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SEMI-NORMAL LOG CENTRES AND DEFORMATIONS OF PAIRS

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Abstract We show that some of the properties of log canonical centres of a log canonical pair also hold for certain subvarieties that are close to being a log canonical centre. As a consequence, we obtain that, in working with deformations of pairs where all the coefficients of the boundary divisor are bigger than $\frac{1}{2}$, embedded points never appear on the boundary divisor.

Keywords: log canonical centre; semi-normality; deformation of pairs

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The philosophy of Shokurov [**20**] stresses the importance of understanding the log canonical (lc) centres of an lc pair (X, Δ) (see Definition 1). After the initial work of [**9**], a systematic study was started in [**2**]. For extensions, surveys and comprehensive treatments see [**3**, **6**]. The following are two of the principal results of [**2**, **3**, **6**].

- Any union of log canonical centres is semi-normal (see Definition 10).
- Any intersection of log canonical centres is also a union of log canonical centres.

The aim of this note is to extend these results to certain subvarieties of an lc pair (X, Δ) that are close to being a log canonical centre. To state our results, we need a definition. (See [**5**, **17**] for basic concepts and results relating to the minimal model program (MMP). As in the above papers, we also work over a field of characteristic 0.)

Definition 1. Let (X, Δ) be lc and let $Z \subset X$ be an irreducible subvariety. Following Shokurov and Ambro, the *minimal log discrepancy* of Z is the infimum of the numbers $1 + a(E, X, \Delta)$ as E runs through all divisors over X whose centre is Z [**1**]. (Here, $a(E, X, \Delta)$ denotes the discrepancy of E with respect to (X, Δ) (see [**17**, 2.25]).) The minimal log discrepancy is denoted by $\operatorname{mld}(Z, X, \Delta)$.

An irreducible subvariety $Z \subset X$ is called a *log centre* of (X, Δ) if $\operatorname{mld}(Z, X, \Delta) < 1$. If $Z \subset X$ is a divisor, then Z is a log centre if and only if it is an irreducible component of Δ, and then its coefficient is $1 - \operatorname{mld}(Z, X, \Delta)$.

A *log canonical centre* is a log centre whose minimal log discrepancy equals 0.

Our first aim is to prove the following. (See Definition 10 for semi-normality.)

Theorem 2. *Let (X, Δ) be an lc pair and let $Z_i \subset X$ be log centres for $i = 1, \ldots, m$.*

(1) *If $\operatorname{mld}(Z_i, X, \Delta) < \frac{1}{6}$ for every i, then $Z_1 \cup \cdots \cup Z_m$ is semi-normal.*

(2) *If $\sum_{i=1}^{m} \operatorname{mld}(Z_i, X, \Delta) < 1$, then every irreducible component of $Z_1 \cap \cdots \cap Z_m$ is a log centre with minimal log discrepancy less than or equal to $\sum_{i=1}^{m} \operatorname{mld}(Z_i, X, \Delta)$.*

A result of this type is not entirely surprising. By Shokurov's conjecture on the boundedness of complements (see [**20**, § 5] or [**19**, Chapter 4]), if $(X, \sum a_i D_i)$ is lc and the a_i are close enough to 1, then there exists another lc pair $(X, \Delta' + \sum D_i)$ where the D_i all appear with coefficient 1. Thus, the D_i are log canonical centres of $(X, \Delta' + \sum D_i)$; hence, their union is semi-normal and Du Bois [**16**]. In particular, there should exist a function $\epsilon(n) > 0$ such that the union of the D_i with $a_i > 1 - \epsilon(\dim X)$ is semi-normal and Du Bois. The function $\epsilon(n)$ is not known, but it must converge to 0 at least doubly exponentially. (See [**12**, § 8] for the conjectured optimal value of $\epsilon(n)$ and for examples.)

Thus, it is somewhat unexpected that, at least for semi-normality, the bound in Theorem 2 (1) is independent of the dimension.

