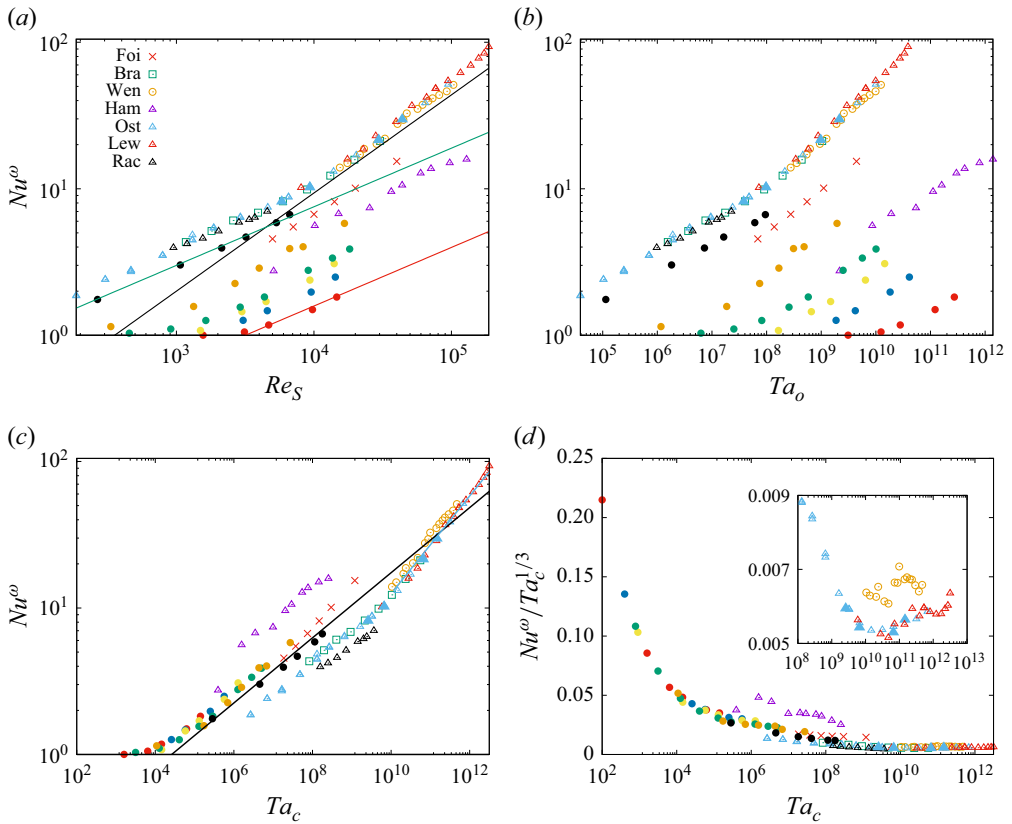


Figure 4. Total torque coefficient (Nu^ω) as a function of: (a) Re_S ; (b) Ta_o ; (c) Ta_c and (d) compensated $Nu^\omega / Ta_c^{1/3}$ versus Ta_c . In panel (a), the inset shows reference data from the literature: Brauckmann & Eckhardt (2013a), Wendt (1933), Froitzheim *et al.* (2019), Hamede, Merbold & Egbers (2023), Ostilla-Mónico *et al.* (2013, table 1), Ostilla-Mónico *et al.* (2014, table 1), Lewis & Swinney (1999) and Racina & Kind (2006). A large part of these data has been extracted by Brauckmann & Eckhardt (2013a, figure 11a). The values of η corresponding to the present simulations are listed in table 1. Solid triangles denote our results for four cases with the same geometry of Ostilla-Mónico *et al.* (2014). In panel (a), the green and red lines indicate the trends $Re_S^{0.4}$, while the black line represents $Re_S^{0.67}$. In panel (c), the black lines indicate the trend $Ta_c^{2/9}$ and the short blue line is $27Ta_c^{1/3}$.

influence of grid resolution at higher Reynolds numbers ($Re = 5000$). When the resolution is insufficient, as shown in figure 3(c), undulations in the profile of Nu^ω appear near both walls. These undulations are primarily attributed to variations in the turbulent contribution Nu_T , and their amplitude increases with larger values of η . In contrast, the results from higher-resolution simulations, presented in figure 3(d), show a clear reduction in the amplitude of these undulations and a lower overall value of Nu^ω for $\eta = 0.5$. At $\eta = 0.0244$, the magnitude of the two contributions to Nu^ω , shown in both figure 3(c) and 3(d), does not significantly change with resolution. However, the more refined simulation yields a more uniform Nu^ω profile across the entire gap.

