Symmedians of a Triangle and their concomitant Circles

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**NOTATION**

\(A\ B\ C\) = vertices of the fundamental triangle

\(A'\ B'\ C'\) = mid points of \(BC\ CA\ AB\)

\(D\ E\ F\) = points of contact of sides with incircle

\(D_i\ E_i\ F_i\) = points of contact of sides with first excircle

\(G\) = centroid of \(ABC\)

\(I\) = incentre of \(ABC\)

\(I_1\ I_2\ I_3\) = 1st 2nd 3rd excentres of \(ABC\)

\(J\) = quartet of points defined in the text

\(J_1\ J_2\ J_3\) = insymmedian point of \(ABC\)

\(K\) = insymmedian point of \(ABC\)

\(K_1\ K_2\ K_3\) = 1st 2nd 3rd exsymmedian points of \(ABC\)

\(L\ M\ N\) = projections of \(K\) on the sides of \(ABC\)

\(L_i\ M_i\ N_i\) = ,, ,, \(K_i\) ,, ,, ,, And so on

\(O\) = circumcentre of \(ABC\)

\(R\ S\ T\) = feet of the insymmedians

\(R'\ S'\ T'\) = ,, ,, exsymmedians

\(X\ Y\ Z\) = ,, ,, perpendiculars from \(A\ B\ C\)
INTRODUCTORY

DEFINITION. The isogonals * of the medians of a triangle are called the symmedians †

If the internal medians be taken, their isogonals are called the internal symmedians ‡ or the insymmedians; if the external medians be taken, their isogonals are called the external symmedians, or the exsymmedians.

The word symmedians, used without qualification or prefix, may, as in the title of this paper, be regarded as including both insymmedians and exsymmedians (cyclists include both bicyclists and tricyclists); frequently however when used by itself it denotes insymmedians, just as the word medians denotes internal medians.

It is hardly necessary to say that as medians and symmedians are particular cases of isogonal lines, the theorems proved regarding the latter are applicable to the former. Medians and symmedians however have some special features of interest, which are easier to examine and recognise than the corresponding ones of the more general isogonals.

DEFINITION. Two points D D' are isotomic § with respect to BC when they are equidistant from the mid point of BC.

It is a well-known theorem (which may be proved by the theory of transversals) that

If three concurrent straight lines AD BE CF be drawn from the vertices of ABC to meet the opposite sides in D E F, and if D' E' F' be isotomic to D E F with respect to BC CA AB, then AD' BE' CF' are concurrent.

DEFINITION. If O and O' be the points of concurrency of two such triads of lines, then O and O' are called reciprocal points §.

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* See Proceedings of Edinburgh Mathematical Society, XIII. 166-178 (1895)
† This name was proposed by Mr Maurice D'Ocagne as an abbreviation of la droite symétrique de la médiane in the Nouvelles Annales, 3rd series, II 451 (1883). It has replaced the previous name antiparallel median proposed by Mr E. Lenoine in the Nouvelles Annales, 2nd series, XII 364 (1873). Mr D'Ocagne has published a monograph on the Symmedian in Mr De Longchamps's Journal de Mathématiques Élémentaires, 2nd series, IV. 173-175, 193-197 (1885)
‡ The names symédiane intérieure and symédiane extérieure are used by Mr Clément Thiry in Le troisième livre de Géométrie, p. 42 (1887)
§ Mr De Longchamps in his Journal de Mathématiques Élémentaires, 2nd series, V. 110 (1886).
Instead of saying that $D$ and $D'$ are isotomic points with respect to $BC$, it is sometimes said that $AD$ and $AD'$ are isotomic lines with respect to angle $A$.

§1

Construction for an insymmedian

**Figure 12**

Let $ABC$ be the triangle.

Draw the internal median $AA'$ to the mid point of $BC$; and make $\angle BAR = \angle CAA'$.

$AR$ is the insymmedian from $A$.

The angle $CAA'$ is described clockwise, and the angle $BAR$ counter-clockwise; consequently $AA'$ and $AR$ are symmetrically situated with respect to the bisector of the interior angle $BAC$.

Hence since $AA'$ is situated inside triangle $ABC$, $AR$ is inside $ABC$.

The following construction* leads to a simple proof of a useful property of the insymmedians.

**Figure 13**

From $AC$ cut off $AB_1$ equal to $AB$ and $AC_1$ equal to $AC$.

If $B_1C_1$ be drawn, it will intersect $BC$ at $L$, the foot of the bisector of the interior angle $BAC$.

Hence if $AA'$, which is obtained by joining $A$ to the point of internal bisection of $BC$, be the internal median from $A$, the corresponding insymmedian $AR$ is obtained by joining $A$ to the point of internal bisection of $B_1C_1$.

*Mr Maurice D'Oeugne in *Journal de Mathématiques Élémentaires et Spéciales*, IV. 539 (1880). This construction, which recalls Euclid's *pons asinorum*, is substantially equivalent to a more complicated one given by Const. Harkema of St Petersburg in Schlömilch's *Zeitschrift*, XVI. 168 (1871)
(1) BC and B_1C are antiparallel with respect to angle A

(2) Since the internal median AA' bisects internally all parallels to BC, therefore the insymmedian AR bisects internally all antiparallels to BC

(3) The insymmedians of a triangle bisect the sides of its orthic triangle

(4) The projections of B and C on the bisector of the interior angle BAC are P and Q. If through P a parallel be drawn to AB, and through Q a parallel be drawn to AC, these parallels will intersect on the insymmedian from A

[The reader is requested to make the figure]

Let A'' be the point of intersection of the parallels, and A' the mid point of BC

It is well known that A'P is parallel to AC, that A'Q is parallel to AB, and that

\[ A'P = \frac{1}{2}(AC - AB) = A'Q \]

Hence the figure A'PA''Q is a rhombus, and A'' is the image of A' in the bisector of angle A

Now since A' lies on the median from A, A'' must lie on the corresponding symmedian

(5) The three internal medians are concurrent at a point, called the centroid; hence, by a property of Isogonals, the three insymmedians are concurrent at a point

[Other proofs of this statement will be given later on]

Various names have been given to this point, such as minimum-point, Grebe's point, Lemoine's point, centre of antiparallel medians. The designation symmedian point, suggested by Mr Tucker, is the one now most commonly in use.

* Dr Franz Wetzig in Schlomilch's Zeitschrift, XII. 288 (1867)
† Mr Maurice D'Ocagne in the Nouvelles Annales, 3rd series, II. 464 (1883)
‡ See Proceedings of the Edinburgh Mathematical Society, XIII. 39 (1899)
§ Proceedings of the Edinburgh Mathematical Society, XIII. 172 (1895)
|| Educational Times, XXXVII. 211 (1884)
The symmedian point has three points harmonically associated with it; when it is necessary to distinguish it from them, the name insymmedian point will be used.

The insymmedian point and the centroid of a triangle are isogonally conjugate points.

(6) If XYZ be the orthic triangle of ABC the insymmedian points of the triangles AYZ XBZ XYC are situated on the medians* of ABC.

(7) The insymmedian from the vertex of the right angle in a right-angled triangle coincides with the perpendicular from that vertex to the hypotenuse†, and the three insymmedians intersect at the mid point of this perpendicular ‡.

The first part of this statement is easy to establish. The second part follows from the fact that the orthic triangle of the right-angled triangle reduces to the perpendicular.

§ 1’

Construction for an exsymmedian

Figure 14

Let ABC be the triangle
Draw the external median AA* parallel to BC;
and make $\angle BAR' = \angle CAA*$
AR' is the exsymmedian from A
The angle CAA* is described counterclockwise, and the angle BAR' clockwise;
consequently AA* AR' are symmetrically situated with respect to the bisector of the exterior angle BAC.
Hence since AA* is situated outside triangle ABC,
AR' is outside ABC.

The following construction leads to a simple proof of a useful property of the exsymmedians.

* Dr Franz Wetzig in Schlömilch's Zeitschrift, XII. 288 (1867)
† C. Adams's Eigenschaften des... Dreiecks, p. 2 (1846)
‡ Mr Clément Thiry's Le troisième livre de Géométrie, p. 42 (1887)
From CA cut off $AB_1$ equal to $AB$
and $BA_1$, $AC_1$, $AC$
If $B_1C_1$ be drawn, it will intersect $BC$ at $L'$ the foot of the bisector of the exterior angle $BAC$

Hence if $AA_x$, which is obtained by joining $A$ to the point of external bisection of $BC$ (that is, by drawing through $A$ a parallel to $BC$) be the external median from $A$, the corresponding exsymmedian $AR'$ is obtained by joining $A$ to the point of external bisection of $B_1C_1$ (that is, by drawing through $A$ a parallel to $B_1C_1$)

(1') $BC$ and $B_1C_1$ are antiparallel with respect to angle $A$
(2') Since the external median $AA_x$ bisects externally all parallels to $BC$, therefore the exsymmedian $AR'$ bisects externally all antiparallels to $BC$
(3') The exsymmedians of a triangle are parallel to the sides of its orthic triangle
(4') The projections of $B$ and $C$ on the bisector of the exterior angle $BAC$ are $P'$ and $Q'$. If through $P'$ a parallel be drawn to $AB$, and through $Q'$ a parallel be drawn to $AC$, these parallels will intersect on the insymmedian from $A$

The proof follows from the fact that $A'P'$ is parallel to $AC$, that $A'Q'$ is parallel to $AB$, and that

$$A'P' = \frac{1}{2}(AC + AB) = A'Q'$$

(5') The external medians from any two vertices and the internal median from the third vertex are concurrent at a point; hence, by a property of Isogonals, the corresponding exsymmedians and insymmedian are concurrent at a point

[Other proofs of this statement will be given later on]
Three points are thus obtained, and they are sometimes called the \textit{exsymmedian} points.

The three points obtained by the intersections of the external medians of $ABC$ are the vertices of the triangle formed by drawing through $A\ B\ C$ parallels to $BC\ CA\ AB$; that is, they are the points anticomplementary\(^a\) to $A\ B\ C$.

Hence the exsymmedian points of a triangle are \textit{isogonally conjugate} to the anticomplementary points of the vertices of the triangle.

\((6')\) \textit{The tangents to the circumcircle of a triangle at the three vertices are the three exsymmedians of the triangle} \(\dagger\).

\textbf{Figure 14}

For $\angle BAR' = \angle CAA$,

\begin{align*}
&= \angle ACB;
\end{align*}

\text{therefore } AR' \text{ touches the circle } ABC \text{ at } A.

\textbf{(7')} \textit{When the triangle is right-angled two of the exsymmedians are parallel, or they intersect at infinity on the perpendicular drawn from the vertex of the right angle to the hypotenuse.}

\(\S\ 2\)

\textit{The distances of any point in an insymmedian from the adjacent sides are proportional to those sides} \(\ddagger\).  

\textbf{Figure 16}

Let $AA'$ be the internal median, $AR$ the insymmedian from $A$.

From $R$ draw $RV\ RW$ perpendicular to $AC\ AB$;

and $A',\ A'P\ A'Q$.

Then,

\begin{align*}
RW : RV &= A'P : A'Q \\
&= AB : AC
\end{align*}


\(\ddagger\) C. Adams's \textit{Eigenschaften des Dreiecks}, p. 5 (1846).


\[RW : RV = \sin C : \sin B\]
§2'

The distances of any point in an exsymmedian from the adjacent sides are proportional to those sides

**FIGURE 17**

Let $AA_n$ be the external median, $AR'$ the exsymmedian from $A$. From $R'$ draw $R'V' R'W'$ perpendicular to $AC AB$; and from $A_1$ any point in the external median, draw $A_1P' A_1Q'$ perpendicular to $AC AB$.

Then $R'W' : R'V' = A_1P' : A_1Q' = AB : AC$

§3

The segments into which an insymmedian from any vertex divides the opposite side are proportional to the squares of the adjacent sides

**FIGURE 16**

Let $AR$ be the insymmedian from $A$.
Draw $RV RW$ perpendicular to $AC AB$.

Then $AB : AC = RW : RV$

Therefore $AB^2 : AC^2 = AB \cdot RW : AC \cdot RV$

$= ABR : ACR$

$= BR : CR$

Another demonstration, by Mr. Clément Thiry, will be found in *Annuaire Scientifique du Cercle des Normaliens* (published at Gand, no date given), p. 104


$BR : CR = \sin^2 C \cdot \sin^2 B$
§ 3’

The segments into which an exsymmedian from any vertex divides the opposite side are proportional to the squares of the adjacent sides*

**Figure 17**

Let $AR'$ be the exsymmedian from $A$

Draw $R'V$ $R'W$ perpendicular to $AC$ $AB$

Then $AB : AC = R'W : R'V$

therefore $AB^2 : AC^2 = AB \cdot R'W : AC \cdot R'V$

$= ABR' : ACR'$

$= BR' : CR'$

§ 4

The insymmedians of a triangle are concurrent

**First Demonstration**

**Figure 18**

Let $AR$ $BS$ $CT$ be the insymmedians

Then $BR : CR = AB^2 : AC^2$

$CS : AS = BC^2 : BA^2$

$AT : BT = CA^2 : CB^2$

therefore $BR \cdot CS \cdot AT \over CR \cdot AS \cdot BT = -1$

since of the ratios $BR : CR$ $CS : AS$ $AT : BT$ all are negative;

therefore $AR$ $BS$ $CT$ are concurrent

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*C. Adams's *Eigenschaften des ... Dreiecks, pp. 3-4 (1846). Pappus in his Mathematical Collection, VII. 119 gives the following theorem as a lemma for one of the propositions in Apollonius's *Loci Plani*:

If $AB^2 : AC^2 = BR' : CR'$

then $BR' \cdot CR' = AR'^2$

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SECOND DEMONSTRATION

FIGURE 19

Let BK CK the insymmedians from B C cut each other at K:
to prove that K lies on the insymmedian from A

Through K draw

EF' antiparallel to BC with respect to A
FD' ,, ,, CA ,, ,, ,, B
DE' ,, ,, AB ,, ,, ,, C

Because FD' is antiparallel to CA
therefore BK bisects FD'
Similarly CK bisects DE'

Now \( \angle D'DK = \angle A = \angle D'D'K \);
therefore \( KD = KD' \),
therefore \( KD = KD' = KE' = KE' \),

Again \( \angle E'E'K = \angle B = \angle EE'K \);
therefore \( KE = KE' \),
Similarly \( KF = KF' \),
therefore \( KE = KF' \);
therefore K is on the insymmedian from A

THIRD DEMONSTRATION*

FIGURE 20

On the sides of ABC let squares X Y Z be described either all
outwardly to the triangle or all inwardly. Produce the sides of the
squares Y Z opposite to AC and AB to meet in A'; the sides of
the squares Z X opposite to BA and BC to meet in B'; the sides
of the squares X Y opposite to CB and CA to meet in C'

* E. W. Grebe in Gruner’s Archiv, IX. 258 (1847)
Let BB' and CC' meet at K

Then BB' is the locus of points whose distances from AB and BC are in the ratio \( r : a \);
CC' is the locus of points whose distances from AC and BC are in the ratio \( b : a \);
therefore the ratio of the distances of K from AB and AC is \( c : b \)
that is, K lies on AA'.