Note that we do not assert that these Z_i are log canonical centres of some other lc pair (X, Δ'); this is actually not true. In particular, unlike log canonical centres, the Z_i are not in general Du Bois (see Example 5 (5)).

As Example 5 (1)–(3) show, the value $\frac{1}{6}$ is optimal. There is, however, one important special case when it can be improved to $\frac{1}{2}$. The precise statement is given in Theorem 11; here, we mention a consequence that was the main reason of this project. The result implies that if we consider the moduli of lc pairs (X, Δ), where all the coefficients in Δ are greater than $\frac{1}{2}$, then we do not have to worry about embedded points on Δ. (Examples of Hassett show that embedded points do appear when the coefficients in Δ are less than or equal to $\frac{1}{2}$. See [**14**, § 6] for an overview and the forthcoming [**15**] for details.)

Corollary 3. *Let $(X, \Delta = \sum_{i \in I} b_i B_i)$ be lc. Let $f\colon X \to C$ be a morphism to a smooth curve such that $(X, X_c + \Delta)$ is lc for every fibre $X_c := f^{-1}(c)$. Let $J \subset I$ be any subset such that $b_j > \frac{1}{2}$ for every $j \in J$, and set $B_J := \bigcup_{j \in J} B_j$.*

Then, $B_J \to C$ is flat with reduced fibres.

The extension of these results to the semi-log-canonical case requires additional considerations; these are treated in [**15**, Chapter 7].

The proof of Theorem 2 uses the following recently established result of Birkar [**4**] and Hacon and Xu [**8**]. For $\dim X \leqslant 4$, it also follows from earlier results of Shokurov [**21**].

Theorem 4. *Let $g\colon X \to S$ be a projective, birational morphism and let Δ', Δ'' be effective $\mathbb{Q}$-divisors on X such that $(X, \Delta' + \Delta'')$ is divisorially log terminal (dlt), $\mathbb{Q}$-factorial and $K_X + \Delta' + \Delta'' \sim_{\mathbb{Q},g} 0$. The (X, Δ'')-MMP with scaling over S then terminates with a $\mathbb{Q}$-factorial minimal model.*

One of the difficulties in [**2**, **6**] comes from making the proof independent of MMP assumptions. The proof in [**8**] uses several delicate properties of log canonical centres, including some of the theorems of [**2**, **6**]. Thus, although the statement of Theorem 2 sharpens several of the theorems of [**2**, **6**] on lc centres, it does not give a new proof.

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Example 5. The following examples show that the numerical conditions of Theorem 2 are sharp.

(1) $(\mathbb{A}^2, \frac{5}{6}(x^2 = y^3))$ is lc, the curve $(x^2 = y^3)$ is a log centre with mld $= \frac{1}{6}$, but it is not semi-normal.

(2) Consider $(\mathbb{A}^3, \frac{11}{12}(z - x^2 - y^3) + \frac{11}{12}(z + x^2 + y^3))$. One can check that this is lc. The irreducible components of the boundary are smooth, but their intersection is a cuspidal curve, hence not semi-normal. It is, again, a log centre with mld $= \frac{1}{6}$.

(3) The image of $\mathbb{C}^2_{uv}$ by the map $x = u$, $y = v^3$, $z = v^2$, $t = uv$ is a divisor $D_1 \subset X := (xy - zt) \subset \mathbb{C}^4$, and $\mathbb{C}^2 \to D_1$ is an isomorphism outside the origin. Note that the zero set of $(y^2 - z^3)$ is $D_1 + 2(y = z = 0)$. Let D_2, D_3 be two general members of the family of planes in the linear system $|(y = z = 0)|$. We claim that $(X, \frac{5}{6}D_1 + \frac{5}{6}D_2 + \frac{5}{6}D_3)$ is lc. Here, D_1 is a log centre with mld $= \frac{1}{6}$, but semi-normality fails in codimension 3 on X.