In figure 4, we consider possible parametrisations of the torque coefficient, using fully resolved simulations with $\Gamma = 2$ and without any rotational symmetry for flows with $\eta \leq 0.5$ at low and high values of Reynolds numbers. Brauckmann & Eckhardt (2013a) conducted numerical simulations of TC flow and compiled both numerical and experimental data, showing in their figure 11(a) a satisfactory collapse of Nu^ω as a function

of the shear Reynolds number Re_S . It is important to note that their dataset corresponds to $\eta \approx 0.7$. For these cases, at relatively low Reynolds numbers, they reported a scaling law of the form $Nu^\omega = Re_S^{0.67}$. They further noted that this exponent is close to the value 0.69 obtained by Lewis & Swinney (1999) in experiments conducted at much higher Reynolds numbers and discussed other values of the exponent reported in the literature.

Several previous studies have examined TC flow at different values of η . For instance, Hamede *et al.* (2023) carried out experiments at the relatively small $\eta = 0.1$, close to the present configuration, and their results as a function of Re_S are shown in figure 4(a). These results at high Reynolds number do not align with our simulations at $\eta = 0.1$ and low Re (green solid circles in figure 4a). Numerical results at $\eta = 0.317$, experimental data from figure 2 of Froitzheim *et al.* (2019) and the classic measurements of Wendt (1933) at very high Reynolds numbers for $\eta > 0.68$ are also reported in figure 4(a). These comparisons indicate that the present simulations, covering both low and high Reynolds numbers, provide an opportunity to assess whether a transitional value of Re_S can be identified and correlated with η . As shown in figure 4(a), for $\eta = 0.5$ (slightly smaller than the value considered by Brauckmann & Eckhardt (2013a)), simulations with $\Gamma = 2$ at high Re yield the scaling $Nu^\omega \approx Re_S^{0.67}$. This trend is also consistent with the data of Froitzheim *et al.* (2019) and, despite some scatter, with the low- Re experiments of Hamede *et al.* (2023). Notably, our results for $\eta = 0.5$ align with the classical scaling law $Nu^\omega = Re_S^{0.67}$, which at high Re_S is supported by the experimental measurements of Wendt (1933) and Lewis & Swinney (1999), as well as by the high- Re simulations of Brauckmann & Eckhardt (2013a). The results of Ostilla-Mónico *et al.* (2014) at large Reynolds numbers also agree well with those of Wendt (1933), and up to $Re_S = 10^5$, they coincide with the measurements of Lewis & Swinney (1999). In the same figure, our simulation at $\eta = 0.714$ and $Re = 37\,400$ falls within the range where the scaling $Nu^\omega \approx Re_S^{0.67}$ holds. In this regime, the relation $Nu^\omega \approx Ta^{1/3}$ emerges and at higher values, a gradual tendency towards the ultimate regime becomes apparent, as discussed later. At lower Reynolds numbers ($Re < 5000$), the scaling becomes $Nu^\omega \approx Re_S^{0.40}$. These latter results, obtained for $800 < Re < 2400$, correspond to flows where $Nu_T \leq Nu_V$ in the central gap region, although both contributions remain non-uniform. In contrast, in cases where $Nu^\omega \approx Re_S^{0.67}$, both the turbulent and viscous contributions are approximately constant across the central region, indicating fully developed turbulence unaffected by the coherent structures that typically form near the inner cylinder. This aspect will be revisited later with the aid of flow visualisations. Additionally, figure 4(a) shows that at $\eta = 0.0244$, the torque scales as $Nu^\omega \approx Re_S^{0.40}$, suggesting that the flow is in a transitional regime – consistent with the red profiles presented in figure 3. The key conclusion from figure 4(a) is that the shear Reynolds number Re_S does not provide a reliable parameter for collapsing the values of non-dimensional torque across all η . However, it appears to perform reasonably well for cases with $\eta > 0.5$.

