BOUNDDED GENERATORS IN LINEAR TOPOLOGICAL SPACES

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1. Ito and Seidman in [5] define a BG space as a locally convex space in which there exists a bounded set with a dense span. In this note we extend the idea to a class of not necessarily locally convex linear topological spaces (l.t.s.). We note the link between the idea of a BG space and Weston’s characterization in [7] of separable Banach spaces. Finally we examine σ-BG spaces; here the bounded set in the definition of a BG space is replaced by the union of a sequence of bounded sets.

2. Let $A$ be a subset of a linear space $E$. If $k \geq 2$ and $A + A \subseteq kA$, then $A$ is called a $k$-convex set. An l.t.s. which has a base of neighbourhoods of the origin consisting of balanced $k$-convex sets (for some fixed $k$) is called a $k$-convex l.t.s.

Every locally convex space is a $k$-convex l.t.s. for any $k \geq 2$. Also, a locally bounded space (i.e. an l.t.s. which has a bounded neighbourhood) is a $k$-convex l.t.s. for some $k \geq 2$. Thus a $k$-convex l.t.s. need not be a locally convex space. If for some fixed $k$, $E_\alpha$ is a $k$-convex l.t.s. for each $\alpha$ in an index set $\Psi$, then the product space $X(E_\alpha: \alpha \in \Psi)$ is a $k$-convex l.t.s. However the product of a sequence of complete Hausdorff locally bounded spaces need not be a $k$-convex l.t.s. for any $k$ (see for example, p. 179 of [6]).

In [6, p. 170] Simons defines the notion of a $\lambda$-pseudometric. It is clear from [6, Theorem 4] that if $(E, \mu)$ is a $k$-convex l.t.s., then for some $\lambda$, there is a family $(\rho_\gamma)$ of $\mu$-continuous $\lambda$-pseudometrics which give the topology $\mu$.

Let $E$ be a $k$-convex l.t.s. As in [5], let $\Phi(E)$ denote the set of all families $\varphi = (\varphi_\gamma)$ of continuous $\lambda$-pseudometrics (for some fixed $\lambda$) which give the topology of $E$, and if $\varphi = (\varphi_\gamma) \in \Phi(E)$, let

$$S_\varphi = \{x : x \in E, \sup \varphi_\gamma(x) < \infty\}.$$

The proof of the equivalence of (A) and (B) in Theorem 1 of [5] goes through for a $k$-convex l.t.s. if we replace "seminorm" by "$\lambda$-pseudometric" throughout. We use the fact [6, Theorem 6] that if $(\varphi_\gamma) \in \Phi(E)$, then a subset $A$ of $E$ is bounded if and only if $\varphi_\gamma(A)$ is bounded for each $\gamma$.

If we call an l.t.s. which contains a bounded set with dense span, a BG space, we immediately have the following generalization of the corollary of Theorem 1 of [5].

**Theorem 1.** With the notation above, a Hausdorff $k$-convex l.t.s. $E$ is a BG space if and only if there is $\varphi$ in $\Phi(E)$ such that $S_\varphi$ is dense in $E$.

**Lemma 1.** If $A$ is a balanced $k$-convex bounded set in an l.t.s. $(E, \mu)$, then the family $(k^{-n}A : n = 1, 2, \ldots)$ of sets is a base of neighbourhoods for a locally bounded $k$-convex topology $\nu_A$ on the linear span $E_A$ of $A$ which is finer than the $\mu$-induced topology. The space $(E_A, \nu_A)$ is Hausdorff and complete if $(E, \mu)$ is Hausdorff sequentially complete and $A$ is $\mu$-closed.

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For the situation of a locally convex space \((E, u)\), Lemma 1 is well known (see for example the proof of Chapter III, Section 3, No. 4, Lemma 1 of [1]).

It is easy to see that in a \(k\)-convex l.t.s., any bounded set is contained in a balanced \(k\)-convex bounded set.

**Theorem 2.** A (sequentially complete) Hausdorff \(k\)-convex l.t.s. \((E, u)\) is a BG space if and only if there is a one-to-one continuous linear map \(t\) from a (complete) Hausdorff locally bounded \(k\)-convex l.t.s. \(F\) into \((E, u)\) such that \(t(F)\) is \(u\)-dense.