In order to check the claim, blow up the ideal (x, z). On $\mathbb{C}^2_{uv}$ this corresponds to blowing up the ideal (u, v^2).

On one of the charts we have the coordinates $x_1 := x/z$, y, z, and the birational transform D_1' of D_1 is given by $(y^2 = z^3)$. On the other chart we have the coordinates x, $z_1 := z/x$, t, and D_1' is given by $(z_1 x^3 = t^2)$. Thus, we see that $(B_{(x,z)}X, \frac{5}{6}D_1')$ is lc. The linear system $|(y = z = 0)|$ becomes base-point free on the blow-up; hence, $(B_{(x,z)}X, \frac{5}{6}D_1' + \frac{5}{6}D_2' + \frac{5}{6}D_3')$ is lc and so is $(X, \frac{5}{6}D_1 + \frac{5}{6}D_2 + \frac{5}{6}D_3)$.

(4) Assume that $(X, \sum_{i\in I} a_i D_i)$ has a simple normal crossing and that $a_i \leqslant 1$ for every i. Let $J \subset I$ be a subset such that $a_j > 0$, for every $j \in J$, and $\sum_{j\in J} a_j > |J| - 1$. Every irreducible component of $\bigcap_{j\in J} D_j$ is then a log centre of $(X, \sum_{i\in I} a_i D_i)$ with mld $= \sum_{j\in J}(1 - a_j) = |J| - \sum_{j\in J} a_j$. In particular, D_i is a log centre of $(X, \sum_{i\in I} a_i D_i)$ with mld $= 1 - a_i$. Thus, Theorem 2 (2) is sharp. By [**17**, § 2.3], every log centre of $(X, \sum_{i\in I} a_i D_i)$ arises in this way.

(5) Let X be a smooth variety and let $D \subset X$ be a reduced divisor. Then, D is Du Bois if and only if (X, D) is lc. (See [**16**, **18**] for much stronger results.) In particular, $D := (x^2 + y^3 + z^7 = 0) \subset \mathbb{A}^3$ is a log centre of the lc pair $(\mathbb{A}^3, \frac{41}{42}D)$ with mld $= \frac{1}{42}$, but D is not Du Bois and it cannot be an lc centre of any lc pair (X, Δ). On the other hand, D is normal, hence semi-normal.

Log centres and birational maps

Let $g\colon (Y, \Delta_Y) \to (X, \Delta_X)$ be a proper birational morphism between lc pairs (with Δ_X, Δ_Y not necessarily effective) such that $K_Y + \Delta_Y \sim_{\mathbb{Q}} g^*(K_X + \Delta_X)$ and $g_*\Delta_Y = \Delta_X$.

If $Z \subset Y$ is a log centre of (Y, Δ_Y), then $g(Z)$ is also a log centre of (X, Δ_X) with the same mld. Moreover, every log centre of (X, Δ_X) is the image of a log centre of (Y, Δ_Y).

Thus, for any (X, Δ_X), we can use a log resolution $g\colon (Y, \Delta_Y) \to (X, \Delta_X)$ to reduce the computation of log centres to the simple normal crossing case considered in Example 5 (4).

This implies that an lc pair (X, Δ) has only finitely many log centres, and the union of all log centres of codimension greater than or equal to 2 is the smallest closed subscheme $W \subset X$ such that $(X \setminus W, \Delta|_{X\setminus W})$ is canonical.

Proof of the divisorial case of Theorem 2. We prove Theorem 2 in the special case when (X, Δ') is dlt for some Δ' and the $Z_i =: D_i$ are $\mathbb{Q}$-Cartier divisors.

Since (X, Δ') is dlt, X is Cohen–Macaulay and so is $\sum D_i$ [**17**, 5.25]. In particular, $\sum D_i$ satisfies Serre's condition S_2. An S_2-scheme is semi-normal if and only if it is semi-normal at its codimension 1 points. By localization at codimension 1 points, we are reduced to the case when $\dim X = 2$.