Considerable effort in previous studies has focused on identifying a scaling relationship between Nu^ω and the Taylor number. As a first step, we plotted the simulation data for various η and Reynolds numbers against Ta_o . Figure 4(b) shows that the data do not collapse: for each value of η , the relation $Nu^\omega \approx Ta_o^n$ holds with a different proportionality constant and with exponent $n = 2/9$. The data in table 1 of Ostilla-Mónico *et al.* (2013), together with our results at intermediate Ta_o and $\eta = 0.714$, also follow the scaling $Nu^\omega \propto Ta_o^{2/9}$, but with a larger prefactor than observed in our simulations at $\eta \leq 0.5$. These observations suggest that the dependence on η might, in principle, be absorbed through an η -dependent correction factor. Alternatively, using the Taylor number definition proposed by Chandrasekhar (1961), figure 4(c) shows that Nu^ω collapses well for all η at $Ta_c \leq 10^7$.

A modest scatter is still visible in our data relative to the reference line $0.13, Ta_c^{2/9}$ at intermediate Re . A similar degree of scatter appears in the data of Brauckmann & Eckhardt (2013a) and Ostilla-Mónico *et al.* (2013), both proportional to $Ta_c^{2/9}$ but with proportionality constants differing from that of our results at $Re = 5000$. The measurements of Hamede *et al.* (2023) lie to the left of our data, whereas those of Wendt (1933) extend further to the right. This discrepancy may partly arise because the data of Hamede *et al.* (2023) were digitised from their published figures without accounting for error bars, which are substantial owing to the experimental challenges associated with torque measurements (Brauckmann & Eckhardt 2013a). The measurements of Wendt (1933) at $\eta = 0.68$ and of Lewis & Swinney (1999) at $\eta = 0.724$ agree well with both our results and those of Ostilla-Mónico *et al.* (2014) at high Reynolds numbers with $\eta = 0.714$. In this range, they closely follow the blue line ($27, Ta_c^{1/3}$), which characterises the Taylor–Couette regime where the torque is dominated by turbulent stresses (figure 3d). In this regime, the azimuthal velocity in wall units exhibits a logarithmic profile, with a coefficient that depends on η , as discussed later. Figure 4(d) further shows that the normalised torque in figure 4(c) collapses onto a single curve tending towards a constant value (denoted C_N) when scaled with $Ta_c^{1/3}$. The inset of figure 4(d) – including data at $\eta = 0.68$ (Wendt 1933), $\eta = 0.724$ (Lewis & Swinney 1999) and $\eta = 0.714$ (present data and Ostilla-Mónico *et al.* 2014) – suggests a consistent value of $C_N = 0.06$. Finally, it is important to recall again that our conclusions regarding the parametrisation of the torque coefficient may depend critically on the chosen flow configuration, in which the Reynolds number of the inner cylinder is fixed while the radius ratio is varied. Under these conditions, both the shear and rotation Reynolds numbers vary simultaneously (Dubrulle *et al.* 2005).

3.2. Flow visualisations

To our knowledge, TC flow with rotating inner cylinders at small η has not been numerically or experimentally investigated. It is thus instructive to investigate the resulting flow structures within the gap. Di Prima & Swinney (1981) presented flow structures at various Reynolds numbers, obtained by Coles (1965), who introduced small flat flakes into the fluid. These particles were transported by the azimuthal velocity, revealing the formation of azimuthal waves on the laminar rollers, which transition to a fully turbulent regime at higher Reynolds numbers. These visualisations stemmed from a set-up featuring $\eta = 0.88$. DNS allows to probe the azimuthal fluctuating vorticity component ω'_θ , which is a good indicator of the shape of the coherent flow structures, which we present in figure 5. In the case of a relatively large inner cylinder ($\eta = 0.5$), at $Re = 800$ (figure 5a), δ -sized rollers are visible which retain symmetry about the z -axis, indicating that the structures do not exhibit azimuthal instability. The vorticity layers are found to be strong near the internal wall than at the outer wall. Additionally, in the central region, two adjacent sheets of opposite-sign vorticity form. At $Re = 1600$, the azimuthal instability reported by Coles (1965) is clearly evident in figure 5(b), where different orientations of patches of azimuthal vorticity are visible.