**Proof.** If \((E, u)\) is a BG space, let \(A\) be a balanced \(k\)-convex bounded set which has a dense span. With \((E_A, v_A)\) as in Lemma 1, let \(t\) be the identity map from \((E_A, v_A)\) into \((E, u)\).

Given a subset \(A\) of an l.t.s. \(E\), and \(k \geq 2\) a fixed real number, the intersection \(C\) of the non-empty set of all (closed), balanced \(k\)-convex subsets of \(E\) containing \(A\) is (closed) balanced and \(k\)-convex. The set \(C\) is called the (closed) balanced \(k\)-convex envelope of \(A\).

**Lemma 2.** In a Hausdorff \(k\)-convex l.t.s. \(E\), the balanced \(k\)-convex envelope \(C\) of a precompact set \(A\) is precompact.

**Proof.** Let \(U\) be an open balanced \(k\)-convex neighbourhood in \(E\). Since \(A\) is precompact, there is a finite subset \(B\) of \(E\) such that \(A \subseteq B + U\), and therefore \(C \subseteq B' + U\), where \(B'\) is a compact set, being the closed absolutely convex envelope of the finite set \(B\). As \(B'\) is compact and \(U\) is open, there is a finite subset \(D\) of \(B'\) such that \(C \subseteq D + U\).

Let us call a linear map from one l.t.s. \(G\) into another \(H\) a precompact (compact) map if there is a neighbourhood which is mapped into a precompact (compact) set in \(H\).

Weston in [7] proves that a Banach space \((E, u)\) is separable if and only if there is a one-to-one compact map \(t\) say, from a Banach \(F\) into \((E, u)\) such that \(t(F)\) is \(u\)-dense. It is shown in [2] that this result is still valid if “Banach space” is replaced by “complete Hausdorff locally bounded space”.

**Theorem 3.** A (complete metrizable) metrizable \(k\)-convex l.t.s. \((E, u)\) is separable if and only if there is a one-to-one (compact) precompact linear map \(t\) say, from a (complete) Hausdorff locally bounded \(k\)-convex l.t.s. \(F\) into \((E, u)\) such that \(t(F)\) is \(u\)-dense.

**Proof.** Let \((E, u)\) be a separable metrizable \(k\)-convex l.t.s. and let \((U_n)\) be a shrinking base of \(u\)-neighbourhoods. If \((x_n: n = 1, 2, \ldots)\) is a countable \(u\)-dense subset of \(E\), then for each \(n\), there is a non-zero real number \(a_n\) such that \(a_n x_n \in U_n\). As \((U_n)\) is shrinking, the sequence \((a_n x_n)\) thus converges to zero in \((E, u)\). By Lemma 2, the balanced \(k\)-convex envelope \(A\) of \((a_n x_n: n = 1, 2, \ldots)\) is precompact; its closure is compact if \((E, u)\) is complete. We now apply Lemma 1. With \(F = (E_A, v_A)\), the identity map \(t\) from \(F\) into \((E, u)\) is precompact, being compact if \((E, u)\) is complete.

**Corollary.** A separable infinite dimensional Fréchet space contains a dense subspace on which there is a finer Fréchet space topology.

**Theorem 4.** A complete metrizable \(k\)-convex l.t.s. \((E, u)\) is finite dimensional if and only if \(t(F)\) is closed in \((E, u)\) whenever \(t\) is a continuous linear map from a complete metrizable \(k\)-convex l.t.s. \(F\) into \((E, u)\).
Proof. Suppose first that \((E, u)\) is separable. Then by Theorem 3, there is a one-to-one compact (and therefore continuous) linear map \(t\) say, from a complete metrizable \(k\)-convex l.t.s. \(F\) into \((E, u)\) such that \(t(F)\) is \(u\)-dense. If \(t\) has a closed range, \(t(F) = E\) and \(t\) is a topological isomorphism by Banach's inversion theorem. Therefore \((E, u)\) has a compact neighbourhood and is thus finite dimensional.