Then, X has a quotient singularity at every point of $\sum D_i$, and Reid's covering method [**10**, 20.3] reduces the claim to the smooth case. It is now an elementary exercise to see that if $(\mathbb{A}^2, \sum a_i D_i)$ is lc and $a_i > \frac{5}{6}$, then $\sum D_i$ has only ordinary nodes, hence is semi-normal.

We next prove Theorem 2 (2), assuming that $m = 2$ and $Z_i =: D_i$ are $\mathbb{Q}$-Cartier divisors. Every irreducible component of $D_1 \cap D_2$ then has codimension 2; thus, it is again enough to check the smooth surface case. The exceptional divisor of the blow-up of $x \in D_1 \cap D_2$ shows that x is a log centre with mld $\leqslant (1 - a_1) + (1 - a_2)$. □

Any argument along these lines breaks down completely if we only assume that $(X, \sum a_i D_i)$ is lc. In general, the D_i are not S_2, not even if $a_i = 1$. Thus, semi-normality at codimension 1 points does not imply semi-normality.

Instead, we choose a suitable dlt model (Y, Δ_Y) of (X, Δ), use the proof of the divisorial case of Theorem 2 on it, and then descend semi-normally from Y to X. The next two lemmas construct (Y, Δ_Y).

Lemma 6. *Let (X, Δ) be lc. There then exists a projective, birational morphism $g\colon (Y, \Delta_Y) \to (X, \Delta)$ such that*

(1) *(Y, Δ_Y) is dlt, $\mathbb{Q}$-factorial (and Δ_Y is effective);*

(2) *$K_Y + \Delta_Y \sim_{\mathbb{Q}} g^*(K_X + \Delta)$; and*

(3) *for every log centre $Z \subset X$ of (X, Δ) there exists a divisor $D_Z \subset Y$ such that $g(D_Z) = Z$ and D_Z appears in Δ_Y with coefficient $1 - \mathrm{mld}(Z, X, \Delta)$.*

Proof. This is well known. Under suitable MMP assumptions, a proof is given in [**10**, 17.10]. One can remove the MMP assumptions as follows.

A method of Hacon (see [**16**, 3.1]) constructs a model satisfying (1) and (2). Since there are only finitely many log centres, it is enough to add the divisors D_Z one at a time. This is explained in [**13**, 37]. A simplified proof can be found in [**7**, § 4]. □

Lemma 7. *Let $g\colon Y \to X$ be a projective, birational morphism and let Δ_1, Δ_2 be effective $\mathbb{Q}$-divisors on Y. Assume that $(Y, \Delta_1 + \Delta_2)$ is dlt, $\mathbb{Q}$-factorial and that $K_Y + \Delta_1 + \Delta_2 \sim_{\mathbb{Q},g} 0$. By Theorem 4, a suitable (Y, Δ_2)-MMP over X terminates with a $\mathbb{Q}$-factorial minimal model $g^m\colon (Y^m, \Delta_2^m) \to X$. Then,*

(1) *$-\Delta_1^m$ is g^m-nef,*

(2) *$g(\Delta_1) = g^m(\Delta_1^m)$ and*

(3) $\mathrm{Supp}(g^m)^{-1}(g^m(\Delta_1^m)) = \mathrm{Supp}\,\Delta_1^m$.

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Proof. Since $K_Y + \Delta_1 + \Delta_2 \sim_{\mathbb{Q},g} 0$, we see that $K_{Y^m} + \Delta_1^m + \Delta_2^m \sim_{\mathbb{Q},g^m} 0$. Thus, $-\Delta_1^m \sim_{\mathbb{Q},g^m} K_{Y^m} + \Delta_2^m$ is g^m-nef. Since g^m has connected fibres and Δ_1^m is effective, every fibre of g^m is either contained in $\operatorname{Supp}\Delta_1^m$ or is disjoint from it. This proves (3).