In the case of a very small inner cylinder ($\eta = 0.0244$), at $Re = 800$, the vorticity contours (see figure 5c) exhibit four weak and thin layers. At $Re = 1600$, the number of axial instabilities increases and figure 5(d) conveys that these instabilities remain axisymmetric in the azimuthal direction. The vorticity patches do not fill the entire gap, remaining concentrated near the inner cylinder. This change of the flow organisation with the Reynolds number may explain why Nu_T increases less in figure 4(d) than in figure 4(c) for $\eta = 0.0244$. Although the vorticity patches in figure 5(d) appear stronger,

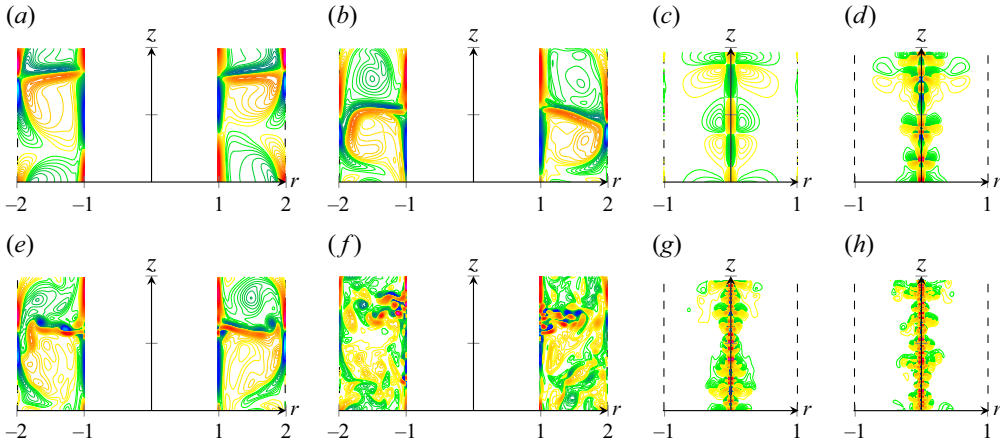


Figure 5. Contours of fluctuating azimuthal vorticity ($\omega'_\theta/|\omega'_\theta|_{\max}$) in transverse (r - z) planes, (a,b,e,f) for $\eta = 0.5$ and (c,d,g,h) $\eta = 0.0244$. (a,c) $Re = 800$, with $|\omega'_\theta|_{\max} = 6.70$ in panel (a) and 11.45 in panel (c); (b,d) $Re = 1600$, with $|\omega'_\theta|_{\max} = 8.76$ in panel (b) and 37.42 in panel (d); (e,g): $Re = 2400$, with $|\omega'_\theta|_{\max} = 11.05$ in panel (e) and 65.54 in panel (g); (f,h): $Re = 5000$, with $|\omega'_\theta|_{\max} = 16.28$ in panel (f) and 131.40 in panel (h). Green, blue and cyan denote negative values; yellow, red and magenta denote positive values, with increments $\Delta = 0.02$ in panels (a), (b), (e), (f) and $\Delta = 0.05$ in panels (c), (d), (g), (h).

they do not significantly affect Nu^ω , which remains close to its laminar value. For $\eta = 0.5$ the vorticity patches near the inner cylinder become very thin at $Re = 2400$, as shown in figure 5(e). At $Re = 5000$, these patches break into small, turbulent-like structures, as shown in figure 5(f), which are concentrated near the inner cylinder. Even for $\eta = 0.024$ at $Re = 2400$, the vorticity field in figure 5(g) remains axisymmetric, similar to figure 5(d); however, a higher number of instabilities is found along the axial direction. Axisymmetry is lost at $Re = 5000$ (see figure 5(h)), where the non-dimensional torque exceeds unity (see figure 3(b)) with $Nu_T > Nu_V$. The formation of azimuthal vorticity patches near the inner cylinder, and not near the outer cylinder, explains the larger difference in the gradients of Nu_T and Nu_V near the inner and the outer cylinder.

3.3. Velocity profiles

In TC flow, scaling in wall units using the friction velocity at the inner cylinder may not always be the most appropriate approach for evaluating flow statistics. This issue was examined by Bilson & Bremhorst (2007), who performed direct numerical simulations of Taylor–Couette flow at $\eta = 0.617$ and $Re = 3200$, with a spatial resolution comparable to that of the present study. In figure 20 of their work, velocity profiles in wall units were shown near both the inner and outer cylinders, each scaled with the respective friction velocity. The authors observed that the velocity departs from the linear profile at $y^+ \approx 2$, a value much closer to the wall than typically found in canonical wall-bounded flows such as pipes and channels. Our results are consistent with those of Bilson & Bremhorst (2007) and further indicate that deviations from the linear region become increasingly pronounced as the inner cylinder radius decreases.

