ON PROPERTIES POSSESSED BY SOLVABLE
AND NILPOTENT GROUPS

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The object of this note is to study two properties of groups, which we will denote by (*) and (**). The property (*) is possessed by solvable groups (and in fact, by groups which have a solvable invariant system) and the property (**) is possessed by nilpotent groups (and in fact, by groups which have a central system).

It is quite easy to show that if a group satisfies (*) locally, then it satisfies (*); this gives a short proof of Malcev's theorem that a locally solvable group cannot be simple unless it is cyclic of prime order. It should be remarked, however, that the proof given is simply an adaption of Malcev's proof — its only virtue is that it is short and easy.

Theorem 2 states that a finitely generated group $G$ satisfying (*) and the minimum condition for normal subgroups is finite and solvable, and Theorem 3 studies the connection between property (*) and a property studied by Ore.

Theorem 5 states that if the group $G$ — with hypercentre $C$ — satisfies (**), then $G/C$ satisfies (**); from this we deduce that if $G$ satisfies (**) and the minimum condition for normal subgroups, $G$ is hypercentral.

Notations

$[a, b] = a^{-1}b^{-1}ab.$

$n(U) =$ normalizer of the subgroup $U$ in $G.$

$Z(G) =$ centre of the group $G.$

$A \leq B: =$ $A$ is a subgroup of $B.$

$A < B: =$ $A$ is a proper subgroup of $B.$

$A \triangleleft B: =$ $A$ is a normal subgroup of $B.$

$E =$ trivial subgroup (consisting of the identity element).

Following Kurosh we call $G$ an $SI$-group ($SN$-group) if it has an invariant (normal) system with abelian factors (see Kurosh [5, p. 171—73

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and we call $G$ a $Z$-group if it has a central system — see Kurosh [5, p. 218]. We say that $G$ is a $ZA$-group if the upper central chain for $G$, possibly continued transfinitely, leads up to $G$ — see Kurosh [5, p. 218—19]. (Baer calls such a group hypercentral and uses the equivalent definition that $G$ is hypercentral if every epimorphic image ($\neq E$) has a non-trivial centre.) $G$ is an $SI^*$-group if it has a solvable ascending invariant series (this is what Baer calls hyperabelian; again an equivalent definition is that the group $G$ is hyperabelian if every epimorphic image ($\neq E$) has a non-trivial normal abelian subgroup).

The property (*)

**Definition 1.** The group $G$ satisfies (*) if: given elements $a, b (\neq 1, 1)$ in $G$, there is a normal subgroup $C = C(a, b)$ of $G$ such that $[a, b]$ is in $C$ but not both $a$ and $b$ are in $C$.

**Remark.** If $G$ satisfies (*), and $a, b (\neq 1, 1)$ are elements of $G$, we can define

$$C_{a,b} = \{n \in G : [a, b] \in C \text{ and not both } a \text{ and } b \text{ are in } C\}.$$

Clearly $C_{a,b}$ is normal in $G$, $[a, b]$ is in $C_{a,b}$ but not both $a$ and $b$ are in $C_{a,b}$. $C_{a,b}$ is the unique smallest normal subgroup of $G$ with these properties.

**Lemma 1.** (i) If $S$ is a subgroup of the group $G$ and if $G$ satisfies (*), then $S$ satisfies (*).

(ii) If $N$ is a normal subgroup of the group $G$ and if $G$ satisfies (*), then given elements $a, b (\neq 1, 1)$ in $N$ there exists a normal subgroup $C$ of $G$ such that $C < N$, $[a, b] \in C$ and not both $a$ and $b$ are in $C$.

Thus if $G$ has a local system each of whose subgroups satisfies (*), the finitely generated subgroups of $G$ satisfy (*).

**Proposition 1.** If $G$ is an $SI$-group, then $G$ satisfies (*). In particular, if $G$ is solvable, $G$ satisfies (*).

**Proof.** Let $\Sigma$ be an invariant system for $G$ with abelian factors. Let $a, b (\neq 1, 1)$ be any two elements of $G$ and define

$$\bar{C} = \{\cap N : N \in \Sigma, a \text{ and } b \text{ both } \in N\},$$

$$C = \{\cup K : K \in \Sigma, \text{ not both } a \text{ and } b \in K\}.$$

Then $C < \bar{C}$ is a jump in $\Sigma$; hence $\bar{C}/C$ is abelian so that $[a, b] \in C$. Clearly $C$ is a normal subgroup of $G$ and not both $a$ and $b$ belong to $C$.