In order to establish (2), we prove by induction that, at every intermediate step $g^i\colon (Y^i, \Delta_2^i) \to X$ of the MMP, we have that $g(\Delta_1) = g^i(\Delta_1^i)$. This is clear for $Y^0 := Y$. As we go from i to $i+1$, the image $g^i(\Delta_1^i)$ is unchanged if $Y^i \dashrightarrow Y^{i+1}$ is a flip. Thus, we need to show that $g^{i+1}(\Delta_1^{i+1}) = g^i(\Delta_1^i)$ if $\pi_i\colon Y^i \to Y^{i+1}$ is a divisorial contraction with exceptional divisor E^i. Let $F^i \subset E^i$ be a general fibre of $E^i \to X$. It is clear that

$$g^{i+1}(\Delta_1^{i+1}) \subset g^i(\Delta_1^i),$$

and equality fails only if E^i is a component of Δ_1^i but no other component of Δ_1^i intersects F^i. Since π_i contracts a $(K_{Y^i} + \Delta_2^i)$-negative extremal ray, $-\Delta_1^i \sim_{\mathbb{Q},g^i} K_{Y^i} + \Delta_2^i$ shows that Δ_1^i is π_i-nef. However, an exceptional divisor has negative intersection with some contracted curve, which is a contradiction. □

Proof of Theorem 2 (1). Set $\epsilon_i := \operatorname{mld}(Z_i, X, \Delta)$. As in Lemma 6, let $g\colon (Y, \Delta_Y) \to (X, \Delta)$ be a $\mathbb{Q}$-factorial dlt model and let $D_i \subset Y$ be divisors such that $a(D_i, X, \Delta) = -1 + \epsilon_i$ and $g(D_i) = Z_i$. Set $D := \sum_{i=1}^m D_i$; then $g(D) = Z$.

Pick $1 > c \geqslant 0$ such that $1 - \epsilon_i \geqslant c$ for every i and write $\Delta_Y = cD + \Delta_2$, where Δ_2 is effective. (It may have common components with D.) Apply Lemma 7 to get a $\mathbb{Q}$-factorial model $g^m\colon Y^m \to X$ such that

(1) $(Y^m, cD^m + \Delta_2^m)$ is lc,

(2) (Y^m, Δ_2^m) is dlt,

(3) $K_{Y^m} + cD^m + \Delta_2^m \sim_{\mathbb{Q},g^m} 0$,

(4) $-D^m$ is g^m-nef and

(5) $\operatorname{Supp} D^m = \operatorname{Supp}(g^m)^{-1}(Z)$, and hence $g^m(D^m) = Z$.

If $\epsilon_i < \frac{1}{6}$ for every i, then we can assume that $c > \frac{5}{6}$. As we noted in the proof of the divisorial case of Theorem 2, in this case D^m is semi-normal and Lemma 8 shows that $g^m_* \mathcal{O}_{D^m} = \mathcal{O}_Z$. Thus, Z is semi-normal by Lemma 13. □

Lemma 8. *Let Y, X be normal varieties and let $g\colon Y \to X$ be a proper morphism such that $g_* \mathcal{O}_Y = \mathcal{O}_X$. Let D be a reduced divisor on Y and let Δ'' be an effective $\mathbb{Q}$-divisor on Y. Fix some $0 < c \leqslant 1$. Assume that*

(1) *$(Y, cD + \Delta'')$ is lc,*

(2) *(Y, Δ'') is dlt,*

(3) *$K_Y + cD + \Delta'' \sim_{\mathbb{Q},g} 0$ and*

(4) *$-D$ is g-nef (and hence $D = g^{-1}(g(D))$).*

Then $g_* \mathcal{O}_D = \mathcal{O}_{g(D)}$.