Ostilla-Mónico *et al.* (2014) examined the mean azimuthal velocity profiles for $\eta = 0.714$, and found that they fitted reasonably well to a logarithmic law, with Kármán constant (κ_u) and additive constant (B_u) depending on the Reynolds number. Their study was inspired by Grossmann, Lohse & Sun (2014), who analysed the differences between

η	$Re(\times 10^{-4})$	Ta_c	$Nu^\omega(\times 10^{-1})$	$1/\kappa_u$	B_u	$1/\kappa_o$	B_o
0.024	0.9762	0.6257×10^5	0.1483	NA	NA	NA	NA
0.048	0.9546	0.2511×10^6	0.1955	NA	NA	NA	NA
0.070	0.9348	0.5680×10^6	0.2350	NA	NA	NA	NA
0.100	0.9091	0.1260×10^7	0.2776	NA	NA	NA	NA
0.100	1.8105	0.5038×10^7	0.3880	1.335	3.808	1.836	3.230
0.200	0.8333	0.6782×10^7	0.4040	1.307	4.014	1.819	3.350
0.333	0.7500	0.3164×10^8	0.5402	1.229	4.939	1.795	3.903
0.429	0.7000	0.8441×10^8	0.6243	1.213	5.482	1.775	4.330
0.500	0.6667	0.1778×10^9	0.6629	1.144	6.439	1.729	5.092
0.714	4.3875	0.1467×10^{12}	2.9745	1.076	8.906	1.570	7.013
0.714	2.9525	0.6641×10^{11}	2.1475	1.038	9.016	1.484	7.500
0.714	0.9335	0.6640×10^{10}	1.0165	0.579	10.252	1.175	8.407
0.714	0.5834	0.2594×10^{10}	0.8173	0.403	9.957	1.041	8.049

Table 3. Flow parameters: η , Re_s , Ta_c , Nu^ω , κ_u , B_u , κ_o , B_o . The log-law constants for $\eta = 0.5$ and $\eta = 0.714$ and $Re_S = 2.952 \times 10^4$ have been used to plot the dashed lines in the inset of [figure 6\(a\)](#) and [6\(c\)](#).

the azimuthal velocity, $U^+ = \langle V_i - v_\theta \rangle / u_\tau |_i$, and the angular velocity, $\Omega^+ = \langle V_i / r_i - v_\theta / r \rangle r_i / u_\tau |_i$. They showed that, for large values of $\rho^+ = (r - r_i)^+$, both quantities follow logarithmic profiles reasonably well. The Kármán constant is expected to depend on the flow conditions: at fixed η , it may vary with Re and at fixed Re , it may vary with η . This is indeed the case when the flow is fully turbulent. The present simulations make it possible to verify this dependence and demonstrate that TC flow differs from the usual behaviour of wall-bounded turbulence, in which at high Re , the constants κ_u and B_u are independent of Reynolds number. To determine these constants, the diagnostic function $\mathcal{E} = dF^+ / d \log(\rho^+)$ was evaluated. The value of κ_u was obtained from the minimum of $\mathcal{E}$ for $\rho^+ > 15$ and, from the location of this minimum, the corresponding B_u was estimated. For flows that do not exhibit a minimum in $\mathcal{E}$, a logarithmic fit cannot be established, indicating that the Reynolds number must be increased to achieve a fully turbulent azimuthal velocity profile. This condition was indeed met by performing a simulation at $\eta = 0.1$ and $Re = 10\,000$, which yielded logarithmic profiles whose constants are reported in [table 3](#),

$$U^+ = 1/\kappa_u \log \rho^+ + B_u, \quad \Omega^+ = 1/\kappa_o \log \rho^+ + B_o. \quad (3.5)$$

We focus on the U^+ profiles since, as seen in the inset of [figure 6\(a\)](#) and in [figure 6\(c\)](#), they appear straighter than the corresponding Ω^+ profiles, consistent with the results of Grossmann *et al.* (2014). The values of $1/\kappa_u$ denoted as NA in [table 3](#) indicate that the profiles bend downward, as shown in [figure 6\(c\)](#) for $\eta \leq 0.1$. For other η at the same Re , the profiles are nearly parallel and, consequently, κ_u increases slightly with increasing η , while a small increase in B_u causes the profiles in [figure 6\(c\)](#) to converge. At $\eta = 0.714$ and $Re = 25\,000$, the present profiles agree closely with those of Ostilla-Mónico *et al.* (2014) and do not differ significantly from the profiles at $Re = 37\,600$. In this case, [table 3](#) shows that κ_u increases with Re at intermediate Reynolds numbers, indicating a transition regime from $Nu \approx Ta_c^{2/9}$ to $Nu \approx Ta_c^{1/3}$. At higher Reynolds numbers, the U^+ profiles exhibit $\kappa_u \approx 1$ and further increases in Re merely extend the logarithmic region.