The eight varieties of position which the squares may occupy relatively to the sides of the triangle may be thus enumerated:

1. X outwardly Y outwardly Z outwardly
2. X inwardly Y inwardly Z inwardly
3. X inwardly Y outwardly Z outwardly
4. X outwardly Y inwardly Z inwardly
5. X outwardly Y inwardly Z outwardly
6. X inwardly Y outwardly Z inwardly
7. X inwardly Y inwardly Z outwardly
8. X outwardly Y outwardly Z inwardly

If the construction indicated in the enunciation of the third demonstration be carried out on these eight figures

1 and 2 will give the insymmedian point K
3, 4, 5, first exsymmedian K₁
6, 7, 8, second K₂
7, 8, third K₃

Now that the existence of the insymmedian point is established,
it may be well to give that property of the point which was the first
to be discovered.

The sum of the squares of the distances of the insymmedian point
from the sides is a minimum*

*“Yanto” in Leybourn's Mathematical Repository, old series, III. 71 (1803).
See Proceedings of the Edinburgh Mathematical Society, XI. 92-102 (1803)
In the identity
\[(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) - (ax + by + cz)^2 = (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2\]
let \(a\ b\ c\) denote the sides of the triangle,
\(x\ y\ z\) the distances of any point from the sides

Then the left side of the identity is a minimum when the right side is a minimum.

But \(a^2 + b^2 + c^2\) is fixed, and so is \(ax + by + cz\), since it is equal to \(2\Delta\); therefore \(x^2 + y^2 + z^2\) is a minimum when the right side is 0.

Now the right side is the sum of three squares, and can only be 0 when each of the squares is 0; therefore
\[bz - cy = cx - az = ay - bx = 0\]
therefore
\[\frac{x}{a} = \frac{y}{b} = \frac{z}{c}\]

Hence the point which has the sum of the squares of its distances from the sides a minimum is that point whose distances from the sides are proportional to the sides.

[The proof here given is virtually that of Mr Lemoine, in his paper communicated to the Lyons meeting (1873) of the Association Française pour l'avancement des Sciences. Another demonstration by Professor Neuberg will be found in Ronché et de Comberousse's Traité de Géométrie, First Part, p. 455 (1891)]

§ 4'

The insymmedian from any vertex of a triangle and the exsymmedians from the two other vertices are concurrent*

First Demonstration

Figure 21

Let \(AR\) be the insymmedian from \(A\), and \(BS'\ CT'\) the exsymmedians from \(B\ C\)

* C. Adams's Eigenschaften des...Dreiecks, pp. 3-4 (1846)
Then \( BR : CR = AB^2 : AC^2 \)
\( CS' : AS' = BC^2 : BA^2 \)
\( AT' : BT' = CA^2 : CB^2 \)
therefore \( \frac{BR}{CR} \cdot \frac{CS'}{AS'} \cdot \frac{AT'}{BT'} = -1 \)

since of the ratios \( BR : CR, CS' : AS', AT' : BT' \) two are positive and one negative;
therefore \( AR, BS', CT' \) are concurrent

Hence also \( AR', BS, CT' \); \( AR', BS', CT \) are concurrent

The points of concurrency of
\( AR, BS', CT' ; AR', BS, CT' ; AR', BS', CT \)
will be called the 1st 2nd 3rd exsymmedian points, and
will be denoted by \( K_1, K_2, K_3 \)

**Second Demonstration**

**Figure 22**

About \( ABC \) circumscribe a circle; draw \( BK_1, CK_1 \) tangents to it at \( B, C \)

Then \( BK_1, CK_1 \) are the exsymmedians from \( B, C \) : to prove \( AK_1 \) to be the insymmedian from \( A \)

Through \( K_1 \) draw \( DE \) antiparallel to \( BC \), and let \( AB, AC \) meet it at \( D, E \)

Then \( \angle BDK_1 = \angle ACB = \angle DBK_1 \);
therefore \( BK_1 = DK_1 \)
Similarly \( CK_1 = EK_1 \);
therefore \( DK_1 = EK_1 \)

therefore \( AK_1 \) is the insymmedian from \( A \)

From this mode of demonstration it is clear that if \( K_1 \) be taken as centre and \( K_1B \) or \( K_1C \) as radius and a circle be described, that circle will cut \( AB, AC \) at the extremities of a diameter

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* Professor J. Neuberg in *Matheesis*, I, 173 (1881)
(1) The insymmedians of a triangle pass through the poles of the sides of the triangle with respect to the circumcircle.

For \( K_1 \) is the pole of \( BC \) with respect to the circumcircle.

(2) The six internal and external symmedians of a triangle meet three and three in four points which are collinear in pairs with the vertices.

**Figure 25**

(3) If triangle \( ABC \) be acute-angled, the points

\[
\begin{align*}
& A & B & C \\
& K_2K_3 & K_3K_1 & K_1K_2;
\end{align*}
\]

and the circle \( ABC \) will be the incircle of triangle \( K_1K_2K_3 \).

If, however, triangle \( ABC \) be obtuse-angled, suppose at \( C \), then the point \( A \) will be situated on \( K_2K_3 \) produced

\[
\begin{align*}
& B & B & K_2K_1 \\
& C & C & K_1K_2;
\end{align*}
\]

and the circle \( ABC \) will be an excircle of triangle \( K_1K_2K_3 \).

**Figure 26**

(4) Hence the relation in which triangle \( ABC \) stands to \( K_1K_2K_3 \) will, if \( ABC \) be acute-angled, be that in which triangle \( DEF \) stands to \( ABC \); or, if \( ABC \) be obtuse-angled, it will be that in which one of the triangles \( D'E_1F_1 \), \( D'E_2F_2 \), \( D'E_3F_3 \) stands to \( ABC \).

(5) If \( DEF \) be considered as the fundamental triangle, then \( A \), \( B \), \( C \) are the first, second, and third exsymmedian points, and the concurrent triad \( AD \) \( BE \) \( CF \) meet at the insymmedian point of \( DEF \).

If \( D'E_1F_1 \) be considered as the fundamental triangle, then \( A \), \( B \), \( C \) are the first, second, and third exsymmedian points, and the concurrent triad \( AD \) \( BE \) \( CF \) meet at the insymmedian point of \( D'E_1F_1 \).

Similarly for triangles \( D'E_2F_2 \), \( D'E_3F_3 \).

(6) The points of concurrency* of the triads.

*The concurrency may be established by the theory of transversals.
AD BE CF
AD₁ BE₁ CF₁
AD₂ BE₂ CF₂
AD₃ BE₃ CF₃

Γ being called* the Gergonne point of ABC, and Γ₁ Γ₂ Γ₃ the associated Gergonne points

Hence the Gergonne point and its associates are the insymmedian points of the four DEF triangles

(7) With respect to BC
D and D₁ are isotomic points, so are D₂ and D₃; and a similar relation holds for the E points with respect to CA, and for the F points with respect to AB. Hence the triads

AD₁ BE₂ CF₃
AD BE CF₂
AD₃ BE CF₁
AD₂ BE₁ CF

which are concurrent† at

J
J₁
J₂
J₃

furnish the four pairs of reciprocal points,

Γ Γ₁ Γ₂ Γ₃ (Gergonne points)
J J₁ J₂ J₃ (Nagel points)

(8) Since AD passes through J₁
BE " " J₂
CF " " J₂

therefore Γ is situated on each of the straight lines AJ₁ BJ₂ CJ₃; in other words, the triangles ABC J₁J₂J₃ are homologous and have Γ for centre of homology

* By Professor J. Neuberg. J. D. Gergonne (1771-1859) was editor of the Annales de Mathématiques from 1810 to 1831
† Many of the properties of the J points were given by C. H. Nagel in his Untersuchungen über die wichtigsten zum Dreiecke gehörigen Kreise (1836). This pamphlet I have never been able to procure. Since 1836 some of these properties have been rediscovered several times
Since AD, passes through \( \Gamma_1 \)
\[ BE_2 \; \Gamma_2 \]
\[ CF_3 \; \Gamma_3 ; \]
therefore \( J \) is situated on each of the straight lines \( \Delta\Gamma_1, \Delta\Gamma_2, \Delta\Gamma_3 \); in other words, the triangles \( \Delta ABC, \Gamma_1\Gamma_2\Gamma_3 \) are homologous and have \( J \) for centre of homology.

Similarly \( \Delta ABC \) is homologous with
\[ J_1 J_2 J_3 \; J_0 J_1 J_2 J_3 J \]
the centres of homology being respectively
\[ \Gamma_1 \; \Gamma_2 \; \Gamma_3 ; \]
and \( \Delta ABC \) is homologous with
\[ \Gamma \; \Gamma_2 \Gamma_3 \; \Gamma_3 \Gamma_1 \; \Gamma_2 \Gamma_1 \Gamma \]
the centres of homology being respectively
\[ J_1 \; J_2 \; J_3 \]

(9) If \( \Gamma', \Gamma'_1 \) \... \ are the points of concurrency of lines drawn from \( A', B', C' \), the mid points of the sides, parallel to the triads of angular transversals which determine the points
\[ \Gamma' \; \Gamma'_1 \; \Gamma'_2 \; \Gamma'_3 \]
then
\[ \Gamma \Gamma' \; \Gamma'_1 \Gamma'_1 \; \Gamma'_2 \Gamma'_2 \; \Gamma'_3 \Gamma'_3 \]
are concurrent at the centroid of \( \Delta ABC \).

The points \( \Gamma', \Gamma'_1 \) \... as belonging to triangle \( A'B'C' \) correspond to the points \( \Gamma \; \Gamma_1 \) \... as belonging to triangle \( \Delta ABC \); hence as \( \Delta ABC, A'B'C' \) are similar and oppositely situated and have \( G \) for their homothetic centre, \( \Gamma \Gamma', \Gamma_1 \Gamma'_1 \) \... pass through \( G \).

(10) The \( \Gamma' \) points are complementary to the \( \Gamma \) points, and the tetrad
\[ \Pi' \; I_1 \Gamma'_1 \; I_2 \Gamma'_2 \; I_3 \Gamma'_3 \]
are concurrent at the insymmedian point* of \( \Delta ABC \).

* William Godward in the *Lady's and Gentleman's Diary* for 1867, p. 63. He contrasts this point, in reference to one of its properties, with the centroid of \( \Delta ABC \), and recognises it as the point determined by Mr Stephen Watson in 1865. See § 8 (2) of this paper.
(11) The J points are anticomplementary to the I points, and the tetrad* 
\[ \text{IJ} \quad \text{I}_1\text{J}_1 \quad \text{I}_2\text{J}_2 \quad \text{I}_3\text{J}_3 \]
are concurrent at G the centroid of ABC

(12) J₁, J₂, J₃, J form an orthic tetrastigm *

**Figure 26**

(13) A₁I₁B₁C₁I₁ intersect E₁F₁F₁D₁D₁E₁ at the feet of the medians of triangle DEF;
AD₁BE₁CF₁ intersect E₁F₁F₁D₁D₁E₁ at the feet of the internal symmedians
A₁I₁B₁C₁I₁ intersect E₂F₂F₂D₂D₂E₂ at the feet of the medians of triangle D₁E₁F₁;
AD₁BE₁CF₁ intersect E₂F₂F₂D₂D₂E₂ at the feet of the internal symmedians
Similarly for triangles D₂E₂F₂, D₃E₃F₃

(14) The external symmedians of any triangle are also the external symmedians of three other associated triangles

**Figure 26**

Let DEF be the triangle
Circumscribe a circle about DEF, and draw tangents to it at D E F. Let these tangents intersect at A B C. Then D₁E₁F₁, D₂E₂F₂ D₃E₃F₃ are the three triangles associated with DEF
To determine their vertices it is not necessary to find I₁, I₂, I₃ and to draw perpendiculars to BC CA AB
Make CD₁ = BD, CE₁ = CD₁ and BF₁ = BD₁ and the triangle D₁E₁F₁ is determined
Similarly for D₂E₂F₂, D₃E₃F₃

*William Godward in *Mathematical Questions from the Educational Times* II, 87, 88 (1865)
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(15) \( I_A' I_B' I_C' \) are concurrent* at the insymmedian point of \( I_1I_2I_3 \)

**Figure 26**

For BC is antiparallel to \( I_2I_3 \) with respect to \( \angle I_2I_1I_3 \) and \( A' \) is its mid point;
therefore \( I_1A' \) is the insymmedian of \( I_1I_2I_3 \) from \( I_1 \)
Similarly \( I_2B' \) " " " " I_2 and \( I_3C' \) " " " " I_3

\[
\begin{align*}
(I_A') & & (I_B') & & (I_C') \\
(I_2A') & & (I_2B') & & (I_2C') \\
(I_3A') & & (I_3B') & & (I_3C')
\end{align*}
\]

are concurrent †
respectively at the insymmedian points of the triangles
\( I_1I_2I_3 \) \( I_1I_1I_1I_1 \) \( I_1I_3I_3 \)

§ 5

The internal and external symmedians from any vertex are conjugate harmonic rays with respect to the sides of the triangle which meet at that vertex ‡

**Figure 25**

For \( BR : CR = AB^2 : AC^2 \)
\[ = BR' : CR' \]
therefore \( B R C R' \) form a harmonic range
and \( AR \) \( AR' \) are conjugate harmonic rays with respect to \( AB \) \( AC \)

* Geometricus (probably Mr William Godward) in *Mathematical Questions from the Educational Times*, III. 29-31 (1865). The method of proof is not his.

Mr W. J. Miller adds in a note that \( I_1A' \) divides \( I_2I_3 \) into parts which have to one another the duplicate ratio of the adjacent sides of the triangle \( I_1I_2I_3 \) and similarly for \( I_2B' \) \( I_3C' \); and that the point of concurrency is such that the sum of the squares of the perpendiculars drawn therefrom on the sides of the triangle \( I_1I_2I_3 \) is a minimum, and these perpendiculars are moreover proportional to the sides on which they fall.

† Professor Johann Döttl in his *Neue merkwürdige Punkte des Dreiecks*, p. 14 (no date) states the concurrency, but does not specify what the points are.