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Proof. By pushing forward the exact sequence

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0,$$

we obtain that

$$\mathcal{O}_X = g_*\mathcal{O}_Y \to g_*\mathcal{O}_D \to R^1 g_*\mathcal{O}_X(-D).$$

Note that

$$-D \sim_{\mathbb{Q},g} K_Y + \Delta'' + (1-c)(-D),$$

and the right-hand side is of the form $K + \Delta + (g - \text{nef})$. Let $W \subset Y$ be an lc centre of (Y, Δ''). Then W is not contained in D, since then $(Y, cD + \Delta'')$ would not be lc along W. In particular, D is disjoint from the general fibre of $W \to X$ by (4). Thus, from Theorem 9 we conclude that none of the associated primes of $R^1 g_*\mathcal{O}_Y(-D)$ is contained in $g(D)$. On the other hand, $g_*\mathcal{O}_D$ is supported on $g(D)$; hence, $g_*\mathcal{O}_D \to R^1 g_*\mathcal{O}_Y(-D)$ is the zero map.

This implies that $\mathcal{O}_X \to g_*\mathcal{O}_D$ is surjective. This map factors through $\mathcal{O}_{g(D)}$; hence, $g_*\mathcal{O}_D = \mathcal{O}_{g(D)}$. □

A curious property of log centres

Assume that (X, Δ) is klt and let $Z \subset X$ be a union of arbitrary log centres. As in the proof of Theorem 2 (1) we construct $(Y, cD + \Delta'')$, which is klt. Thus, as we apply Lemma 8, the higher direct images $R^i g_*\mathcal{O}_Y$ and $R^i g_*\mathcal{O}_Y(-D)$ are 0 for $i > 0$. Thus, D is a reduced Cohen–Macaulay scheme D such that

$$g_*\mathcal{O}_D = \mathcal{O}_Z \quad \text{and} \quad R^i g_*\mathcal{O}_D = 0 \quad \text{for } i > 0.$$

Moreover, D is a divisor on a $\mathbb{Q}$-factorial klt pair.

This looks like a very strong property for a reduced scheme Z, but so far I have been unable to derive any useful consequences from it. In fact, I do not know how to prove that not every reduced scheme Z admits such a morphism $g\colon D \to Z$.

We have used the following form of [**2**, 3.2, 7.4] and [**6**, 2.52].

Theorem 9. *Let $g\colon Y \to X$ be a projective morphism and let M be a line bundle on Y. Assume that $M \sim_{\mathbb{Q},g} K_Y + L + \Delta$, where (Y, Δ) is dlt and, for every log canonical centre $Z \subset Y$, the restriction of L to the general fibre of $Z \to X$ is semi-ample.*

Every associated prime of $R^i g_* M$ is then the image of a log canonical centre of (Y, Δ).

Proof of Theorem 2 (2). By induction on m, it is enough to prove Theorem 2 (2) for the intersection of two log centres.

Let $g\colon (Y, \Delta_Y) \to (X, \Delta)$ be a $\mathbb{Q}$-factorial dlt model and let $D_1, D_2 \subset Y$ be divisors such that $a(D_i, X, \Delta) = -1 + \operatorname{mld}(Z_i, X, \Delta)$ and $g(D_i) = Z_i$. Set $D := D_1 + D_2$. Pick any $c > 0$ such that $\Delta_Y = cD + \Delta_2$, where Δ_2 is effective, and apply Lemma 7. Thus, we get a $\mathbb{Q}$-factorial model $g^m\colon Y^m \to X$ such that

(1) $K_{Y^m} + cD^m + \Delta_2^m \sim_{\mathbb{Q},g^m} 0$ and

(2) $\operatorname{Supp} D^m = \operatorname{Supp}(g^m)^{-1}(Z_1 \cup Z_2)$.

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By (2), every irreducible component $V_j \subset Z_1 \cap Z_2$ is dominated by an irreducible component of $W_j \subset D_1^m \cap D_2^m$. By the proof of the divisorial case of Theorem 2, each W_j is a log centre of $(Y^m, cD^m + \Delta_2^m)$ with mld $\leqslant \operatorname{mld}(Z_1, X, \Delta) + \operatorname{mld}(Z_2, X, \Delta)$. Thus, V_j is a log centre of (X, Δ) with the same minimal log discrepancy. □

Definition 10. Let X be a reduced scheme and let $U \subset X$ be an open subscheme. We say that X is *semi-normal relative to* U if every finite, universal homeomorphism $\pi\colon X' \to X$ that is an isomorphism over U is an isomorphism.