Near the outer cylinder, only at $\eta = 0.5$ does a short, straight layer with a logarithmic U^+ profile appear, as shown in [figure 6\(d\)](#). For all other η , the profiles remain largely

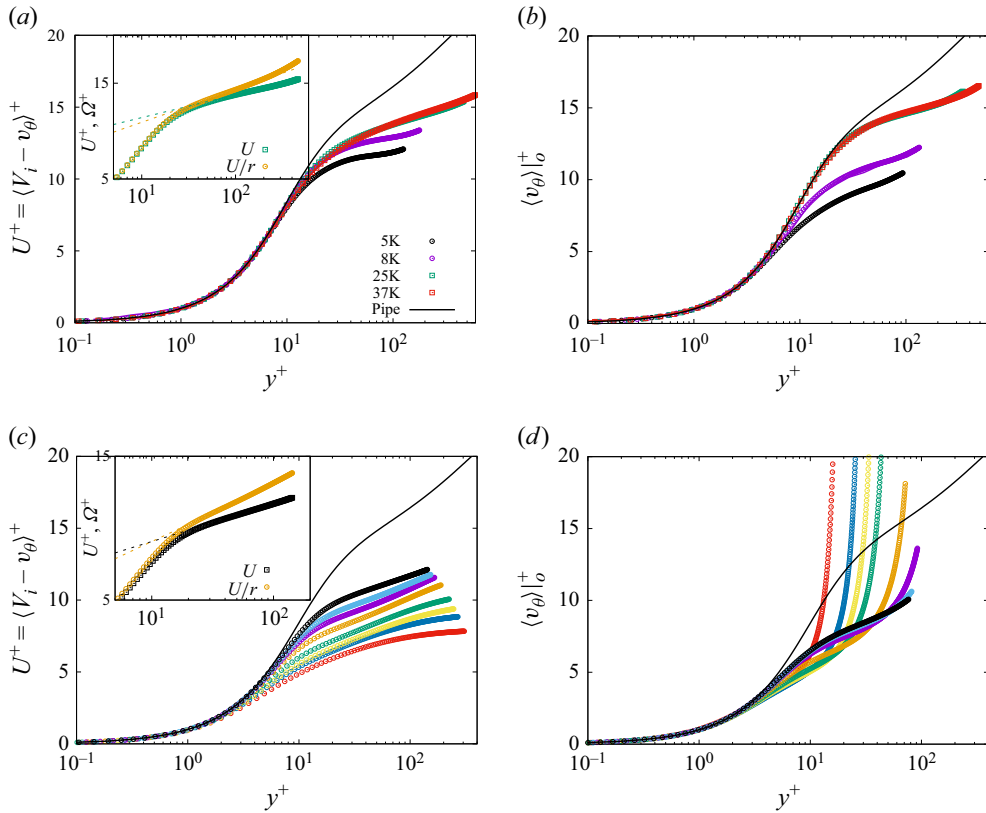


Figure 6. Profiles of mean relative velocity (*a,c*) near the inner cylinder $U^+ = \langle V_i - v_\theta \rangle^+$ and (*b,d*) near the outer cylinder $\langle v_\theta \rangle_o^+$. Panels (*a*) and (*b*) correspond to $\eta = 0.714$, at the Reynolds numbers indicated in the inset, where lines denote the results of Ostilla-Mónico *et al.* (2014) and open symbols denote the present results. In panels (*c*) and (*d*), we report the velocity profiles at various η as in table 1. The solid lines denote DNS data of pipe flow at $Re_\tau = 550$ (Pirozzoli *et al.* 2021). The insets of panels (*a*) and (*c*) report profiles of U^+ and Ω^+ plotted at $Re = 25\,000$ in panel (*a*) and at $\eta = 0.5$ in panel (*c*), the profiles of U^+ and Ω^+ (symbols), and the fitting logarithmic profiles, with constants given in table 3.

laminar-like. In contrast, at $\eta = 0.714$, figure 6(*b*) shows a pronounced Re -dependence at intermediate Re , which vanishes in the $Ta_c^{1/3}$ regime, where the flow becomes independent of Re . At present, we have not been able to determine the log-law constants for $\eta < 0.5$, as reaching sufficiently high Reynolds numbers demands a very large computational effort.