‡ C. Adams’s *Eigenschaften des...Dreiecks*, p. 5 (1846)
Hence also for C S A S' and A T B T'

(1) The following triads of points are collinear:
   \[ R'ST; R'S'T; RST'; R'S'T' \]

(2) The following are harmonic ranges *
   \[ A K R K_1; B K S K_2; C K T K_2 \]
   \[ A K_2 R' K_2; B K_1 S' K_2; C K_1 T' K_2 \]

For B R C R' is a harmonic range;
therefore A.BR.CR' is a harmonic pencil;
and its rays are cut by the transversals BKSK₂ and B K₁ S' K₂;
therefore B K S K₂ B K₁ S' K₃ are harmonic ranges

(3) If D E F be the points in which AK BK CK cut the
circumcircle of ABC, then the following are harmonic ranges
   \[ A R D A' \]
   \[ B S E A' \]
   \[ C T F K_1 \]

FIGURE 25

For K₁B K₁C are tangents to the circle ABC, and K₁DRA is a
secant through K₁;
therefore this secant is cut harmonically † by the chord of contact
BC and the circumference

(4) \( R' \) is the pole of \( AK \), with respect to the circumcircle ‡

Since \( AR' \) is the tangent at A
therefore \( AR' \) is the polar of A
Now \( BR' \) " " " " K₁;
therefore \( R' \) is the pole of \( AK₁ \)
Similarly for \( S' \) and \( T' \)

(5) \( R'D \) S'E T'F are tangents to the circumcircle
For AD is the polar of \( R' \) with respect to the circumcircle, that
is, AD is the chord of contact of the tangents from \( R' \)

---

* The first of these is mentioned by Dr Franz Wetzig in Schlömilch's Zeitschrift, XII. 289 (1867)
† This is one of Apollonius's theorems. See his Conics, Book III., Prop. 37-40
‡ C. Adams's Eigenschaften des... Dreiecks, pp. 3-4 (1846)
(6) The straight line $R'S'T''$ is the polar of $K$ with respect to the circumcircle

For $AK_1$, $BK_2$, $CK_3$ pass through $K$; therefore their respective poles $R'$, $S'$, $T'$ will lie on the polar of $K$.

(7) $R'S'T'$ is perpendicular to $OK$, and its distance from $O$ is equal to $R^2/OK$, where $R$ denotes the radius of the circumcircle. $R'S'T'$ is sometimes called Lemoine's line.

(8) $R'S'T'$ is the trilinear polar of $K$, or it is the line harmonically associated with the point $K$.

For $ST$, $TR$, $RS$ meet $BC$, $CA$, $AB$ at $R'$, $S'$, $T'$; therefore $R'S'T'$ is the trilinear polar of $K$.

(9) The three triangles $ABC$, $RST$, $K_1K_2K_3$ taken in pairs will have the same axis of homology, namely the trilinear polar of $K$.

(10) The following triads of points are collinear

$R'E'F'$; $S'F'D'$; $T'D'E'$

For $BCEF$ is an encyclic quadrilateral, and $BE$, $CF$ intersect at $K$; therefore $EF$ intersects $BC$ on the polar of $K$. Now the polar of $K$ intersects $BC$ at $R'$; therefore $EF$ passes through $R'$.

(11) If $BF$, $CE$ intersect at $D'$

$CD$, $AF$ " " $E'$

$AE$, $BD$ " " $F'$

then $A$, $D$, $D'$; $B$, $E$, $E'$; $C$, $F$, $F'$ are collinear; and so are $D'$, $E'$, $F'$.

Since $BE$, $CF$ intersect at $K$; therefore $BF$, $CE$ intersect at a point on the polar of $K$.

Similarly for $CD$, $AF$ and for $AE$, $BD$; therefore $D'$, $E'$, $F'$ are collinear.

*Mr J. J. A. Mathieu in Nouvelles Annales, 2nd series, IV. 404 (1865)
Again BCEF is an encyclic quadrilateral, and
BE CF intersect at K
BC EF " " R'
BF CE " " D';
therefore triangle KR'D' is self-conjugate with respect to the
circumcircle;
therefore KD' is the polar of R'
But AK " " " " R';
therefore A D D' are collinear

(12) The following triads of lines are concurrent:
AK BF CE; BK CD AF; CK AE BD
at D' ; E' ; F'
and D' E' F' are situated on R'S'T'

(13) The straight lines which join the mid point of each side of a
triangle to the mid point of the corresponding perpendicular of the
triangle are concurrent at the insymmedian point

\textbf{Figure 23}

Let K K₁ be the insymmedian and first exsymmedian points of
ABC;
let A' be the mid point of BC and let A'K meet the perpendicular
AX at P₁
Join A'K₁
Then A'K₁ is parallel to AX
Now since A K R K₁ is a harmonic range
therefore A'.AKR₁ is a harmonic pencil;
therefore AX which is parallel to the ray A'K₁ is bisected by the
conjugate ray A'K

§ 6

\textit{If L M N be the projections of K on the sides, then K is the}
centroid \( \dagger \) of triangle LMN

* Dr Franz Wetzig in Schlömilch's Zeitschrift, XII. 289 (1867)
\( \dagger \) E. W. Grebe in Grunert's Archiv, IX. 253 (1847)
Through L draw a parallel to MK, meeting NK produced in K'  
Join K'M  
Then triangle KLK' has its sides respectively perpendicular  
 to BC CA AB:  
therefore KL : LK' = BC : CA  
But  
KL : KM = BC : CA  
therefore LK' = KM;  
therefore KLK'M is a parallelogram;  
therefore KK bisects LM,  
that is, KN is a median of LMN  
Similarly KL KM are medians;  
therefore K is the centroid of LMN  

Another demonstration by Professor Neuberg will be found in Mathesis,  
I. 173 (1881)  

(1) The sides of LMN are proportional to the medians of ABC,  
and the angles of LMN are equal to the angles which the medians of  
ABC make with each other*  
Since KL KM KN are two-thirds of the respective medians  
of LMN, and are proportional to BC CA AB;  
therefore the medians of LMN are proportional to BC CA AB;  
therefore the sides of LMN are proportional to the medians of ABC  

See Proceedings of the Edinburgh Mathematical Society, I. 26 (1894)  
The second part of the theorem follows from (37) on p. 25 of  
the preceding reference, and from the fact that the angles  

CGB' AGC' BGA'  
are respectively equal to the angles  

CBG + GCB ACG + GAC BAG + GBA  

Or it may be proved as follows:  
Since G is the point isogonally conjugate to K, therefore  
AG BG CG are respectively perpendicular to MN NL LM  

See Proceedings of the Edinburgh Mathematical Society, XIII. 178 (1895)  

* Dr Franz Wetzig in Schlimilch's Zeitschrift, XII. 297 (1867)
If LMN be considered as the fundamental triangle, K its centroid, and if at the vertices L M N perpendiculars be drawn to the medians KL KM KN a new triangle ABC is formed having K for its insymmedian point.

The sum of the squares of the sides of the triangle LMN inscribed in ABC is less than the sum of the squares of the sides of any other inscribed triangle.*

The proof of this statement depends on the following lemma:

Given two fixed points M N and a fixed straight line BC; that point L on BC for which NL² + LM² is a minimum is the projection on BC of the mid point of MN.

If through every two vertices and the centroid of a triangle circles be described, the triangle formed by joining their centres will have for centroid and insymmedian point the circumcentre and the centroid of the fundamental triangle†.

* Mr. Emile Lemoine in the Journal de Mathématiques Élémentaires, 2nd series, III. 52.3 (1884).
† This theorem and the proof of it have been taken from Professor W. Fuhrmann's Synthetische Beweise planimetrischer Sätze, pp. 101-2 (1890).
Again if through A B C perpendiculars be drawn to the medians AG BG CG these perpendiculars will form a triangle UVW whose vertices will be situated on the circumferences* of $O_1, O_2, O_3$ and which will be similar to the triangle $O_1O_2O_3$. Also the triangles UVW $O_1O_2O_3$ have G for their centre of similitude.

Now triangle UVW has G for its insymmedian point; therefore G is also the insymmedian point of triangle $O_1O_2O_3$.

\[(5)\text{ If } L_1 M_1 N_1, L_2 M_2 N_2, L_3 M_3 N_3\text{ be the projections on } BC, CA, AB\text{ of}
\]

\[K_1, K_2, K_3\text{ then } K_1N_1L_1M_1, K_2L_2M_2N_2, K_3M_3N_3L_2\text{ are parallelograms}^\dagger\]

**Figure 40**

The points $K_1, L_1, B, N_1$ are concyclic therefore $\angle L_1K_1N_1 = \angle ABC$.

The points $K_1, M_1, C, L_1$ are concyclic therefore $\angle K_1L_1M_1 = \angle K_1C M_1$

\[= \angle K_2C A\]

\[= \angle ABC\]

therefore $K_1N_1$ is parallel to $L_1M_1$.

Similarly $K_1M_1, \ldots, L_1N_1$.

The point $K_1$ is the first of the three points harmonically associated with the centroid of the triangle $L_1M_1N_1$; the point $K_2$ is the second of the three points harmonically associated with the triangle $L_2M_2N_2$; and the point $K_3$ is the third of the three points harmonically associated with the triangle $L_3M_3N_3$.

* For proof of some of the statements made here, see Proceedings of the Edinburgh Mathematical Society, I. 36-7 (1894)

† Dr Franz Wetzig in Schlömilch's Zeitschrift, XII. 298 (1867)
§ 7

If $AK$, $BK$, $CK$ be produced to meet the circumcircle in $D$, $E$, $F$ the triangle $DEF$ has the same insymmedians as $ABC$

**FIRST DEMONSTRATION**

**FIGURE 29**

From $K$ draw $KL$, $KM$, $KN$ perpendicular to $EC$, $CA$, $AB$ and join $MN$, $NL$, $LM$

Since the points $B$, $L$, $K$, $N$ are concyclic therefore \[ \angle KLN = \angle KBN = \angle EBA = \angle EDA \]

Since the points $C$, $L$, $K$, $M$ are concyclic therefore \[ \angle KLM = \angle KCM = \angle FCA = \angle FDA \]

Hence \[ \angle MLN = \angle EDF \]

Similarly \[ \angle LMN = \angle DEF \text{ or } \angle MNL = \angle EFD \]

and triangles $LMN$, $DEF$ are directly similar

But since \[ \angle KLN = \angle KDE \]

and \[ \angle KLM = \angle KDF \]

therefore the point $K$ in triangle $LMN$ corresponds to its isogonally conjugate point in triangle $DEF$

Now $K$ is the centroid of triangle $LMN$; therefore $K$ is the insymmedian point of triangle $DEF$

**SECOND DEMONSTRATION**

**FIGURE 25**

Let $AR'$, $BS'$, $CT'$ be the exsymmedians.

Since $AK$ is the polar of $R'$, and $BC$, $EF$ both pass through $R'$ not only will the tangents to the circumcircle at $B$, $C$ meet on the polar of $R'$ but also the tangents at $E$, $F$

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But the tangents at E F meet on the insymmedian of DEF from D; therefore the insymmedian AD is common to triangles ABC DEF

Similarly for the insymmedians BE CF.

The cosymmedian triangles ABC DEF are homologous, the insymmedian point K being their centre of homology, and R'S'T' their axis of homology.

(1) If two triangles be cosymmedian the sides of the one are proportional to the medians of the other.*

For triangle DEF is similar to triangle LMN

Or thus:

Let G be the centroid of ABC

Join GB GC

Then \[ \angle EDF = \angle EDA + \angle ADF \]
\[ = \angle KBA + \angle KCA \]
\[ = \angle GBC + \angle GCB \]

since the points G K are isogonally conjugate.

Similarly \[ \angle DEF = \angle GCA + \angle GAC \]
and \[ \angle EFD = \angle GAB + \angle GAB \]

A reference to the Proceedings of the Edinburgh Mathematical Society, I. 25 (1894) will show that this proves the theorem.

(2) The ratio of the area of ABC to that of its cosymmedian triangle DEF is \[ \frac{(-a^2 + 2b^2 + 2c^2)(2a^2 - b^2 + 2c^2)(2a^2 + 2b^2 - c^2)}{27a^2b^2c^2} \]

Let \( \triangle' \) be the triangle whose sides are the medians of ABC and which is similar to DEF; and let R' be the radius of its circumcircle.

Then \[ \triangle' = \frac{3}{4} \triangle ABC = \frac{3}{4} \]
and \[ \frac{\triangle}{\triangle'} = R' : R'' \]

* Dr John Casey in Proceedings of the Royal Irish Academy, 2nd series, IV. 546 (1886)
† Rev. T. C. Simmons in Milne's Companion to the Weekly Problem Papers, p. 150 (1888)
Hence
\[
\text{DEF} = \frac{3 \Delta R^2}{4 R'^2} = \frac{3 \Delta}{4} \cdot \frac{a^2 b^2 c^2}{16 \Delta^2} \cdot \left(\frac{3 \Delta}{m_1 m_2 m_3}\right)^2
\]
\[
= \frac{27 \Delta a^2 b^2 c^2}{64 m_1^2 m_2^2 m_3^2}
\]

The values of \(m_1, m_2, m_3\) are given in the Proceedings of the Edinburgh Mathematical Society, I. 29 (1894)

(3) If BD CD be joined, DR DR' are an insymmedian and an exsymmedian of triangle * DCB

**Figure 24**

Draw \(\Delta A_1\) parallel to BC to meet the circumcircle at \(A_1\) and let \(A_1 K_1\) meet the circle at \(D_1\).

Then triangle \(A_1 CK_1\) is congruent to \(A_1 B K_1\)

therefore \[\angle C A D = \angle B A_1 D_1\]

therefore \(D_1 D\) is parallel to BC

Now since BC is the polar of \(K_1\) and \(\Delta A_1\) DD\(_1\) are parallel,
then \(\Delta D A_1\) \(A_1 D\) intersect on BC at its mid point \(A'\)

Again \[\angle C D R = \angle C D A\]
\[= \angle B D A_1\]
\[= \angle B D A'\]

therefore DR is isogonal to the median DA'.

But DR' is a tangent to the circumcircle at D;
then DR' is an insymmedian

(4) Hence BR BK\(_1\) are an insymmedian and an exsymmedian of triangle BDA;
CR CK\(_1\) an insymmedian and an exsymmedian of triangle CAD

(5)

\[A R'^2 + B K_1^2 = K_1 R'^2\]

Let O be the centre of the circumcircle ABC

* C. Adams in his *Eigenschaften des Dreiecks*, pp. 4-5 (1846) gives (3)—(7)
Then \[ AR'^2 = OR'^2 - OA^2 \]
\[ = OA'^2 + A'R'^2 - OA^2 \]
\[ BK_1^2 = A'B^2 + A'K_1^2 \]
\[ = OB^2 - OA'^2 + A'K_1^2 \]

Therefore \[ AR'^2 + BK_1^2 = A'R'^2 + A'K_1^2 \]
\[ = K_1R'^2 \]

(6) \textit{OR is perpendicular to } K_1R' \textit{.}

For \( R' \) is the pole of \( AK_1 \)
and \( K_1 \), """, """, BC
therefore \( K_1R' \) is the polar of R
therefore \( OR \) is perpendicular to \( K_1R' \)

(7) \textit{AR' is a mean proportional between } A'R' \textit{ and } RR' \textit{.}

Since \( B \ R \ C \ R' \) form a harmonic range,
and \( A' \) is the mid point of BC
therefore \[ B \ R' : A'R' = RR' : CR' \]
therefore \[ A'R' \cdot RR' = BR' \cdot CR' \]
\[ = AR'^2 \]

(8) \[ AB \cdot CD = AC \cdot BD = \frac{1}{2} AD \cdot BC \]

\textbf{Figure 24}

For \[ AB^2 : AC^2 = BR : CR \]
\[ = BD^2 : CD^2 \]
therefore \[ AB : AC = BD : CD \]

The last property follows from Ptolemy's theorem

that \[ AB \cdot CD + AC \cdot BD = AD \cdot BC \]

(9) The distances of R from the four sides of the quadrilateral ABDC are proportional to those sides.
This follows from § 2

\textbf{Definition.} The four points A B D C form a harmonic system of points on the circle; and hence ABDC is called a \textit{harmonic quadrilateral}. 

https://doi.org/10.1017/50013091500031758 Published online by Cambridge University Press
This name was suggested to Mr Tucker by Professor Neuberg in 1885.