If \((E, u)\) is not necessarily separable, let \(E_0\) be a subspace of \(E\) of countable dimension. Let \(E_1\) be the closure of \(E_0\) in \(E\) and let \(u_1\) be the \(u\)-induced topology on \(E_1\). Then \((E_1, u_1)\) is a separable complete metrizable \(k\)-convex l.t.s. If \(t(F)\) is closed in \((E, u)\) whenever \(t\) is a continuous linear map from a complete metrizable \(k\)-convex l.t.s. \(F\) into \((E, u)\), then by the argument above, the dimension of \(E_1\) is necessarily finite. The dimension of \(E\) must then be finite, otherwise, we could choose \(E_0\) as above to have countably infinite dimension.

Corollary. If \(E\) is a Fréchet space and every continuous linear map from any Fréchet space into \(E\) has a closed range then \(E\) is finite dimensional.

Ito and Seidman in [5, p. 287] call a Hausdorff locally convex space \(E\) a HBG space if every closed linear subspace of \(E\) is a BG space. Let \((E, u)\) be a normed linear space of infinite dimension. If \(v\) is the weak topology associated with \(u\), then it follows from [5, Theorem 2(D)] that \((E, v)\) is a BG space. As \((E, v)\) is not quasibarrelled, \((E, v)\) is not the quotient of a product of normed linear spaces.

Cf. [5, p. 287, questions 2 and 3].

3. Let \(E\) be an l.t.s. We call \(E\) a \(\sigma\)-BG space if there is a sequence of bounded sets, the union of which spans a dense subspace of \(E\). Every BG space is a \(\sigma\)-BG space. Also, every separable l.t.s. is a \(\sigma\)-BG space.

If \(E\) is a linear space of countably infinite dimension, then under its finest locally convex topology \(\tau(E, E^*)\), \(E\) is separable (and complete) and therefore the space \((E, \tau(E, E^*))\) is a \(\sigma\)-BG space. As each \(\tau(E, E^*)\)-bounded set is contained in some finite dimensional linear subspace of \(E\), \((E, \tau(E, E^*))\) is not a BG space.

It follows from [1, Ch. III, section 2, exercise 5] that a metrizable \(k\)-convex l.t.s. is a BG space if and only if it is a \(\sigma\)-BG space. The example of Ito and Seidman [5, p. 286] then shows that a Fréchet space need not be a \(\sigma\)-BG space. However as in Theorem 2 of [5], a product of \(\sigma\) (\(\sigma\)-BG) spaces is a \(\sigma\) (\(\sigma\)-BG) space, and the image under a continuous linear map of a \(\sigma\) (\(\sigma\)-BG) space is of the same sort.

For a fixed \(k \geq 2\) and each positive integer \(n\), let \((E_n, u_n)\) be a \(k\)-convex l.t.s. such that \(E_n \subseteq E_{n+1}\). If \(E = \bigcup_n (E_n)\), then there is a finest linear topology \(u\) say, on \(E\) such that each identity map \((E_n, u_n) \to E\) is continuous [4, Definition 2.1]. By an application of Proposition 2.2 of [4], we see that \((E, u)\) is a \(k\)-convex l.t.s., and that if each \((E_n, u_n)\) is locally convex, so is \((E, u)\). The space \((E, u)\) is called the generalized strict \(k\)-convex inductive limit of \((E_n, u_n)\). If in addition, each \(u_n\) coincides with the topology induced on \(E_n\) by \(u_{n+1}\), then \((E, u)\) is called the strict \(k\)-convex inductive limit of \((E_n, u_n)\).

If \((E, u)\) is the strict \(k\)-convex inductive limit of \((E_n, u_n)\), then the topology \(u\) coincides with \(u_n\) on each \(E_n\). (\(E, u\) is Hausdorff if each \((E_n, u_n)\) is [4, Proposition 2.7, Cor. 1], and in this case if each \((E_n, u_n)\) is complete, \((E, u)\) is also complete [4, Proposition 2.8, Cor.], but is not metrizable [4, Proposition 2.9, Cor.].) We shall prove:
Theorem 5. The strict $k$-convex inductive limit of a sequence of complete Hausdorff $k$-convex $\sigma$-BG spaces is also a $\sigma$-BG space.