3.4. TKE budgets

In canonical wall-bounded flows, the inner-scaled velocity profiles are typically independent of the Reynolds number. However, the peak value of TKE in the buffer layer increases slightly with Re , and there is still no consensus on whether this growth saturates as $Re \rightarrow \infty$. Based on the simulations of Pirozzoli *et al.* (2021), the Re -dependence of K^+ appears to arise primarily from the increase in the streamwise stress. Figure 6 shows that, for Taylor–Couette flow at intermediate Reynolds numbers, the velocity profiles depend strongly on η for $\eta \leq 0.5$. At larger η , however, the Reynolds-number dependence weakens and tends to vanish at high Re . It is therefore of interest to examine how this behaviour influences the TKE profiles and the corresponding energy budgets. In figure 7(*a*), the TKE

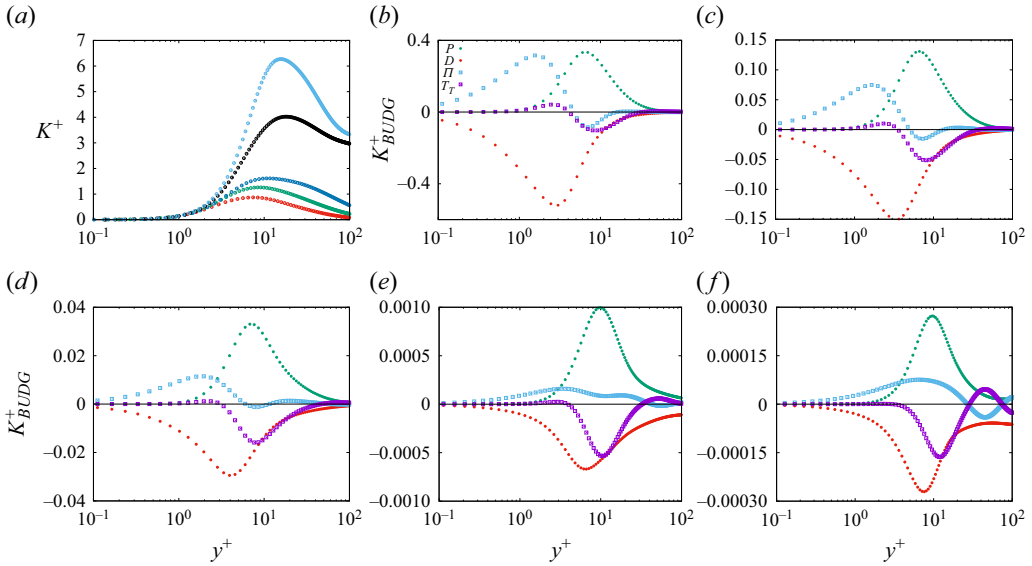


Figure 7. (a) Inner-scaled turbulent kinetic energy K . The other panels report the turbulent kinetic energy budgets for TC flow: (b) $\eta = 0.0244$; (c) $\eta = 0.049$; (d) $\eta = 0.1$; (e) $\eta = 0.5$; (f) $\eta = 0.714$. The symbols denote production (green), total dissipation (red), pressure–velocity correlation (blue) and turbulent transfer (magenta).

profiles in wall units display a marked increase in the K^+ peak with η , further amplified by the rise of u_τ with η . Despite this variation, the location of the peak changes only weakly with η .

This behaviour is more clearly reflected in the TKE budgets, reported in the other panels of figure 7. In pipe flow (not shown), the budget is characterised by an approximate balance between production ($P = -1/r \langle v'_r v'_\theta \rangle d(\langle v_\theta > r)/dr$) and total dissipation, defined as $\mathcal{D} = \nu u_i \nabla^2 u_i$. In pressure-driven curved channel flow, Orlandi *et al.* (2025) found that the TKE budget near the inner wall exhibits a very different behaviour for small η , with a positive contribution from the pressure–velocity correlation term ($\Pi = -1/r d/dr(r \langle v'_r p' \rangle)$), which is nearly balanced by total dissipation. In that case, the production term is of minor importance and turbulent diffusion is negative, being smaller than the total dissipation.