**Proposition 2.** Let $G$ be a group and assume that for each pair of elements $a, b (\neq 1, 1)$ a normal subgroup $C_{a,b}$ can be chosen so that $[a, b] \in C_{a,b}$, but not both $a$ and $b$ are in $C_{a,b}$ and that in addition these subgroups can be chosen
in such a way that for \( a, b(\neq 1, 1), c, d(\neq 1, 1) \) in \( G \) either \( C_{a,b} \leq C_{c,d} \) or \( C_{c,d} \leq C_{a,b} \) (i.e. in such a way that the subgroups are linearly ordered). Then \( G \) is an SI-group.

**Proof.** Complete the system of normal subgroups \( \{C_{a,b}\} \) to a system \( \Sigma \). We show that if \( K < L \) is a jump in \( \Sigma \), then \( L/K \) is abelian. For suppose not; then there are elements \( a \) and \( b \) in \( L \) with \( [a, b] \) not in \( K \). Now if \( L \leq C_{a,b} \), \( a \) and \( b \) both lie in \( C_{a,b} \), which is impossible. Hence \( C_{a,b} < L \), which implies that \( C_{a,b} \leq K \). But then \( [a, b] \in K \), a contradiction.

**Theorem 1.** If the group \( G \) satisfies (*) locally, then \( G \) satisfies (*).

**Proof.** Let \( \Sigma \) consist of all finitely generated subgroups of \( G \). For \( A \) in \( \Sigma \) and \( a, b(\neq 1, 1) \) in \( A \) let \( C_{a,b}(A) \) be a fixed normal subgroup of \( A \) such that \( a \) and \( b \) are not both in \( C_{a,b}(A) \) but \( [a, b] \in C_{a,b}(A) \).

For \( a, b(\neq 1, 1) \) in \( G \) and \( S \) a finite subset of \( G \) define

\[
K_{a,b}(S) = \{ \cap C_{a,b}(A) \mid A \in \Sigma, \{a, b, S\} \subseteq A \}.
\]

Clearly if \( S_1 \subseteq S_2 \) are finite subsets of \( G \), \( K_{a,b}(S_1) \leq K_{a,b}(S_2) \). Thus for arbitrary finite subsets \( S_1 \) and \( S_2 \) of \( G \), \( K_{a,b}(S_1) \leq K_{a,b}(S_2 \cup S_2) \) for \( i = 1, 2 \).

Let \( H_{a,b} = \{ \cup K_{a,b}(S) \mid S \text{ a finite subset of } G \} \). It is clear that \( H_{a,b} \) is a subgroup of \( G \) which contains \( [a, b] \) but does not contain both \( a \) and \( b \). It remains to verify that \( H_{a,b} \) is normal in \( G \). So let \( c \in H_{a,b} \) and \( d \in G \). Then \( c \in K_{a,b}(S) \) for some finite subset \( S \) of \( G \) and we can assume that \( d \in S \). Now \( c \in C_{a,b}(A) \) for each \( A \) in \( \Sigma \) with \( \{a, b, S\} \subseteq A \). Hence by the normality of \( C_{a,b}(A) \) in \( A \), \( d^{-1}cd \) is in \( C_{a,b}(A) \) for each \( A \) in \( \Sigma \) with \( \{a, b, S\} \subseteq A \). Hence \( d^{-1}cd \in K_{a,b}(S) \) and this implies that \( d^{-1}cd \in H_{a,b} \).

**Corollary 1.** If \( G \) is locally solvable and not cyclic of prime order, then \( G \) is not simple.

As noted in the introduction the proof of Theorem 1 is just Malcev's proof adapted to the case considered. Malcev's Theorem states that if a group has the property SI locally then it is an SI-group. For a proof see Kurosh [5, p. 183—87].

**Definition 2.** Let \( V \) be a maximal normal subgroup of the group \( U \); then \( U/V \) is a tor of \( U \).

**Lemma 2.** Let \( G \) be a group which satisfies (*) and the minimum condition for normal subgroups. Then if \( K \) is a normal subgroup of \( G \), any tor of \( K \) is abelian.