The first systematic study of harmonic quadrilaterals was made by Mr Tucker. In his article "Some properties of a quadrilateral in a circle, the rectangles under whose opposite sides are equal," read to the London Mathematical Society on 12th February 1885, he states that in his attempt to extend the properties of the Brocard points and circle to the quadrilateral he "was brought to a stand at the outset by the fact that the equality of angles does not involve the similarity of the figures for figures of a higher order than the triangle. Limiting the figures, however, by the restriction that they shall be circumscribable" he arrived at a large number of beautiful results all of which cannot unfortunately be given here.

Starting with the encyclic quadrilateral ABCD whose diagonals intersect at E, and investigating the condition that a point P can be found such that

\[ \angle PAB = \angle PBC = \angle PCD = \angle PDA \]

he finds, by analytical considerations, that a condition for the existence of such a point is that the rectangles under the opposite sides of the quadrilateral must be equal. He then shows that if there be one Brocard point P for the quadrilateral there will be a second P'; that the lines

\[ PA \; PB \; PC \; PD \; ; \; P'A \; P'B \; P'C \; P'D \]

intersect again in four points which, with P P' lie on the circumference of a circle with diameter OE, where O is the centre of the circle ABCD.

Next, if through E parallels be drawn to the sides of the quadrilateral, these parallels will meet the sides in eight points which lie on a circle concentric with the previous one.

Lastly he shows that the symmedian points \((\rho_1, \rho_2)\) of ABD BCD lie on AC; the symmedian points \((\sigma_1, \sigma_2)\) of ABC ADC lie on BD; the lines \(O\rho_1\; O\rho_2\; O\sigma_1\; O\sigma_2\) are the diameters of the Brocard circles of the triangles ABD BCD ABC ACD respectively; the centres of the four Brocard circles lie two and two on straight lines, parallel to AC BD; the circles themselves intersect two and two on the diagonals AC BD at their mid points, that is, where the Brocard circle of the quadrilateral meets the diagonals.

Mr Tucker's researches were taken up by Messrs Neuberg and...
Tarry, Dr Casey, and the Rev. T. C. Simmons, and there now exists a tolerably extensive theory of harmonic polygons. The reader who wishes to pursue this subject may consult


Professor Neuberg *Sur le Quadrilatère Harmonique* in *Mathesis*, V. 202–204, 217–221, 241–248, 265–269 (1885)

Dr John Casey’s memoir (read 26th January 1886) “On the harmonic hexagon of a triangle” in the *Proceedings of the Royal Irish Academy*, 2nd series, Vol. IV. pp. 545–556

A memoir by Messrs Gaston Tarry and J. Neuberg *Sur les Polygones et les Polyèdres Harmoniques* read at the Nancy meeting (1886) of the *Association Française pour l’avancement des sciences*. See the Report of this meeting, second part, pp. 12–26


Dr Casey’s *Sequel to the First Six Books of the Elements of Euclid*, 6th edition, pp. 220–238 (1892)

§ 8

**The Cosine or Second Lemoine Circle**

If through the insymmedian point of a triangle, antiparallels be drawn to the three sides, the six points in which they meet the sides are concyclic

**Figure 19**

Let K be the insymmedian point of ABC

and through K let there be drawn $EF'$ $FD'$ $DE'$ respectively antiparallel to BC CA AB

the D points being on BC, the E's on CA, the F's on AB

Then $EF'$ $FD'$ $DE'$ are each bisected at K

Now $\angle KDD' = \angle A = \angle KD'D$

therefore $KD = KD'$ and $DE' = FD'$

Hence also $DE' = EF'$

therefore K is equidistant from the six points D D' E E' F F'
[This theorem was first given by Mr Lemoine at the Lyons meeting (1873) of the Association Française pour l'Avancement des sciences, and the circle determined by it has hence been called one of Lemoine's circles (the second).

The existence of the circle however, and the six points through which it passes were discovered by Mr Stephen Watson of Haydonbridge in 1865, and its diameter expressed in terms of the sides of the triangle. See Lady's and Gentleman's Diary for 1865, p. 89, and for 1866, p. 55

In the same publication Mr Thomas Milbourn in 1867 announced a neat relation connecting the diameter of this circle with the diameter of the circum-circle, and here, as far as the Diary is concerned, the inquiry seemed to have stopped]

(1) The figures DD'E'F EE'F'D FF'D'E are rectangles *

It may be interesting to give the way in which these three rectangles made their first appearance.

(2) Three rectangles may be inscribed in any triangle so that they may have each a side coincident in direction with the respective sides of the triangle, and their diagonals all intersecting in the same point, and one circle may be circumscribed about all the three rectangles †

**Figure 30**

Let ABC be the triangle

Draw AX perpendicular to BC;

and produce CB to Q making BQ equal to CX

About ACQ circumscribe a circle cutting AB at P

Join PC; and draw BE parallel to PC and meeting AC at E

From E draw ED' parallel to AB and EF perpendicular to AB,

and let these lines meet BC AB at D' F

About D'EF circumscribe a circle cutting BC CA AB again in D E' F'. The six points D D' E E' F F' are the vertices of the required rectangles

Draw CZ perpendicular to AB,

and let ED' meet CP at R

The similar triangles ABX CBZ give

\[ \frac{AX}{CZ} = \frac{AB}{CB} = \frac{BQ}{BP} = \frac{CX}{BP} \]

therefore

\[ \frac{AX}{CX} = \frac{CZ}{BP} \]

* Mr Lemoine at the Lille meeting (1874) of the Association Française pour l'Avancement des sciences

† Mr Stephen Watson in the Lady’s and Gentleman's Diary for 1865, p. 89, and for 1866, p. 55
But \[ EF : CZ = AE : AC \]
\[ = AB : AP \]
\[ = ED' : ER \]
\[ = ED' : BP ; \]
therefore \[ EF : ED' = CZ : BP ; \]
therefore \[ AX : CX = EF : ED' ; \]
therefore the right-angled triangles \( AXC \) and \( FED' \) are similar

Hence \[ \angle XAC = \angle EFD' = \angle D'E'E ; \]
therefore \( D'E' \) is parallel to \( AX \)

Now \[ \angle FE'D' = \angle FED' = \text{a right angle} ; \]
therefore \( FE' \) is parallel to \( BC \),
and \( DD'E'F \) is one of the rectangles

Again because \( \angle EFF' \) is right,
therefore \( EF' \) is a diameter ;
therefore \( \angle F'D'E \) is right, as well as \( \angle F'DE \) and \( \angle F'E'E ; \)
therefore \( EE'F'D \) and \( FF'D'E \) are the other rectangles

(3) To find the diameter* of the circle \( DEF \)

**Figure 30**

\[
AB^2 = BC^2 + CA^2 - 2BC \cdot CX \\
= BC^2 + CA^2 - 2BC \cdot BQ \\
= BC^2 + CA^2 - 2AB \cdot BP ;
\]
therefore \[ 2AB \cdot BP = BC^2 + CA^2 - AB^2 \]
Add \( 2AB^2 \) to both sides ;
then \[ 2AB \cdot AP = BC^2 + CA^2 + AB^2 \]
therefore \[ AP = \frac{a^2 + b^2 + c^2}{2c} \]

But \[ EF : CZ = AB : AP \]
therefore \[ AP = \frac{AB \cdot CZ}{EF} = \frac{2\triangle}{EF} \]

* Mr Stephen Watson in the Lady's and Gentleman's Diary for 1866, p. 55
Hence \[ \frac{2 \Delta}{EF} = \frac{a^2 + b^2 + c^2}{2c} \]

and \[ EF = \frac{4c \Delta}{a^2 + b^2 + c^2} \]

Lastly \[ D'F : EF = AC : AX \]

therefore \[ D'F : \frac{4c \Delta}{a^2 + b^2 + c^2} = b : \frac{2 \Delta}{a} \]

therefore \[ D'F = \frac{2abc}{a^2 + b^2 + c^2} = \text{the diameter} \]

The following is another proof

**Figure 19**

Triangles AEF' ABC are similar
and AK is a median of AEF';
therefore \[ EF' : AK = BC : m_1 \]
therefore \[ EF = \frac{AK \cdot BC}{m_1} \]

\[ = \frac{2abc}{a^2 + b^2 + c^2} \]

For the value of AK, namely, \[ \frac{2abm_1}{a^2 + b^2 + c^2} \]
see Formulae connected with the Symmedians, at the end of this paper.

(4) *If d denote the diameter of circle DEF,*
and \[ D "]  "  "  "  "  "  \ A B C, 
then* \[ \frac{1}{d^2} + \frac{1}{D^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \]

For \[ \frac{1}{d} = \frac{a^2 + b^2 + c^2}{2abc} \]
\[ D = \frac{4 \Delta}{2abc} \]

therefore \[ \frac{1}{d^2} + \frac{1}{D^2} = \frac{(a^2 + b^2 + c^2)^2 + (4 \Delta)^2}{4a^2b^2c^2} \]

* Mr Thomas Millbourn in the *Lady's and Gentleman's Diary* for 1867, p. 71, and for 1868, p. 75
\[
\frac{(a^2 + b^2 + c^2)^2 + 4c^2a^2 - (a^2 - b^2 + c^2)^2}{4a^2b^2c^2} = \frac{4c^2a^2 + 2(c^2 + a^2)2b^2}{4a^2b^2c^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}
\]

(5) The centre of the circle DEF is the insymmedian point of the triangle ABC

Because \( \angle EFD' = \angle XAC \)

therefore their complements are equal

that is \( \angle D'FB = \angle ACX \);

therefore D'F is antiparallel to CA with respect to B

Hence E'D'', '' AB'', '' C

and F'E'', '' BC'', '' A

But these antiparallels are all bisected at the centre of the circle DEF;

therefore the centre of the circle is the insymmedian point K

(6) The intercepts DD', EE', FF' made by the circle DEF on the sides of ABC are proportional to the cosines of the angles of ABC

For triangle DD'E' is right-angled;

therefore \( DD' = DE'\cos D'DE' \)

\( = DE'\cos A \)

Similarly \( EE' = EF'\cos B \)

\( FF' = FD'\cos C \)

and DE', EF', FD' are all equal

Hence the name cosine circle, given to it by Mr Tucker

(7) The triangles EFD F'D'E' are directly similar to ABC, and congruent to each other
FIGURE 19

For \[ \angle \text{DEF} = \angle \text{DD}' \text{F} \]
and \[ \angle \text{EFD} = \angle \text{EE}' \text{D} \]
therefore \(EFD\) is similar to \(ABC\)

In like manner for \(F'D'E'\)

Now since \(EFD\) \(F'D'E'\) are similar to each other and are inscribed in the same circle they are congruent

(8) The angles which \(F'D'E'\) make with \(A\ B\ B\ C\ C\ A\)
are equal to the angles which \(F'D'E'\) make with \(B\ C\ C\ A\ A\ B\)

§9

THE TRIPlicate RATIO OR FIRST LEMOINE CIRCLE

If through the insymmedian point of a triangle parallels be drawn to the three sides, the six points in which they meet the sides will be concyclic

FIRST DEMONSTRATION *

FIGURE 31

Let \(K\) be the insymmedian point of \(ABC\), and through \(K\) let there be drawn \(EF'E'\) \(FD'D'\) \(DE'E'\) respectively parallel to \(BC\ CA\ AB\), the \(D\) points being on \(BC\), the \(E\)'s on \(CA\), the \(F\)'s on \(AB\)

Then \(AFKE'\) is a parallelogram;
therefore \(AK\) bisects \(FE'\);
therefore \(FE'\) is antiparallel to \(BC\);
therefore \(FE'\) ″ ″ ″ \(EF'\);

* This mode of proof is due to Mr R. F. Davis. See Fourteenth General Report (1888) of the Association for the Improvement of Geometrical Teaching, p. 39.
therefore the points E E' F F' are concyclic
Hence ,, ,, F F' D D' ,, ,, 
and ,, ,, D D' E E' ,, ,, 

Now if these three circles be not one and the same circle, their 
radical axes, which are DD' EE' FF' or BC CA AB, must meet 
in a point, the radical centre 
But BC CA AB do not meet in a point; 
therefore these three circles are one and the same, 
that is, the six points D D' E E' F F' are concyclic

SECOND DEMONSTRATION

FIGURE 32

Let O be the circumcentre of ABC, and let OA OK be joined.

Because AFKE' is a parallelogram 
therefore AK bisects FE' at U ;
therefore FE' is antiparallel to BC
therefore OA is perpendicular to FE'
If therefore through U a perpendicular be drawn to FE'
it will be parallel to OA, and will pass through O', 
the mid point of OK

Hence also the perpendiculars to DF' and ED' through their mid 
points V and W will pass through O' ;
that is, O' the mid point of OK is the centre of a circle which passes 
through D D' E E' F F'

This theorem also was first given by Mr Lemoine at the Lyons meeting 
(1873) of the Association Francaise, and the circle determined by it has 
been called one of Lemoine's circles (the first). In his famous article "Sur quelques 
propriétés d'un point remarquable d'un triangle" (famous for having given the 
impulse to a long series of researches, Mr Lemoine's own being not the least 
prominent among them all) he states that the centre of the circle is the mid 
point of the line joining the centre of antiparallel medians (or as it is now 
called, the symmedian point) to the circumcentre; and that the intercepts 
made by the circle on the sides of the triangle are proportional to the cubes of 
the sides to which they belong.

Ten years later Mr Tucker, unaware of Mr Lemoine's researches, redis-
covered the circle with many of its leading properties, and gave to it the name of 
triplicate ratio circle. See his papers "The Triplicate-Ratio Circle" in the 
Quarterly Journal of Mathematics, XIX. 342-348 (1883) and in the Appendix 
to the Proceedings of the London Mathematical Society, XIV. 316-321 (1883) 
and "A Group of Circles" in the Quarterly Journal, XX. 57-59 (1884).
The parallels drawn through \( K \), the symmedian point, to the sides of \( ABC \) are often called Lemoine's parallels, and the hexagon they determine \( DD'EE'FF' \) Lemoine's hexagon.

1. If \( E'F'F'D'D'E \) be produced to meet and form a triangle, then the incircle of this triangle will have \( O' \) for its centre, and its radius will be half the radius of the circumcircle of \( ABC \).