If this holds with $U = \emptyset$, then X is called *semi-normal*. For more details, see [**11**, § I.7.2].

If X satisfies Serre's condition S_2, then semi-normality depends only on the codimension 1 points of X. That is, X is semi-normal relative to U if and only if there exists a closed subset $Z \subset X$ of codimension greater than or equal to 2 such that $X \setminus Z$ is semi-normal relative to U.

With this definition, we can state the theorem behind Corollary 3 as follows.

Theorem 11. *Let $(X, S + \Delta)$ be an lc pair, where S is a reduced $\mathbb{Q}$-Cartier divisor. Let $Z_i \subset X$ be log centres of (X, Δ) for $i = 1, \dots, m$.*

If $\operatorname{mld}(Z_i, X, \Delta) < \frac{1}{2}$ for every i, then $S \cup Z_1 \cup \dots \cup Z_m$ is semi-normal relative to $X \setminus S$.

Proof. By passing to a cyclic cover and using Lemma 12, we may assume that S is Cartier. Note that none of the Z_i is contained in S.

We next closely follow the proof of Theorem 2 (1). Let $g\colon (Y, S_Y + \Delta_Y) \to (X, S + \Delta)$ be a $\mathbb{Q}$-factorial dlt model and let $D_i \subset Y$ be divisors such that $a(D_i, X, \Delta) = -1 + \operatorname{mld}(Z_i, X, \Delta)$ and $g(D_i) = Z_i$. Pick $c > \frac{1}{2}$ such that $1 - \operatorname{mld}(Z_i, X, \Delta) \geqslant c$ for every i. Set $D := S_Y + \sum_i D_i$ and write $\Delta_Y = cD + \Delta_2$, where Δ_2 is effective.

Apply Lemma 7 to get a $\mathbb{Q}$-factorial model $g^m\colon Y^m \to X$ such that

(1) $(Y^m, cD^m + \Delta_2^m)$ is lc,

(2) (Y^m, Δ_2^m) is dlt,

(3) $K_{Y^m} + cD^m + \Delta_2^m \sim_{\mathbb{Q}, g^m} 0$,

(4) $-D^m$ is g^m-nef and

(5) $g^m(D^m) = S \cup Z_1 \cup \dots \cup Z_m$.

Using Lemmas 8 and 13 we see that it is enough to prove that D^m is semi-normal relative to $Y^m \setminus S_Y^m$.

Since Y^m is dlt, it is Cohen–Macaulay; hence, D^m is S_2. As we noted in Definition 10, it is sufficient to check semi-normality at the codimension 2 points of Y^m. As in the proof of the divisorial case of Theorem 2, this reduces to the smooth surface case. We see that if F is a smooth surface, $(F, S + cD)$ is lc and $c > \frac{1}{2}$, then, at every point of $S \cap D$, D is smooth and intersects S transversally. Thus, $S + D$ is semi-normal at all points of $S \cap D$. □

Again, note that the bound $\frac{1}{2}$ is sharp; $(\mathbb{A}^2, (x = 0) + \frac{1}{2}(x + y = 0) + \frac{1}{2}(x - y = 0))$ is lc, but its boundary is not semi-normal at the origin.

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Proof of Corollary 3. None of the irreducible components of Δ is contained in a fibre of f; hence, $f\colon B_J \to C$ is flat. The main point is to show that its fibres are reduced.