This theorem will follow immediately from the following result.

Lemma 3. If $(E, u)$ is the strict $k$-convex inductive limit of $(E_n, u_n)$ where each $(E_n, u_n)$ is complete, $k$-convex and Hausdorff, then a subset of $E$ is $u$-bounded if and only if it is contained in some $E_n$ and is $u_n$-bounded.

Proof. Suppose that $A$ is a $u$-closed balanced $k$-convex $u$-bounded set which is not contained in any $E_n$. Then there is a subsequence $(n(i))$ of $(n)$ such that for each $i$, some point of $A \cap E_{n(i)+1}$ is not in $E_{n(i)}$ and $(E, u)$ is the strict $k$-convex inductive limit of $(E_{n(i)}, u_{n(i)})$. Observe that $(E, u)$ is complete and Hausdorff and that each $E_{n(i)}$ is $u$-closed.

As in Lemma 1, let $E_A$ be the linear span of $A$ and $v_A$ the linear topology on $E_A$ with the family $(k^{-m}A : m = 1, 2, \ldots)$ of sets as a base of neighbourhoods. Similarly, let $F_{n(i)}$ be the linear span of $A \cap E_{n(i)}$ and $v_{n(i)}$ the linear topology on $F_{n(i)}$ with the family $(k^{-m}(A \cap E_{n(i)})) : m = 1, 2, \ldots)$ of sets as a base of neighbourhoods. The spaces $(E_A, v_A), (F_{n(i)}, v_{n(i)})$ are complete Hausdorff locally bounded $k$-convex spaces, $F_{n(1)} \subset F_{n(2)} \subset F_{n(3)} \subset \ldots, E_A = \bigcup F_{n(i)}$, and $v_{n(i)}$ coincides with the $v_{n(i+1)}$-induced topology on $F_{n(i)}$. If $(E_A, w)$ is the strict $k$-convex inductive limit of $(F_{n(i)}, v_{n(i)})$, $w$ is finer than the $u$-induced topology on $E_A$, and it follows that the identity map from $(E_A, w)$ onto $(E_A, v_A)$ has a closed graph. By Theorem 4.2 of [3], we see that $v_A = w$, implying that $(E_A, w)$ is metrizable. As this is not possible, the set $A$ must be contained in some $E_n$, and is $u_n$-bounded because $A$ is $u$-bounded and $u$ induces the topology $u_n$ on each $E_n$.

Thus any strict inductive limit of a sequence of Banach or separable Fréchet spaces is a $\sigma$-BG space. Also, if $E$ is the sequence space $l^p(0 < p < 1)$ and $F$ is the algebraic direct sum of countably many copies of $E$, then under the finest linear topology for which the injection maps $E \rightarrow F$ are continuous, $F$ is a $\sigma$-BG space.

There is a parallel to Theorem 2.

Theorem 6. If a (sequentially complete) Hausdorff $k$-convex l.t.s. $(E, u)$ is a $\sigma$-BG space but not a $BG$ space, then there is a one-to-one continuous linear map, $t$ say, from $F$ into $(E, u)$ such that $t(F)$ is $u$-dense, where $F$ is the generalized strict $k$-convex inductive limit of a sequence of (complete) Hausdorff locally bounded spaces.

Proof. Let a Hausdorff $k$-convex l.t.s. $(E, u)$ be a $\sigma$-BG space but not a $BG$ space. Let $(A_n)$ be a sequence of $u$-closed balanced $k$-convex $u$- bounded sets, the union of which spans a dense linear subspace $F$ of $(E, u)$. We may assume that $A_1 \subset A_2 \subset A_3 \subset \ldots$; and since no $A_n$ spans $F$, we may further assume that for each $n$, $A_{n+1} \not\subseteq E_{A_n}$. If $v_{A_n}$ is the topology on $E_{A_n}$ defined as in Lemma 1, let $(F, v)$ be the generalized strict $k$-convex inductive limit of $(E_{A_n}, v_{A_n})$ and let the map $t : F \rightarrow F$ be the identity map.

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