The TKE budgets for TC flow at $\eta = 0.0244$, shown in figure 7(b), exhibit a similar trend to that observed in curved channel flow. However, here the magnitude of the production term is comparable to that of the pressure–velocity correlation. The magnitude of the inner-scaled budget terms decreases with increasing η and a universal behaviour is not achieved. Notably, the location of the production peak, in figure 7, remains almost unchanged as η varies, coinciding with the position where the viscous stress equals the turbulent stress. As η increases, the amplitude of the pressure–velocity correlation term decreases, indicating a change in the pressure distribution near the inner cylinder. At the same time, the turbulent transport term, $T_T = -1/r d/dr(r \langle v'_r v'_i v'_i \rangle)$, increases and compensates for the TKE dissipation. The pronounced increase of Π near the rotating inner cylinder can be attributed to the formation of intense, alternating fluctuations of radial velocity that extend azimuthally in short patches, as shown in figure 5(f). For $\eta > 0.5$, these structures exhibit undulations and strong correlation between v'_r and p' .

4. Conclusions

We have reported a DNS study of TC flows with a small-radius rotating inner cylinder. This study was motivated by the results of Orlandi *et al.* (2025) for curved channel flows with small η , which revealed a TKE budget markedly different from that of canonical wall-bounded flows. That configuration was of purely academic interest and is difficult to reproduce experimentally. Owing to slight modifications in our numerical method, we performed DNS of TC flows for $\eta \leq 0.5$. Our results show both similarities and differences relative to the TKE budgets of Orlandi *et al.* (2025). In TC flow at $\eta = 0.024$, the pressure–work term is large near the inner wall and balanced by total dissipation. In contrast to Orlandi *et al.* (2025), where the production term was negligible, the TC simulations show a consistently significant production term with a peak at nearly the same wall-normal position for all η . The persistence of production enhances the turbulent viscous term as η increases towards the gap width. What remains to be explored in the present work is the link between pressure–work and the flow structures near the inner cylinder. In curved channel flow, Orlandi *et al.* (2025) and Soldati *et al.* (2025) associated the dominant pressure–work contribution with the formation of spanwise rollers rotating around the inner wall at a velocity approximately 0.7 times the bulk velocity. Preliminary observations from the present simulations suggest that, for $\eta \leq 0.2$, alternating, highly coherent pressure fluctuations emerge, which are strongly correlated with radial velocity fluctuations of opposite sign. The relationship between these pressure and velocity structures, and their spectral signatures, will be the subject of a forthcoming study.

The most significant outcome of the present simulations at low η is the demonstration that, among the various definitions of the Taylor number, the formulation proposed by Chandrasekhar (1961) provides the best collapse of both the present and previously published data for the torque–Taylor number relationship. This relation follows the scaling $Nu^\omega \approx Ta_c^{2/9}$ at intermediate values of Ta_c and tends towards $Nu^\omega \approx Ta_c^{1/3}$ at higher Ta_c , as shown by the profile of $Nu^\omega / Ta_c^{1/3}$. Following Grossmann *et al.* (2014), Ostilla-Mónico *et al.* (2014) analysed the velocity profiles scaled in wall units near the inner and outer cylinders, concluding that they depend on the Reynolds number. When the flow reaches the state where $Nu^\omega \approx Ta_c^{1/3}$, the constants of the logarithmic profiles become Re -independent. The present results confirm this finding and further show that the constants depend on η . At $Re = 5000$, the near-wall profiles remain laminar for $\eta \leq 0.1$, whereas doubling the Reynolds number at $\eta = 0.1$ leads to the formation of a logarithmic layer. Our data suggest that the Kármán constant tends towards $\kappa_u = 1$. To verify this result, additional simulations at higher Re are needed, which entail considerable computational cost.

At this stage, these results have been established only for the case of a stationary outer cylinder. It would therefore be of considerable interest to investigate whether the same scaling holds when the outer cylinder is co-rotating or counter-rotating. Preliminary simulations indicate that, for $\eta > 0.2$, the torque contributions do not differ significantly between stationary and counter-rotating outer cylinders. However, noticeable discrepancies appear at smaller η , which warrant further investigation. Although the present study is limited to computationally feasible Reynolds numbers, exploring the potential transition to an ultimate rotational regime would be particularly interesting in future work.

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