**Proof.** Assume that the lemma is false and let \( U \) be a minimal normal subgroup of \( G \) with a non-abelian tor.\(^2\) Hence there exists \( V < U \) such that

\(^2\) i.e. \( U \) is a normal subgroup of \( G \), has a non-abelian tor and is minimal with respect to this property.
$U/V$ is simple non-abelian. Thus there exist elements $a$ and $b$ in $U$ such that $[a, b] \notin V$. Let $C$ be a normal subgroup of $G$ such that $C < U$, $[a, b] \in C$ and not both $a$ and $b \in C$. Then $V \leq VC \leq U$ and $V \neq VC$ since $[a, b] \in C$ but $[a, b] \notin V$. Hence by the maximality of $V, U = VC$.

Now $U/V = VC/V \cong C/V \cap C$. Thus $C$ is a normal subgroup of $G$ with a non-abelian tor and $C < U$. This contradicts the minimality of $U$.

**Theorem 2.** Let $G$ be a finitely generated group which satisfies (*) and the minimum condition for normal subgroups. Then $G$ is a finite, solvable group.

**Proof.** Let $K$ be a normal subgroup of $G$ and assume $K$ is minimal such that $G/K$ is finite and solvable. Assume $K \neq E$. Then since $K$ is finitely generated, it possesses a maximal normal subgroup $M$. By Lemma 2, $K/M$ is abelian and hence cyclic of prime order. Let $\overline{M} = \{ \cap M^{|x|} | x \in G \}$. Since $M$ is of finite index in $G$, $\overline{M}$ is also of finite index in $G$. Furthermore, $\overline{M}$ is normal in $G$ and $G/\overline{M}$ is solvable since $K/\overline{M}$ is solvable. But $\overline{M} < K$ so that the minimality of $K$ is contradicted. Hence $K = E$ and $G$ is finite and solvable.

**Corollary 2.** Let $G$ be a group which satisfies (*) and the minimum condition for subgroups $U$ such that $n(U) > U$. Then $G$ is locally finite and locally solvable. Furthermore, $G$ is an $SI^*$-group.

**Proof.** If $H$ is a finitely generated subgroup of $G$, $H$ satisfies (*) and the minimum condition for normal subgroups. Hence $H$ is finite and solvable. In particular, $H$ is an $SI$-group. By the local theorem for $SI$-groups, $G$ is an $SI$-group and by the minimum condition for normal subgroups, $G$ is an $SI^*$-group.

We now consider a property which Kurosh denotes by $(Q)$, and a somewhat weaker one which will be denoted by $(Q')$. The property $(Q)$ was introduced by Ore (see Kurosh [5, p. 181] and Ore [7, p. 251, Theorem 9]).

**Definition 3.** The subgroup $A$ of $G$ is almost normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $G = AN$ and $A \cap N < G$.

**Definition 4.** The group $G$ satisfies $(Q)$ if $A < B \leq G$, and $A$ maximal in $B$, implies that $A$ is almost normal in $B$.

**Definition 5.** The group $G$ satisfies $(Q')$ if $A < B \leq G$, and $A$ maximal in $B$, implies that either $A < B$, or there exists a proper normal subgroup $N$ of $B$ such that $B = AN$.

It is clear that if $G$ satisfies $(Q)$, it satisfies $(Q')$.

**Theorem 3.** (i) If the group $G$ satisfies $(*)$, it satisfies $(Q')$.

(ii) If the group $G$ satisfies $(*)$ and the minimum condition for subgroups $U$ such that $n(U) > U$, then $G$ satisfies $(Q)$.
(iii) If the group $G$ satisfies $(Q')$ and the minimum condition for subgroups, then $G$ satisfies $(\ast)$.

**Proof.** (i): Let $A < B \subseteq G$ with $A$ maximal in $B$. If $A$ is not normal in $B$, let $a$ and $b$ be elements of $B$ with $[a, b]$ not in $A$. By $(\ast)$ there is a subgroup $C < B$ which does not contain both $a$ and $b$ but which contains $[a, b]$. Then $A \leq AC \leq B$ but $C \not\subseteq A$. Hence by the maximality of $A$, $AC = B$.