For \( O'U = \frac{1}{2}OA \) \( O'V = \frac{1}{2}OB \) \( O'W = \frac{1}{2}OC \);
therefore \( O'U = O'V = O'W \);
and \( O'U \) is perpendicular to \( E'F' \), \( O'V \) to \( F'D' \), \( O'W \) to \( D'E' \).

2. The figures \( DD'EF' \) \( EE'FD' \) \( FF'DE' \) are symmetrical trapeziums;
therefore \( E'F' = F'D' = D'E' \).

3. Triangles \( FDE \) \( E'F'D' \) are directly similar to \( ABC \) and congruent to each other.

For \( \angle FDE = \angle FF'E = \angle B \)
and \( \angle DEF = \angle DD'F = \angle C \)
therefore \( FDE \) is similar to \( ABC \).

In like manner for \( E'F'D' \).

Now since \( FDE \) \( E'F'D' \) are similar to each other and are inscribed in the same circle, therefore they are congruent.

It is not difficult to show that if \( K \) be any point in the plane of \( ABC \) and through it parallels be drawn to the sides, as in the figure, the triangles \( DEF \) \( D'E'F' \) are equal in area.

See Vuibert's *Journal de Mathématiques Élémentaires*, VIII. 12 (1883)

4. The following three triangles are directly similar to \( ABC \):
\[ KDD' \quad E'KE \quad FF'K \]
For \( E'F' \) \( F'D' \) \( DE' \) are parallel to the sides.

5. The following six triangles are inversely similar to \( ABC \):
\[ AE'F \quad KFE' \quad DBF' \quad F'KD \quad D'EC \quad ED'K \]
For \( E'F' \) \( F'D' \) \( D'E' \) are antiparallel to the sides.

6. The triangles* cut off from \( ABC \) by \( E'F' \) \( F'D \) \( D'E' \) are together equal to triangle \( DEF \) or \( D'E'F' \).

* Properties (6)—(9) are due to Mr Tucker. See *Quarterly Journal*, XIX. 344, 346 (1883)
For $AE'F = \frac{1}{2}AE'KF = EKF$
$DBF' = \frac{1}{2}BF'KD = FKD$
$D'EC = \frac{1}{2}CD'KE = DKE$
therefore $AE'F + DBF' + D'EC = DEF = D'E'F'$

(7) The following six angles are equal:
$DFB$ $EDC$ $FEA$ $D'E'C$ $F'D'B$ $E'F'A$
For the arcs $E'F$ $F'D$ $D'E$ are equal

(8) If each of these angles be denoted by $\omega$
$\angle AFE = \angle AE'F' = B + C - \omega$
$\angle BDF = \angle BF'D' = C + A - \omega$
$\angle CED = \angle CD'E' = A + B - \omega$

(9) The following points are concyclic:
$B$ $C$ $E'$ $F$ $C$ $A$ $F'$ $D$ $A$ $B$ $D'$ $E$

(10) The radical axis of the circumcircle and the triplicate ratio circle is the Pascal line of Lemoine's hexagon
Let $FE'$ meet $BC$ at $X$
Since the points $B$ $F$ $E'$ $C$ are concyclic,
therefore $XB\cdot XC = XF\cdot XE'$
therefore $X$ has equal potencies with respect to the circumcircle and the triplicate ratio circle;
therefore $X$ is a point on their radical axis
Hence if $DF'$ meet $CA$ at $Y$, and $ED'$ meet $AB$ at $Z$,
$Y$ and $Z$ are points on the radical axis;
therefore the radical axis is the straight line $XYZ$

(11) The radical axis is the polar of $K$ with respect to the triplicate ratio circle

(12) The diagonals of Lemoine's hexagon
$E'F'$ $DE$ $F'D'$ $EF$ $D'E'$ $FD$
intersect on the polar of $K$ with respect to the triplicate ratio circle
Figure 33

(13) *If the chords
\[ EF' E'F' \quad F'D' F'D' \quad DE D'E' \]
intersect in \( p \quad q \quad r \)
the triangles \( ABC \) \( pqr \) are homologous*

Let \( Bq \) \( Cr \) meet at \( T \), and, for the moment, denote the distances of \( T \) from \( BC \) \( CA \) \( AB \) by \( \alpha \) \( \beta \) \( \gamma \)

Then \( \frac{a}{\gamma} = \text{perp. on BC from } q \quad \text{perp. on } AB \text{ from } q \)
\[
= \frac{DD'}{FF'}
\]
from the similarity of triangles \( DD'q \) \( F'Fg \)

Similarly \( \frac{a}{\beta} = \frac{\alpha^2}{c^2} : \frac{b^2}{c^3} \);
therefore \( \frac{\beta}{\gamma} = \frac{b^2}{c^3} ; \)
therefore the point \( p \) lies on \( AT \)

(14) *The intersections of the antiparallel chords with Lemoine's parallels, that is, of
\[ E'F \quad EF'' \quad F''D \quad F'D' \quad D'E \quad DE' \]
\[
\text{namely } P \quad Q \quad R
\]
are collinear*

The quadrilateral \( EE'FF'' \) is inscribed in the circle \( DEF \), and
\[ EE' \quad FF' \quad EF \quad EF' \quad EF' \quad EF' \]
meet in \( \Lambda \quad p \quad P \)
therefore triangle \( \Lambda p P \) is self-conjugate with respect to circle \( DEF \);
therefore \( P \) is the pole of \( \Lambda p \) with respect to \( DEF \)
Similarly \( Q \) \( " \) \( " \) \( \Lambda q \) \( " \) \( " \) \( " \)
and \( R \) \( " \) \( " \) \( \Lambda r \) \( " \) \( " \) \( " \)
Now \( \Lambda p \) \( Bq \) \( Cr \) are concurrent at \( T \)
therefore \( P \) \( Q \) \( R \) are collinear on the polar of \( T \) with respect to the circle \( DEF \)

* Dr John Casey. See his *Sequel to Euclid*, 6th ed., p. 190 (1892)
(15) If the intersections of
\[
\begin{align*}
DE & \quad F'D' \\
EF & \quad D'E' \\
FD & \quad E'F'
\end{align*}
\]
be \(l\), \(m\), \(n\) the triangles ABC lmn are similar and oppositely situated.

Since the arcs E'F FD D'E are equal
therefore \(\angle E'mF = \angle E'nF = 2\omega\);
therefore the points E' m n F are concyclic;
therefore \(\angle E'nm = \angle E'Fm = \angle E'F'E\);
therefore mn is parallel to EF' and to BC

Similarly for the other sides

(16) The triangles pqr lmn are homologous and K is their centre of homology.

For E'F'D'FED is a Pascal hexagram
therefore the intersections of
\[
\begin{align*}
E'F'' & \quad FE \\
F'D' & \quad ED \\
D'F & \quad DE'
\end{align*}
\]
namely p l K
are collinear

Similarly q m K; r n K are collinear

§10

TUCKER'S CIRCLES *

If triangles ABC A_iB_iC_i be similar and similarly situated and have K the symmedian point for centre of similitude, the six points in which the sides of A_iB_iC_i meet the sides of ABC are concyclic

**Figure 34**

Let the six points be D D' E E' F F'

Since AFA_iE' is a parallelogram
therefore AK bisects E'F;
therefore E'F is antiparallel to BC,
therefore E'F "" "" EF' ;

* See the Quarterly Journal, XX. 57-59 (1884)
therefore the points $E E' F F'$ are concyclic.
Similarly "" "" 'F F' D D' "" ",
and "" "" D D' E E' "" ",
therefore the six points $D D' E E' F F'$ are concyclic.

(1) To find the centre of the circle $DD'EE'FF'$ *

Let $O O_1$ be the circumcentres of $ABC A_1B_1C_1$.

Then $O O_1 K$ are collinear.
and OA is parallel to $O_1A_1$.
Now since $E'F$ is antiparallel to $BC$.
therefore $OA$ is perpendicular to $E'F$.
Hence if $E'F$ meet $AA'$ at $U$;
a line through $U$ parallel to $AO$ will bisect $E'F$ perpendicularly,
and also bisect $OO_1$.
Similarly the perpendicular bisectors of $F'D$ and $D'E$
will bisect $OO_1$;
therefore the centre of the circle is the mid point of $OO_1$.

(2) Triangles $FDE E'F'D'$ are directly similar to $ABC$ and
congruent to each other.

**Figure 34**

Since $F'E' D'F$ $E'D$ are respectively parallel
to $BC CA AB$.
therefore the arcs $E'F$ $F'D$ $D'E$ are equal ;
therefore $\angle EFD = \angle D'FF' = \angle A$.
Similarly $\angle FDE = \angle E'DD' = \angle B$.
therefore $FDE$ is similar to $ABC$.
In like manner for $E'F'D'$.

Now since $FDE E'F'D'$ are similar to each other and are
inscribed in the same circle therefore they are congruent.

* The properties (1), (2), (3), (6), (7) are due to Mr Tucker. See Quarterly
Journal, XX. 59, 57, XIX. 348, XX. 59 (1884, 3)
(3) If $T$ be the mid point of $O_0O_1$, 

$$TU = \frac{1}{2}(OA + O_1A_1)$$

Similarly, if $V$ $W$ be the mid points of $F'D'D'E'$

$$TV = \frac{1}{2}(OB + O_1B_1)$$

$$TW = \frac{1}{2}(OC + O_1C_1)$$

Hence if $E'F'F'D'D'E'$ be produced to meet and form a triangle $A_2B_2C_2$,

$T$ will be the incentre of the triangle $A_2B_2C_2$ and the radius of the incircle will be an arithmetic mean between the radii of the circumcircles of $ABC$ and $A_1B_1C_1$.

(4) The triangle $A_2B_2C_2$ formed by producing $E'F'F'D'D'E'$ will have its sides respectively parallel to those of $K_1K_2K_3$ formed by drawing through $A$ $B$ $C$ tangents to the circumcircle $ABC$.

Figure 35

(5) Triangles $A_3B_3C_3$ $K_1K_2K_3$ have $K$ for their centre of homology.

(6) When the triangle $A_3B_3C_3$ becomes the triangle $ABC$, the Tucker circle $DD'EE'FF'$ becomes the circumcircle.

(7) When the triangle $A_3B_3C_3$ reduces to the point $K$ that is when the parallels $B_3C_3$ $C_3A_3$ $A_3B_3$ to the sides of $ABC$ pass through $K$ the Tucker circle $DD'EE'FF'$ becomes the triplicate ratio or first Lemoine circle.

(8) When the triangle $A_3B_3C_3$ reduces to the point $K$, that is when the antiparallels $B_3C_3$ $C_3A_3$ $A_3B_3$ to the sides of $ABC$ pass through $K$, the Tucker circle $DD'EE'FF'$ becomes the cosine or second Lemoine circle.

(9) If $F'D'D'E'$ meet at $A_3$

$$D'E' \quad E'F' \quad " \quad " \quad B_3$$

$$E'F' \quad F'D' \quad " \quad " \quad C_3$$

then $AK$ $BK$ $CK$ pass through $A_3$ $B_3$ $C_3$.
Let $R$ denote the foot of the symmedian $AK$

Then $\frac{K'F}{KE} = \frac{BD}{CD'}$

and $\frac{RD}{RD'} = \frac{BR - BD}{CR - CD'}$

therefore $F'D$ $KR$ $ED'$ are concurrent

$\S$ 11

**Taylor's Circle**

*The six projections of the vertices of the orthic triangle on the sides of the fundamental triangle are concyclic*

**Figure 86**

Let the projections of $X$ on $CA$ $AB$ be $Y_1$, $Z_1$

" " " " $Y$, $AB$, $BC$, $Z_2$, $X_2$

" " " " $Z$, $BC$, $CA$, $X_3$, $Y_3$

Then $\frac{AZ}{AZ_1} = \frac{AH}{AX} = \frac{AY}{AY_1}$

therefore $YZ$ is parallel to $Y_1Z_1$

Now $Y_3Z_2$ is antiparallel to $YZ$

therefore $Y_3Z_2$ " " " " $Y_1Z_1$

therefore $Y_1$, $Y_3$, $Z_2$, $Z_1$ are concyclic

Similarly $Z_2$, $Z_1$, $X_3$, $X_2$ " " " "

and $X_3$, $X_2$, $Y_3$, $Y_1$ " " " "

therefore the six points $X_3$, $X_2$, $Y_1$, $Y_3$, $Z_2$, $Z_1$ are concyclic

The property, that the six projections of the vertices of the orthic triangle on the sides of the fundamental triangle are concyclic, seems to have been first published in Mr Vuibert's *Journal de Mathématiques Élémentaires* in November 1877. See Vol. II. pp. 30, 43. It is proposed by Eutaris. This name, as my friend Mr Maurice D'Ocagne informs me, was assumed anagramatically by M. Restiau, at that time a répétiteur in the Collège Chaptal, Paris.
The same property, along with three others, is given in Catalan's *Théorèmes et Problèmes*, 6th ed., pp. 132-4 (1879). It occurs also in a question proposed by Professor Neuberg in *Mathesis*, I. 14 (1881), and in a paper by Mr H. M. Taylor in the *Messenger of Mathematics*, XI. 177-9 (1882). A proof by Mr C. M. Jessop, somewhat shorter than that given by Mr Taylor, occurs in the *Messenger*, XII. 36 (1883) and in the same volume (pp. 181-2) Mr Tucker examines whether any other positions of X Y Z on the sides would, with a similar construction, give a six-point circle, and he shows that no other circle is possible under the circumstances.

See also *L'Intérimadiaire des Mathématiciens*, II. 166 (1895).