If $b_j > \frac{1}{2}$, then the corresponding divisor B_j is a log centre and $\operatorname{mld}(B_j, X, \Delta) = 1 - b_j < \frac{1}{2}$. Thus, by Theorem 11, $X_c + B_J$ is semi-normal relative to $X \setminus X_c$ for every $c \in C$. By Lemma 14, this implies that $X_c \cap B_J$ is reduced. □

We have used three easy properties of semi-normal schemes.

Lemma 12. *Let $g\colon Y \to X$ be a finite morphism of normal schemes. Let $Z \subset X$ be a closed, reduced subscheme and let $U \subset X$ be an open subscheme. If $\operatorname{red} g^{-1}(Z)$ is semi-normal relative to $g^{-1}U$, then Z is semi-normal relative to U.*

Proof. We may assume that X, Y are irreducible and affine. Let $\pi\colon Z' \to Z$ be a finite, universal homeomorphism that is an isomorphism over $Z \cap U$. Pick $\phi \in \mathcal{O}_{Z'}$. Since $\operatorname{red} g^{-1}(Z)$ is semi-normal relative to $g^{-1}U$, the pullback $\phi \circ g$ is a regular function on $\operatorname{red} g^{-1}(Z)$. We can lift it to a regular function Φ_X on X. Since Y is normal,

$$\Phi_Y := \frac{1}{\deg X/Y} \operatorname{tr}_{X/Y} \Phi_X$$

is regular on Y and $\Phi_Y|_Z = \phi$. Thus, Z is semi-normal relative to U. □

Lemma 13. *Let $g\colon Y \to X$ be a proper morphism of reduced schemes such that $g_*\mathcal{O}_Y = \mathcal{O}_X$. Let $U \subset X$ be an open subscheme. If Y is semi-normal relative to $g^{-1}U$, then X is semi-normal relative to U.*

Proof. Let $\pi\colon X' \to X$ be a finite, universal homeomorphism that is an isomorphism over U. Set $Y' := \operatorname{red}(Y \times_X X')$ with projection $\pi_Y\colon Y' \to Y$. Then, π_Y is a finite, universal homeomorphism that is an isomorphism over $g^{-1}U$. Thus, π_Y is an isomorphism, so we can factor g as $Y \to X' \to X$. This implies that $\pi_*\mathcal{O}_{X'} \subset g_*\mathcal{O}_Y = \mathcal{O}_X$; hence, π is an isomorphism. □

Lemma 14. *Let X be semi-normal relative to U. Let $X_1, X_2 \subset X$ be closed, reduced subschemes such that $X = X_1 \cup X_2$. Then, $\mathcal{O}_{X_1 \cap X_2}$ has no nilpotent elements whose support is in $X \setminus U$.*

Proof. Let $I \subset \mathcal{O}_{X_1 \cap X_2}$ be the ideal sheaf of nilpotent elements whose support is in $X \setminus U$, and let $r(X_1 \cap X_2) \subset X_1 \cap X_2$ be the corresponding subscheme.

Let $r_i\colon \mathcal{O}_{X_i} \to \mathcal{O}_{X_1 \cap X_2}$ and $\bar{r}_i\colon \mathcal{O}_{X_i} \to \mathcal{O}_{r(X_1 \cap X_2)}$ denote the restriction maps. Then, $\mathcal{O}_X$ sits in an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} + \mathcal{O}_{X_2} \xrightarrow{(r_1, -r_2)} \mathcal{O}_{X_1 \cap X_2} \to 0.$$

The similar sequence

$$0 \to A \to \mathcal{O}_{X_1} + \mathcal{O}_{X_2} \xrightarrow{(\bar{r}_1, -\bar{r}_2)} \mathcal{O}_{r(X_1 \cap X_2)} \to 0$$

defines a coherent sheaf of $\mathcal{O}_X$-algebras A, and $\operatorname{Spec}_X A \to X$ is a finite, universal homeomorphism $\pi\colon X' \to X$ that is an isomorphism over U. Since X is semi-normal relative to U, $A = \mathcal{O}_X$; hence, $X_1 \cap X_2 = r(X_1 \cap X_2)$. □

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