The projections of

- X on BY C Z are Y Z O
- Y , , C Z A X , , Z Z X
- Z , , A X B Y , , X X Y

With regard to the notation it may be remarked that the X points lie on BC and on the perpendicular to it from A

Y , , CA , , , , , , B
Z , , AB , , , , , , C

Let a notation, similar to that which prevails with regard to the sides, the semiperimeter, the radii of the incircle and the excircles of triangle ABC, be adopted for triangle XYZ; that is, let

\[ YZ = x \quad ZX = y \quad XY = z \]

\[ \sigma = \frac{1}{2}(x + y + z) \quad \sigma_1 = \frac{1}{2}(-x + y + z) \quad \sigma_2 = \frac{1}{2}(x - y + z) \quad \sigma_3 = \frac{1}{2}(x + y - z) \]

and let \( \rho, \rho_1, \rho_2, \rho_3 \) be the radii of the incircle and the excircles

If reference be made to the *Proceedings of the Edinburgh Mathematical Society*, XIII. 39-40 (1895), it will be found that various properties are proved with respect to triangle \( I_1 I_2 I_3 \) and its orthic triangle ABC. These properties may be transferred to triangle ABC and its orthic triangle XYZ. The transference will be facilitated by writing down in successive lines the points which correspond. They are

\[
\begin{array}{ccccccccccc}
I & I_1 & I_2 & I_3 & A & B & C & A_1 & A_2 & A_3 & A_4 \\
H & A & B & C & X & Y & Z & Y_2 & Y_1 & Z_0 & Z_1 \\
\end{array}
\]

Hence, from Wilkinson's theorem and corollary, the three following statements relative to Fig 25 may at once be inferred

1. \( Z_1 Y_2 Z_0 Y_1 \quad X_2 Z_3 X_0 Z_2 \quad Y_3 X_1 Y_0 X_3 \)

are three tetrads of collinear points
(2) \[ Y_1Z_1 \quad Z_2X_2 \quad X_3Y_3 \]
or \[ Y_1Z_0 \quad Z_2X_0 \quad X_1Y_0 \]
intersect two by two at the mid points of the sides of XYZ

(3) * \[ Y_1Z_1 = Z_2X_2 = X_3Y_3 = \sigma \]
\[ Y_2Z_0 = Z_3X_2 = X_1Y_0 = \sigma_1 \]
\[ Y_1Z_0 = Z_2X_0 = X_1Y_3 = \sigma_2 \]
\[ Y_2Z_1 = Z_3X_0 = X_1Y_0 = \sigma_3 \]

(4) If \( X' \quad Y' \quad Z' \) be the mid points of
\( YZ \quad ZX \quad XY \)
the sides of triangle \( X'Y'Z' \) intersect the sides of \( ABC \) in six concyclic points

(5) \( \text{Triangles } ABC \quad X'Y'Z' \) are homologous, and the symmedian point \( K \) is the centre of homology
For \( YZ \) is antiparallel to \( BC \),
and \( X' \) is the mid point of \( YZ \);
therefore \( AX' \) is the symmedian from \( A \)
Similarly \( BY' \) \( CZ' \) are the symmedians from \( B \quad C \)

(6) † \[ R \cdot Y_1Z_1 = ABC \quad R \cdot Y_2Z_2 = HCB \]
\[ R \cdot Y_1Z_0 = CHA \quad R \cdot Y_2Z_1 = BAH \]

\textbf{FIGURE 37}

Join \( O \) the circumcentre to \( A \quad B \quad C \)
Then \( OA \quad OB \quad OC \) are respectively perpendicular
to \( YZ \quad ZX \quad XY \)
therefore \( 2AZOY = OA \cdot YZ \)
\( 2BZOY = OB \cdot ZX \)
\( 2CYOZ = OC \cdot XY \)
therefore \[ 2 \Delta = R (YZ + ZX + XY) \]

* The property that \( Y_1Z_1 \) is equal to the semiperimeter of \( XYZ \) occurs in Lhuilier’s \( \text{Eléments d'Analyse} \), p. 231 (1809)
† The first of these equalities is given by Feuerbach, \( \text{Eigenschaften des ... Dreiecks} \), §19, or Section VI., Theorem 3 (1822). The other three are given by C. Hellwig in Grunert’s \( \text{Archiv} \), XIX., 27 (1852). The proof is that of Messrs W. E. Heal and P. F. Mange in Artemas Martin’s \( \text{Mathematical Visitor} \), II. 42 (1883)
(7) The following triangles are isosceles:

\[ X'ZZ, X'ZJ, X'ZY, X'YJY \]

For triangles \( YZZ, YZY \) are right-angled and \( X' \) is the midpoint of their hypotenuse.

Similarly there are four isosceles triangles with vertex \( Y' \) and

\[ "", "", "", "", "", Z' \]

(8) \( Y_1Z \) is antiparallel to \( BC \) with respect to \( A \)

\[ Z_2X_2 ,,, CA ,,, B \]
\[ X_2Y_1 ,,, AB ,,, C \]

(9) \( Y_2Z \) is parallel to \( BC \)

\[ Z_1X_1 ,,, CA \]
\[ X_2Y_1 ,,, AB \]

(10) \( Z_1X_1 \) and \( Y_1X_1 \) intersect on the symmedian from \( A \)

Let \( \Lambda_1 \) be their point of intersection.

Then \( AZ\Lambda_1Y_1 \) is a parallelogram, and \( AA\Lambda_1 \) bisects \( Y_1Z_1 \)

But \( Y_1Z_1 \) is antiparallel to \( BC \) with respect to \( A \); therefore \( AA\Lambda_1 \) is the symmedian from \( A \)

Similarly \( Y_1X_1 \) and \( Z_3Y_3 \) intersect on the symmedian from \( B \); and \( Z_2Y_2 \)

(11) Triangles \( Y_1Z_1X_1 \) and \( Z_2Y_2Z_2 \) are directly similar to \( \triangle ABC \) and congruent to each other

\[ \text{Figure 36} \]

For \( \angle X_3Y_1Z_2 = \angle X_3Y_3Z_2 \)

Now \( X_3Y_3 \) is parallel to \( XY \)

and \( Y_3Z_3 \)

therefore \( \angle X_3Y_3Z_2 = \angle YX \)

\[ = \angle A \]

Similarly \( \angle Y_1Z_2X_2 = \angle B \)

therefore \( Y_1Z_2X_2 \) is similar to \( \triangle ABC \)

In like manner for \( Z_1X_2Y_2 \)

Now since \( Y_1Z_2X_3 \) and \( Z_4X_3Y_3 \) are similar to each other and are inscribed in the same circle, therefore they are congruent.
(12) Since XYZ is the orthic triangle not only of ABC, but also of HCB CHA BAH, if the projections of X Y Z be taken on the sides of the last three triangles, three other circles are obtained.

These circles are:

\[ X_2X_3Y_2Y_3Z_2, \quad \quad X_1X_2Y_3Z_3, \quad \quad X_0X_1Y_0Y_2Z_2. \]

If they be denoted by T_1, T_2, T_3 and the circle

\[ X_2X_3Y_1Y_2Z_2Z_1, \quad \quad \text{"}, \quad \quad \text{"}, \quad \quad \text{"}, \quad \quad \text{"}, \quad \quad \text{"}, \quad \quad \text{"}, \quad \quad \text{"}. \]

then

\[
\begin{align*}
T & \quad T_1 \\
T & \quad T_2 \\
T & \quad T_3 \\
T_2 & \quad T_3 \\
T_3 & \quad T_1 \\
T_1 & \quad T_2
\end{align*}
\]

have for radical axis

\[
\begin{align*}
BC & \\
CA & \\
AB & \\
AX & \\
BY & \\
CZ &
\end{align*}
\]

(13) The centres of the circles T, T_1, T_2, T_3 are the incentre and the excentres of the triangle X'Y'Z'.

For Y,Z_1 Z_2X_2, X_3Y_3 are equal chords in circle T; therefore the centre of T is equidistant from them. But these chords form by their intersection the triangle X'Y'Z'; therefore the centre of T must be the incentre of X'Y'Z'.

Hence T, T_1, T_2, T_3 form an orthic tetraastigm.

(14) The centres of T, T_1, T_2, T_3 are the four points of concurrency of the triads of perpendiculars from X' Y' Z' on the sides of ABC HCB CHA BAH.


(15) The circle T belongs to the group of Tucker circles.*

---

(16) The circle $T$ cuts orthogonally the three excircles of the orthic triangle $XYZ$, and each of the circles $T_1$, $T_2$, $T_3$ cuts orthogonally* the incircle and two of the excircles of $XYZ$.

**Figure 36**

Let $p_1$, $p_2$, $p_3$ denote the perpendiculars from A, B, C on YZ, ZX, XY; these perpendiculars are the radii of the three excircles of $XYZ$.

Since triangles $AYZ$, $ABC$ are similar, therefore $p_1^2 : AX^2 = AZ^2 : AC^2$.

Similarly for the other statements.

(17) The squares of the radii of the circles†

<table>
<thead>
<tr>
<th>Circle</th>
<th>$r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$\frac{1}{4}(\rho^2 + \sigma^2)$</td>
</tr>
<tr>
<td>$T_1$</td>
<td>$\frac{1}{4}(p_1^2 + \sigma_1^2)$</td>
</tr>
<tr>
<td>$T_2$</td>
<td>$\frac{1}{4}(p_2^2 + \sigma_2^2)$</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$\frac{1}{4}(p_3^2 + \sigma_3^2)$</td>
</tr>
</tbody>
</table>

**Figure 36**

The triangle $X'Y'Z'$ is similar to $XYZ$, the ratio of similitude being $1 : 2$.

Therefore the radii of the incircle and excircles of $X'Y'Z'$ are $\frac{1}{2}\rho$, $\frac{1}{2}p_1$, $\frac{1}{2}p_2$, $\frac{1}{2}p_3$.

Now if $T$ be the incentre of $X'Y'Z'$, the perpendicular from $T$ to $Y_1Z_1$ will bisect $Y_1Z_1$, and will be equal to $\frac{1}{2}p$.

Hence if $t$ denote the radius of circle $T$, $t^2 = T_1Y_1^2 = (\frac{1}{2}\rho)^2 + (\frac{1}{2}\sigma)^2 = \frac{1}{4}(\rho^2 + \sigma^2)$.

Similarly if $T_1$ be the first excentre of $X'Y'Z'$, and $t_1$ denote the radius of circle $T_1$, $t_1^2 = T_1Y_1^2 = (\frac{1}{2}p_1)^2 + (\frac{1}{2}\sigma_1)^2 = \frac{1}{4}(p_1^2 + \sigma_1^2)$.

* This corollary and the mode of proof have been taken from Dr John Casey's *Sequel to Euclid*, 6th ed. p. 195 (1892).

† Dr John Casey's *Sequel to Euclid*, 5th ed. p. 195 (1892).
The sum of the squares of the radii of the circles \( T, T_1, T_2, T_3 \) is equal to the square of the radius of the circumcircle of \( ABC \).

In reference to triangle \( ABC \), the following property may be proved to be true

\[
16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 + a^2 + b^2 + c^2
\]

This becomes in reference to triangle \( X'Y'Z' \)

\[
16(R^2) = (\frac{1}{2}p)^2 + (\frac{1}{2}p_1)^2 + (\frac{1}{2}p_2)^2 + (\frac{1}{2}p_3)^2 + (\frac{1}{2}z)^2 + (\frac{1}{2}y)^2 + (\frac{1}{2}x)^2
\]

or

\[
R^2 = \frac{1}{4}(p^2 + p_1^2 + p_2^2 + p_3^2 + x^2 + y^2 + z^2)
\]

In connection with the Taylor circles it may be interesting to compare the properties given in the Proceedings of the Edinburgh Mathematical Society, Vol. I. pp. 88-96 (1894). These properties were worked out before the Taylor circle had attracted much attention.

If \( A'B'C' \) be the complementary, and \( XYZ \) the orthic triangle of \( ABC \), the Wallace lines of the points \( A' B' C' \) with respect to the triangle \( XYZ \) pass through the centre of the circle \( T \).

It is well known that the points \( A' B' C' X Y Z \) are situated on the nine point circle of \( ABC \).

Since \( A' \) is the mid point of \( BC \)

therefore \( A'Y = A'Z \)

therefore the foot of the perpendicular from \( A' \) on \( YZ \) is \( X' \) the mid point of \( YZ \).

Since \( BC \) bisects the exterior angle between \( XY \) and \( ZX \)

the straight line joining the feet of the perpendiculars from \( A' \) on \( XY \) and \( ZX \) will be perpendicular to \( BC \).

Hence the Wallace line \( A' (XYZ) \) passes through \( X' \) and is perpendicular to \( BC \).

that is, it passes through the centre of \( T \).

Similarly for the Wallace lines \( B' (XYZ) \) and \( C' (XYZ) \).
(20) The Wallace lines of the points \(X, Y, Z\) with respect to the triangle \(A'B'C'\) pass through the centre of the circle \(T\)

[The reader is requested to make the figure]

Let the feet of the perpendiculars from \(X\) on \(B'C'\ C'A'\ A'B'\) be \(L\ M\ N\)

Then the points \(A'\ M\ X\ N\) are concyclic

therefore \(\angle A'MN = \angle A'XN\)

\[= 90° - \angle B\]

\[= \angle CAO\]

therefore \(LMN\) is parallel to \(\Delta AO\)

But \(L\) is the mid point of \(\Delta AX\) and \(H_1\) is situated on \(AO\)

therefore \(LMN\) passes through the mid point of \(H_1X\), that is through \(T\)

(21) If \(H_1\ H_2\ H_3\) be the orthocentres of triangles \(AYZ\ ZBX\ XYC\)

the lines \(H_1X\ H_2Y\ H_3Z\)

pass through the centre of circle \(T\) and are there bisected

**Figure 36**

The orthocentre \(H_1\) of triangle \(AYZ\) is the point of intersection of \(ZY_2\ YZ_2\) which are respectively perpendicular to \(CA\ AB\)

The centre \(T\) of the Taylor circle is the point of intersection of \(YT\ ZT\)

which are respectively perpendicular to \(CA\ AB\)

Hence since \(XZ = 2XY'\ XY = 2XZ'\)

the quadrilaterals \(H_1ZXY\ TY'XZ'\) are homothetic;

therefore \(H_1X\) passes through \(T\) and is there bisected

(22) Triangle \(H_1H_2H_3\) is congruent and oppositely situated to triangle \(XYZ\) and \(T\) is their homothetic centre

(23) The centre \(T\) is situated on the straight line which joins \(O\) the circumcentre of \(ABC\) to the orthocentre of \(XYZ\)

For \(O\) is the orthocentre of \(H_1H_2H_3\)
Not only is $\triangle XYZ$ the orthic triangle of $\triangle ABC$ and triangles similar to

$\triangle AYZ \quad \triangle XBZ \quad \triangle XYC$ \quad similar to $\triangle ABC$

but $\triangle XYZ$ is the orthic triangle of $\triangle HCB$ $\triangle CHA$ $\triangle BAH$ and triangles similar to

$\triangle HYZ \quad \triangle XCZ \quad \triangle XYB$ \quad similar to $\triangle HCB$

$\triangle CYZ \quad \triangle XHZ \quad \triangle XYA$ \quad similar to $\triangle CHA$

$\triangle BYZ \quad \triangle XAZ \quad \triangle XYH$ \quad similar to $\triangle BAH$

Let the orthocentres of the second, third, and fourth triads of triangles be denoted by

$H_1', H_2', H_3', H_1'', H_2'', H_3''$

The following results (among several others) will be found to be established in the *Proceedings of the Edinburgh Mathematical Society*, I. 83–87 (1894). They are quoted here, without proof, to save the reader the trouble or the expense of hunting out the reference

(24) The homothetic centre of the triangles

$\triangle XYZ \quad H_1' \quad H_2' \quad H_3'$ \quad is $\quad T_1$

$\triangle XYZ \quad H_1'' \quad H_2'' \quad H_3''$ \quad is $\quad T_1$

and $T_1T_2T_3$ is similar and oppositely situated to $\triangle ABC$

(25) The point $T$ is the centre of three parallelograms

$\triangle YZH \quad H_1$, $\quad \triangle ZKH \quad H_2$, $\quad \triangle XHY \quad H_3$

and similarly $T_1$, $T_2$, $T_3$ are each the centre of three parallelograms

Let the incircle and the excircles of $\triangle XYZ$ be denoted by their centres $H$ $A$ $B$ $C$

(26) The radical axes of

$\triangle T_1, T_2, T_3 \quad T_1T_2 \quad T_1T_3 \quad T_2T_3 \quad T_3T$
(27) The radical centres of
\[ A \quad B \quad C \quad H \quad C \quad B \quad C \quad H \quad A \quad B \quad H \]
are \[ T \quad T_1 \quad T_2 \quad T_3 \]

(28) \( X' \quad Y' \quad Z' \) are the feet of the perpendiculars of the triangle \( T_1T_2T_3 \)

(29) The homothetic centre of the triangles
\[
\begin{align*}
\text{T}_1T_2T_3 & \quad \text{H}_1'H_1''H_1''' \quad \text{is} \quad \text{X} \\
\text{T}_1T_2T_3 & \quad \text{H}_2'H_2''H_2''' \quad \text{,,} \quad \text{Y} \\
\text{T}_1T_2T_3 & \quad \text{H}_3'H_3''H_3''' \quad \text{,,} \quad \text{Z}
\end{align*}
\]

(30) The straight lines \( HT \quad AT_1 \quad BT_2 \quad CT_3 \)
pass through the centroid of \( XYZ \)

(31) If \( G' \) denote this centroid
\[
\frac{HG'}{TG'} = \frac{AG'}{TG} = \frac{BG'}{TG} = \frac{CG'}{TG'} = 2 : 1
\]

(32) If \( HG'T \) be produced to \( J' \) so that \( TJ' = HT \) then \( J' \) will be the incentre \( X',Y',Z' \) the triangle anticomplementary to \( XYZ \)

Similarly \( J_1', J_2', J_3' \) situated on \( AT_1 \quad BT_2 \quad CT_3 \) so that 
\[ T,J_1' = AT_1 \] and so on, will be the first, second, and third excircles \( X_1',Y_1',Z_1' \)

(33) The tetrads of points
\[
\begin{align*}
\text{HGTJ}' & \quad \text{AG'TJ}_1' \\
\text{BG'TJ}_2' & \quad \text{CG'TJ}_3'
\end{align*}
\]
form harmonic ranges

\textbf{§12}

\textbf{Adams's Circle} *

\textit{If D E F be the points of contact of the incircle with the sides of \( \Delta ABC \), and if through the Gergonne point \( \Gamma \) (the point of concurrency of \( AD \ BE \ CF \)) parallels be drawn to \( EF \ FD \ DE \), these parallels will meet the sides of \( \Delta ABC \) in six concyclic points}

* C. Adams's \textit{Die Lehre von den Transversalen}, pp. 77-80 (1843)
Let $X X' Y Y' Z Z'$ be the six points
Join $LL' MM' NN'$

The complete quadrilateral $AFEBC$ has its diagonal $AF$ cut harmonically by $FE$ $BC$;
therefore $A U \Gamma D$ is a harmonic range;
therefore $E A U \Gamma D$ is a harmonic pencil

Now $\Gamma MEM'$ is a parallelogram;
therefore $MM'$ is bisected by $E\Gamma$
therefore $MM'$ is parallel to that ray of the harmonic pencil which is conjugate to $E\Gamma$, namely $EA$

In like manner $NN'$ is parallel to $AB$, and $LL'$ to $BC$

Again, since $YEM'M Y'EMM'$ are parallelograms,
therefore $YE = Y'E$
Similarly $Z'F = ZF$
therefore $YE : Y'E = Z'F : ZF$

Now $YZ$ is parallel to $EF$;
therefore $Y'Z$ is parallel to $EF$

In like manner $Z'X$ is parallel to $FD$ and $XY$ to $DE$
Hence the two hexagons $LL'MM'NN'$ and $XX'YY'ZZ'$ are similar, and the ratio of their corresponding sides is that of 1 to 2

Lastly, since $LL'$ is parallel to $BC$

$$\angle L'LL' = \angle CDE$$
$$= \angle CED$$
$$= \angle MM'T$$

therefore the points $L L' M M'$ are concyclic
Similarly the points $M M' N N'$ are concyclic and the points $N N' L L'$ are concyclic;
therefore all the six points are concyclic*

Hence the six points $X X' Y Y' Z Z'$ are also concyclic

* This method of proof is different from Adams's
(1) The centre of Adams’s circle is the incentre* of ABC

For XX’ YY’ ZZ’ are chords of Adams’s circle, and they are
bisected at D E F;
hence the centre of Adams’s circle is found by drawing through
D E F perpendiculars to XX’ YY’ ZZ’
These perpendiculars are concurrent at I the incentre of ABC

(2) To find the centre of the circle LL’MM’NN’

Since Γ is the homothetic centre of the two circles XX’YY’ZZ’
and LL’MM’NN’, and I is the centre of the first of these circles,
therefore the centre of the second circle is situated on ΓI

If I’ denote the centre of the second circle
then
ΓI : ΓI’ = 2 : 1

(3) Since Γ the Gergonne point of ABC is the insymmedian
point of DEF, the circle LL’MM’NN’ is the triplicate ratio or first
Lemoine circle of DEF

(4) Besides the six-point circle obtained by drawing through Γ
the Gergonne point of ABC parallels to the sides of triangle DEF,
three other six-point circles will be obtained if through the associated
Gergonne points Γ₁ Γ₂ Γ₃ parallels be drawn to the sides of the
triangles D₁E₁F₁ D₂E₂F₂ D₃E₃F₃ respectively

The centres of these three circles are the excentres of ABC
namely I₁ I₂ I₃ and the centres of the three LMN circles which
 correspond to them are the mid points of Γ₁I₁ Γ₂I₂ Γ₃I₃

These three LMN circles are the triplicate ratio circles of the
triangles D₁E₁F₁ D₂E₂F₂ D₃E₃F₃

Formulæ connected with the symmedians

The sides \(a\ b\ c\) are in ascending order of magnitude

\[
\begin{align*}
BR &= \frac{ac^2}{b^2 + c^2} \\
CS &= \frac{ba^2}{c^2 + a^2} \\
AT &= \frac{cb^2}{a^2 + b^2} \\
CR &= \frac{ab^2}{b^2 + c^2} \\
AS &= \frac{bc^2}{c^2 + a^2} \\
BT &= \frac{ca^2}{a^2 + b^2}
\end{align*}
\]

\(1\)

* C. Adams’s Die Lehre von den Transversalen, p. 79 (1843)
Let the three internal medians be denoted by

\[ m_1, m_2, m_3 \]

Their values in terms of the sides are

\[ 4m_1^2 = -a^2 + 2b^2 + 2c^2 \]
\[ 4m_2^2 = 2a^2 - b^2 + 2c^2 \]
\[ 4m_3^2 = 2a^2 + 2b^2 - c^2 \]

\[ AR = \frac{2b cm_1}{b^2 + c^2} \]
\[ BS = \frac{2ca m_2}{c^2 + a^2} \]
\[ CT = \frac{2ab m_3}{a^2 + b^2} \]

\[ \text{Figure 12} \]

Let \( AA' \) be the internal median and symmedian from \( A \)
Then

\[ BR \cdot CR : BA' \cdot CA' = AR^2 : AA'^2 \]

therefore

\[ AR^2 = \frac{BR \cdot CR}{BA' \cdot CA'} \cdot AA'^2 \]

\[ AR' = \frac{abc}{c^2 - b^2} \]
\[ BS' = \frac{abc}{c^2 - a^2} \]
\[ CT' = \frac{abc}{b^2 - a^2} \]

\[ \text{Figure 14} \]

For

\[ AR'^2 = BR' \cdot CR' \]

\[ (AR^2 + BR^2)b^2 + (AR^2 + CR^2)c^2 = 2b^2c^2 \]

and so on

* Mr Clément Thiry, \textit{Applications remarquables du Théorème de Stewart}, p. 20 (1891)
Since
\[\frac{AS}{CS} = \frac{AKB}{CKB}\]
\[\frac{AT}{BT} = \frac{AKC}{BKC}\]

therefore
\[\frac{AS}{CS} + \frac{AT}{BT} = \frac{AKB + AKC}{BKC}\]
\[= \frac{AKB + AKC}{BKR + CKR}\]

Now
\[AK = AKB\]
\[RK = BKR = CKR\]
\[= \frac{AKB + AKC}{BKR + CKR}\]

therefore
\[\frac{AK}{AR} = \frac{b^2 + c^2}{a^2 + b^2 + c^2}\]

\[a \cdot AK : b \cdot BK : c \cdot CK = m_1 : m_2 : m_3\]  \hspace{1cm} (8) \dagger

\[a^2 \cdot AK^2 + b^2 \cdot BK^2 + c^2 \cdot CK^2 = \frac{3a^2b^2c^2}{a^2 + b^2 + c^2}\]  \hspace{1cm} (9) \ddagger
\[
\begin{align*}
AK_1 &= \frac{2bc m_1}{-a^2 + b^2 + c^2} \\
RK_1 &= \frac{2a^2bc m_1}{(b^2 + c^2)(-a^2 + b^2 + c^2)} \\
BK_2 &= \frac{2ca m_2}{a^2 - b^2 + c^2} \\
SK_2 &= \frac{2ab^2c m_2}{(c^2 + a^2)(a^2 - b^2 + c^2)} \\
CK_3 &= \frac{2ab m_3}{a^2 + b^2 - c^2} \\
TK_3 &= \frac{2abc^2 m_3}{(a^2 + b^2)(a^2 + b^2 - c^2)} \\
BK_1 = CK_1 &= \frac{abc}{-a^2 + b^2 + c^2} \\
CK_2 = AK_2 &= \frac{abc}{a^2 - b^2 + c^2} \\
AK_3 = BK_3 &= \frac{abc}{a^2 + b^2 - c^2} \\
KK_1 &= \frac{4a^2bc m_1}{(a^2 + b^2 + c^2)(-a^2 + b^2 + c^2)} \\
KK_2 &= \frac{4ab^2c m_2}{(a^2 + b^2 + c^2)(a^2 - b^2 + c^2)} \\
KK_3 &= \frac{4abc^2 m_3}{(a^2 + b^2 + c^2)(a^2 + b^2 - c^2)} \\
K_2K_2 &= \frac{2a^2bc}{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)} \\
K_3K_1 &= \frac{2ab^2c}{(a^2 + b^2 - c^2)(-a^2 + b^2 + c^2)} \\
K_1K_2 &= \frac{2abc^3}{(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)} \\
\end{align*}
\]

\[\alpha^2 \frac{AK_1}{KK_1} = \frac{b^3 B K_2}{KK_2} = \frac{c^2 C K_3}{KK_3} = \frac{a^2 + b^2 + c^2}{2} \quad (14) \]

\[\frac{KK_1 : KK_2 : KK_3}{AK_1 : BK_2 : CK_3} = a^2 : b^2 : c^2 \quad (15) \]

* Dr Franz Wetzig in Schlömilch's Zeitschrift, XII. 294 (1867)
\[
\begin{align*}
BL &= \frac{a(a^2 - b^2 + 3c^2)}{2(a^2 + b^2 + c^2)} = \frac{a(c^2 + ca \cos B)}{a^2 + b^2 + c^2} \\
CL &= \frac{a(a^2 + 3b^2 - c^2)}{2(a^2 + b^2 + c^2)} = \frac{a(b^2 + ab \cos C)}{a^2 + b^2 + c^2} \\
CM &= \frac{b(3a^2 + b^2 - c^2)}{2(a^2 + b^2 + c^2)} = \frac{b(a^2 + ab \cos C)}{a^2 + b^2 + c^2} \\
AM &= \frac{b(-a^2 + b^2 + 3c^2)}{2(a^2 + b^2 + c^2)} = \frac{b(c^2 + bc \cos A)}{a^2 + b^2 + c^2} \\
AN &= \frac{c(-a^2 + 3b^2 + c^2)}{2(a^2 + b^2 + c^2)} = \frac{c(b^2 + bc \cos A)}{a^2 + b^2 + c^2} \\
BN &= \frac{c(3a^2 - b^2 + c^2)}{2(a^2 + b^2 + c^2)} = \frac{c(a^2 + ca \cos B)}{a^2 + b^2 + c^2}
\end{align*}
\]

\[\text{(16)}\]

Distances of \(K\) from the sides of \(ABC\)

\[
\begin{align*}
KL &= \frac{2a \Delta}{a^2 + b^2 + c^2} = \frac{a \sin A \sin B \sin C}{\sin^2 A + \sin^2 B + \sin^2 C} \\
KM &= \frac{2b \Delta}{a^2 + b^2 + c^2} = \frac{b \sin A \sin B \sin C}{\sin^2 A + \sin^2 B + \sin^2 C} \\
KN &= \frac{2c \Delta}{a^2 + b^2 + c^2} = \frac{c \sin A \sin B \sin C}{\sin^2 A + \sin^2 B + \sin^2 C} 
\end{align*}
\]

\[\text{(17)}\]

Figure 23

Draw \(AX\) perpendicular to \(BC\)

Then \(AR : KR = AX : KL\)

therefore

\[
KL = \frac{KR \cdot AX}{AR} = \frac{a^2 h_1}{a^2 + b^2 + c^2} = \frac{2a \Delta}{a^2 + b^2 + c^2}
\]

* E. W. Grebe in Grunert's Archiv, IX. 252, 250-1 (1847)
The following is another demonstration *

Let \( \alpha \beta \gamma \) denote the distances of \( K \) from \( BC \ CA \ AB \)

Then

\[
\frac{a}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{a \beta + b \gamma + c^2}{a^2 + b^2 + c^2} = \frac{2 \Delta}{a^2 + b^2 + c^2}
\]

\[
KL^2 + KM^2 + KN^2 = \frac{4 \Delta^2}{a^2 + b^2 + c^2}
\]

\[
= \frac{a^2 \sin^2 \beta \sin^2 \gamma}{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma} = \frac{b^2 \sin^2 \gamma \sin^2 \alpha}{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma} = \frac{c^2 \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma} = \frac{\Delta}{\cot \alpha + \cot \beta + \cot \gamma}
\]

**DISTANCES OF \( K_1 \ K_2 \ K_3 \) FROM THE SIDES OF \( ABC \)**

\[
(K_1) \quad \frac{-2a \Delta}{a^2 + b^2 + c^2} \quad \frac{2b \Delta}{a^2 + b^2 + c^2} \quad \frac{2c \Delta}{a^2 + b^2 + c^2}
\]

\[
(K_2) \quad \frac{2a \Delta}{a^2 - b^2 + c^2} \quad \frac{-2b \Delta}{a^2 - b^2 + c^2} \quad \frac{2c \Delta}{a^2 - b^2 + c^2}
\]

\[
(K_3) \quad \frac{2a \Delta}{a^2 + b^2 - c^2} \quad \frac{2b \Delta}{a^2 + b^2 - c^2} \quad \frac{-2c \Delta}{a^2 + b^2 - c^2}
\]

* Mr R. Tucker in *Quarterly Journal of Mathematics*, XIX. 342 (1883)

† The first of these values is given by "Yanto" in Leybourn's *Mathematical Repository*, old series, Vol. III. p. 71 (1803). Lhuilier in his *Elements d'Analyse*, p. 238 (1809) gives the analogous property for the tetrahedron.

The other values are given by E. W. Grebe in Grunert's *Archie*, IX. 251 (1847)
Grebe, _loco citato_, p. 257, gives the distances of $K_3$ from the sides of $ABC$ with the following trigonometrical equivalents

\[
\begin{align*}
\frac{a \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= \frac{1}{2}a \tan C \\
\frac{b \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= \frac{1}{2}b \tan C \\
-\frac{c \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= -\frac{1}{2}c \tan C
\end{align*}
\]

\[(20)\]

If $\Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4$ denote the sum of the squares of the distances from the sides of $ABC$ of $K_1 \ K_2 \ K_3$

\[
\begin{align*}
\Sigma_1 &= \frac{4 \Delta^2}{-a^2 + b^2 + c^2} \\
\Sigma_2 &= \frac{4 \Delta^2}{a^2 - b^2 + c^2} \\
\Sigma_3 &= \frac{4 \Delta^2}{a^2 + b^2 - c^2}
\end{align*}
\]

\[(21)\]

\[
\frac{1}{\Sigma} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}
\]

\[(22)\]

If $\ k \ k_1 \ k_2 \ k_3$ denote the distances from $BC$ of $K_1 \ K_2 \ K_3$

\[
\frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{k_1}
\]

\[(23)\]

This relation holds for any four harmonically associated points

\[
MN^2 + NL^2 + LM^2 = \frac{12 \Delta^2}{a^2 + b^2 + c^2}
\]

\[(24)\]

For the left side $= 3 \ (KL^2 + KM^2 + KN^2)$

\[
\begin{align*}
MN &= \frac{4m_1 \Delta}{a^2 + b^2 + c^2} = \frac{m_1 \Sigma}{\Delta} \\
NL &= \frac{4m_2 \Delta}{a^2 + b^2 + c^2} = \frac{m_2 \Sigma}{\Delta} \\
LM &= \frac{4m_3 \Delta}{a^2 + b^2 + c^2} = \frac{m_3 \Sigma}{\Delta}
\end{align*}
\]

\[(25)\]

* Dr Franz Wetzig in Schlimilch's _Zeitschrift_, XII. 294-295 (1867)

† Both forms are given by E. W. Grebe in _Grunert's Archiv_, IX. 253 (1847)
For $MN$ can be found by applying Ptolemy's theorem to the encyclic quadrilateral $ANKM$

$$LMN = \frac{12A^2}{(a^2 + b^2 + c^2)^2} \quad (26) *$$

**Figure 28**

Since $K$ is the centroid of $LMN$,

$$LMN = 3KLK'$$

Now $KLK'$ has its sides equal to $KL$ $KM$ $KN$ and it is similar to $ABC$

therefore

$$\frac{KLK'}{ABC} = \frac{KL^2 + KM^2 + KN^2}{a^2 + b^2 + c^2}$$

$$= \frac{4A^2}{(a^2 + b^2 + c^2)^2}$$

$$KBC : KCA : KAB = a^2 : b^2 : c^2 \quad (27) \dagger$$

$$K_1BC : K_1CA : K_1AB = -a^2 : b^2 : c^2 \quad (28) \ddagger$$

$$K_2BC : K_2CA : K_2AB = a^2 : -b^2 : c^2$$

$$K_3BC : K_3CA : K_3AB = a^2 : b^2 : -c^2$$

$$AA' \cdot BB' \cdot CC' : AK_1 \cdot BK_2 \cdot CK_3$$

$$= AK \cdot BK \cdot CK : KK_1 \cdot KK_2 \cdot KK_3 \quad (29) \ddagger$$

$$AK_2 : AK_3 = CX : BX \quad (30) \ddagger$$

$$BK_3 : BK_1 = AY : CY$$

$$CK_1 : CK_2 = BZ : AZ$$

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 298 (1867)
† L. C. Schulz von Strasznicki in Baumgartner and D'Ettingshausen's *Zeitschrift für Physik und Mathematik*, II. 403 (1827)
‡ Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 287, 293, 291 (1867)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press
**Figure 23**

Draw CZ perpendicular to AB

Let the tangent at A meet $K_1C$ produced at $K_2$ and draw $K_2B'$ perpendicular to CA

From the similar triangles $K_1CA' CAZ$

$$K_1C : CA' = CA : AZ$$

or

$$K_1C : a = b : 2AZ$$

From the similar triangles $K_2CB' CBZ$

$$K_2C : CB' = CB : BZ$$

or

$$K_2C : b = a : 2BZ$$

Hence

$$K_1C \cdot AZ = \frac{1}{2}ab = K_2C \cdot BZ$$

therefore

$$K_1C : K_2C = BZ : AZ$$

If $k_1, k_2, k_3$ denote the distances of $K$ from BC CA AB

$$k_1^2 + k_2^2 + k_3^2 : a^2 + b^2 + c^2 = \frac{1}{3}LMN : ABC \quad (31)$$

If $k_1', k_2', k_3'$ denote the distances of $K_1$

$$k_1' k_2' k_3' : k_1'' k_2'' k_3'' : k_2' k_3' k_1'' = a^3 : b^3 : c^3 \quad (32)$$

$$K_2K_3K_1K_1K_2 : BC \cdot CA \cdot AB = 2\text{circle}K_1K_2K_3 : \text{circle}ABC \quad (33)$$

**Figure 26**

If the sides of triangle DEF be denoted by $d, e, f$, the following formula is obtained by comparison of the similar triangles $I_1I_2I_3$ DEF

$$\frac{def}{r^2} = \frac{abc}{2R^2}$$

Now, in Fig. 26, triangles DEF $ABC$ stand in the same relation to each other as $ABC$ $K_1K_2K_3$

* Dr Wetzig in Schlämilch's *Zeitschrift*, XII. 298, 296, 292 (1867)
This follows from the preceding since
\[ \frac{abc}{2R^2} = \frac{2\Delta}{R} \]

**Cosine Circle or Second Lemoine Circle**

**Figure 19**

\[
\begin{align*}
AE &= \frac{2bc^2}{a^2 + b^2 + c^2} \\
AF' &= \frac{2bc}{a^2 + b^2 + c^2} \\
BF &= \frac{2ca^2}{a^2 + b^2 + c^2} \\
BD' &= \frac{2ca}{a^2 + b^2 + c^2} \\
CD &= \frac{2ab^2}{a^2 + b^2 + c^2} \\
CE' &= \frac{2ab}{a^2 + b^2 + c^2}
\end{align*}
\]

(35)

For triangles $\triangle AEF'$ $\triangle ABC$ are similar and $AK$ is a median of $\triangle AEF'$

therefore

\[
AE : AB = AK : m_1
\]

\[
\begin{align*}
AE' &= \frac{b(-a^2 + b^2 + c^2)}{a^2 + b^2 + c^2} \\
AF &= \frac{c(-a^2 + b^2 + c^2)}{a^2 + b^2 + c^2} \\
BF' &= \frac{c(a^2 - b^2 + c^2)}{a^2 + b^2 + c^2} \\
BD &= \frac{a(a^2 - b^2 + c^2)}{a^2 + b^2 + c^2} \\
CD' &= \frac{a(a^2 + b^2 - c^2)}{a^2 + b^2 + c^2} \\
CE &= \frac{b(a^2 + b^2 - c^2)}{a^2 + b^2 + c^2}
\end{align*}
\]

(36)

\[
\begin{align*}
DD' &= \frac{a(-a^2 + b^2 + c^2)}{a^2 + b^2 + c^2} \\
EE' &= \frac{b(a^2 - b^2 + c^2)}{a^2 + b^2 + c^2} \\
FF' &= \frac{c(a^2 + b^2 - c^2)}{a^2 + b^2 + c^2}
\end{align*}
\]

(37)

\[
\begin{align*}
FD &= \frac{4a\Delta}{a^2 + b^2 + c^2} \\
DE &= \frac{4b\Delta}{a^2 + b^2 + c^2} \\
EF &= \frac{4c\Delta}{a^2 + b^2 + c^2}
\end{align*}
\]

(38)

For

\[ FD^2 = E'D^2 - E'T^2 = E'D^2 - D'D^2 \]
100

\[ BD' \cdot CD = CE' \cdot AE = AF' \cdot BF \]  \hspace{1cm} (39)

\[ BD' : CE' : AF' = \frac{c}{b} : \frac{a}{c} : \frac{b}{a} \]  \hspace{1cm} (40)

\[ BF : CD : AE = \frac{a}{b} : \frac{b}{c} : \frac{c}{a} \]  \hspace{1cm} (41)

**TriPLICATE RATIO or FIRST LEMOINE CIRCLE**

The whole of the subsequent results are taken from two of Mr R. Tucker's papers in the *Quarterly Journal of Mathematics*, XIX. 342-348 (1883) and XX. 57-59 (1885). The proofs are sometimes different from Mr Tucker's.

**Figure 32**

\[
\begin{align*}
AF &= \frac{b^2c}{a^2 + b^2 + c^2} \quad & AE' &= \frac{bc^2}{a^2 + b^2 + c^2} \\
BD &= \frac{c^2a}{a^2 + b^2 + c^2} \quad & BF' &= \frac{ca^2}{a^2 + b^2 + c^2} \\
CE &= \frac{a^2b}{a^2 + b^2 + c^2} \quad & CD' &= \frac{ab^2}{a^2 + b^2 + c^2}
\end{align*}
\]

For

\[ \frac{ABC}{AFE'} = \frac{AB \cdot AC}{AF \cdot AE'} \]

Therefore

\[ \frac{2\Delta}{AE' \cdot KM} = \frac{bc}{AF \cdot AE'} \]

Therefore

\[ AF = \frac{bc \cdot KM}{2\Delta} = \frac{bc \cdot b}{a^2 + b^2 + c^2} \]

\[ BD' \cdot CD' = CE' \cdot AE' = AF' \cdot BF' \]  \hspace{1cm} (42)
101

$$\begin{align*}
AF' &= \frac{c(b^2 + c^2)}{a^2 + b^2 + c^2} \\
AE &= \frac{b(b^2 + c^2)}{a^2 + b^2 + c^2} \\
BD' &= \frac{a(c^2 + a^2)}{a^2 + b^2 + c^2} \\
BF &= \frac{c(c^2 + a^2)}{a^2 + b^2 + c^2} \\
CE' &= \frac{b(a^2 + b^2)}{a^2 + b^2 + c^2} \\
CD &= \frac{a(a^2 + b^2)}{a^2 + b^2 + c^2}
\end{align*}$$

$$\begin{align*}
DD' &= \frac{a^2}{a^2 + b^2 + c^2} \\
EE' &= \frac{b^2}{a^2 + b^2 + c^2} \\
FF' &= \frac{c^2}{a^2 + b^2 + c^2}
\end{align*}$$

$$\begin{align*}
BD : DD' : D'C &= c^2 : a^2 : b^2 \\
AF : FF' : FB &= c^2 : a^2 : a^2
\end{align*}$$

$$\begin{align*}
EF' &= \frac{a(b^2 + c^2)}{a^2 + b^2 + c^2} \\
FD' &= \frac{b(c^2 + a^2)}{a^2 + b^2 + c^2} \\
DE' &= \frac{c(a^2 + b^2)}{a^2 + b^2 + c^2}
\end{align*}$$

$$\begin{align*}
E'F &= F'D = D'E = \frac{abc}{a^2 + b^2 + c^2}
\end{align*}$$

For \(DE'FF'\) is a symmetrical trapezium

therefore \(E'F^2 = \frac{1}{4}(DE' - FF')^2 + Kn^2\)

$$\begin{align*}
&= \frac{1}{4} \left\{ \frac{c(a^2 + b^2)}{a^2 + b^2 + c^2} - \frac{c^2}{a^2 + b^2 + c^2} \right\}^2 + \left\{ \frac{2c\Delta}{a^2 + b^2 + c^2} \right\}^2 \\
&= \frac{c^2}{4(a^2 + b^2 + c^2)^2} \left\{ (a^2 + b^2 - c^2)^2 + 16\Delta^2 \right\} \\
&= \frac{4a^2b^2c^2}{4(a^2 + b^2 + c^2)^2}
\end{align*}$$

$$\begin{align*}
DF &= \frac{c}{a^2 + b^2 + c^2} \sqrt{b^2c^2 + c^2a^2 + a^2b^2} \\
FE &= \frac{b}{a^2 + b^2 + c^2} \sqrt{b^2c^2 + c^2a^2 + a^2b^2} \\
ED &= \frac{a}{a^2 + b^2 + c^2} \sqrt{b^2c^2 + c^2a^2 + a^2b^2}
\end{align*}$$

* It was this property which suggested to Mr Tucker the name "triplicate-ratio circle."
For $D E' F F'$ are concyclic and $DE'FF'$ is a symmetrical trapezium; therefore

\[ DF \cdot E'F' = DE' \cdot FF' + DF' \cdot FE' \]

that is

\[ DF^2 = DE' \cdot FF' + DF'^2 \]

\[ KE \cdot KF' = KD' \cdot KE' \]

\[ = E'F'^2 = F'D^2 = D'E^2 \]

\[ (49) \]

For $KE \cdot KF' = CD' \cdot BD$

The minimum chord through $K$

\[ = 2E'F = 2F'D = 2D'E \]

\[ (50) \]

For $KE \cdot KF' = E'F'^2$

\[ DD' \cdot EE' \cdot FF' = E'F' \cdot F'D \cdot D'E \]

\[ = \frac{a^2b^2c^3}{(a^2 + b^2 + c^2)^3} \]

\[ (51) \]

The hexagon $DD'EE'FF'$ has its

perimeter

\[ = \frac{a^3 + b^3 + c^3 + 3abc}{a^2 + b^2 + c^2} \]

\[ (52) \]

area

\[ = \frac{\triangle (a^4 + b^4 + c^4 + b^4c^2 + c^4a^2 + a^4b^2)}{(a^2 + b^2 + c^2)^2} \]

\[ = AFE + BDF + CED \]

\[ = AEF' + BFD' + CDE' \]

\[ (54) \]

The circle $DD'EE'FF'$ has its

radius

\[ = \frac{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}{a^2 + b^2 + c^2} \cdot R \]

\[ (55) \]

If $EF'$ cuts $FD$ $D'E'$ at $L$ $L'$

$FD'$ ,, $DE$ $E'F'$ ,, $M$ $M'$

$DE'$ ,, $EF$ $F'D'$ ,, $N$ $N'$
then

\[
\begin{align*}
FL : DL &= c^2 : a^2 \\
D'L' : E'L' &= a^2 : b^2 \\
DM : EM &= a^2 : b^2 \\
EM' : FM' &= b^2 : c^2 \\
EN : FN &= b^2 : c^2 \\
FN' : DN' &= c^2 : a^2
\end{align*}
\]

\( (56) \)

**Figure 33**

For

\[
FL : DL = FK : D'K = AE' : CE
\]

and

\[
D'L' : E'L' = CE : E'E'
\]