(ii): By Corollary 2, $G$ is an SI*-group and from this fact it follows that $G$ satisfies $(Q)$ (see Kurosh [5, p. 183]). However, it is easy to give a proof which does not use the local theorem for SI-groups (which is needed for Corollary 2): Let $A < B \leq G$ with $A$ maximal in $B$. Since the normal subgroups of $B$ satisfy the minimum condition, we can choose a minimal subgroup $K$ such that $K < B$ and $B = AK$. Now $A \cap K \neq A$; if $A \cap K < K$, then $A \cap K < B$. So assume that $A \cap K$ is not normal in $K$ and let $a$ and $b$ be elements of $K$ such that $[a, b] \notin A \cap K$. By $(\ast)$ there exists a subgroup $C$ of $K$ such that $C < B$, $[a, b] \in C$, but not both $a$ and $b$ are in $C$. Hence $A < AC \leq B$ since $[a, b] \notin A$. Thus $B = AC$ and the minimality of $K$ is contradicted.

(iii): Assume that the group $G$ satisfies the hypotheses of (iii) but does not satisfy $(\ast)$. Let $H$ be a minimal subgroup of $G$ which does not satisfy $(\ast)$. If $H$ is not finitely generated, all the finitely generated subgroups of $H$ satisfy $(\ast)$; but this implies that $H$ satisfies $(\ast)$ by Theorem 1. Hence $H$ is finitely generated.

If $H$ contains a maximal subgroup $M$ which is normal, then $H/M$ is cyclic of prime order. Hence $M$ is finitely generated and satisfies $(\ast)$ by the minimality of $H$. Therefore, by Theorem 2, $M$ is (finite and) solvable. But this implies that $H$ is solvable so that by Proposition 1, $H$ satisfies $(\ast)$ — a contradiction.

So assume that every maximal subgroup of $H$ is not normal and let $A$ be a maximal subgroup of $H$. Then by $(Q')$ there is a proper normal subgroup $N$ of $H$ such that $H = NA$. Let $M$ be a maximal normal subgroup of $H$ containing $N$. Then $H = MA$. $H/M$ is simple and non-abelian. Also $H/M = MA/M \cong A/M \cap A$ so that $A$ has a non-abelian tor. But $A$ satisfies $(\ast)$ since it is a proper subgroup of $H$, and hence by Lemma 2, any tor of $A$ is abelian. Thus we have a contradiction and the theorem is proved.

**Corollary 3.** Let $G$ be a group which satisfies the minimum condition for subgroups. Then the following are equivalent:

1. $G$ is solvable.
2. $G$ satisfies $(\ast)$.
3. $G$ satisfies $(Q)$.
4. $G$ satisfies $(Q')$.  


PROOF. By Proposition 1, (1) implies (2). (2) implies (3) by Theorem 3 (ii). Clearly (3) implies (4). So assume (4). Then by Theorem 3 (iii) $G$ satisfies (*). Hence by Corollary 2, $G$ is an $SI^*$-group. Therefore, by a theorem of Cernikov, $G$ is solvable (see Kurosh [5, p. 191]).

REMARK: Since submitting this paper it has been drawn to my attention that Baer has two papers to appear shortly ([1] and [2]) in which he considers among other things the properties $(Q)$ and $(Q')$. The main theorem of [1] gives a number of criteria for a group $G$ to be artinian and solvable. One of these is:

(a) Abelian subgroups of $G$ are artinian.

(b) If $F$ is a finitely generated subgroup of $G$, then the normal subgroups of $F$ satisfy the minimum condition.

(b') If $S$ is a maximal subgroup of $F$, then $S$ is almost normal in $F$.

This criterion implies that if $G$ is artinian, then $G$ is solvable if, and only if $G$ satisfies $(Q)$. But, of course, it is a much stronger result.

In the same spirit we could prove: $G$ is artinian and solvable if, and only if

(a) Abelian subgroups of $G$ are artinian.

(b) If $F$ is a finitely generated subgroup of $G$, then the normal subgroups of $F$ satisfy the minimum condition.

(b') $F$ satisfies (*).

This follows from our Theorem 2 and the theorem of Cernikov (see [4]) which states: Let $G$ be locally finite and locally solvable. Then if abelian subgroups of $G$ are artinian, $G$ is artinian and solvable.

In Baer’s paper ‘Normalizatorreiche Gruppen’ there is another proof of the fact that an artinian group $G$ is solvable if, and only if it satisfies $(Q')$ (see [2] Hilfsatz 3.6).

The property (**)

DEFINITION 6. The group $G$ satisfies (**): given an element $a(\neq 1)$ in $G$, there is a normal subgroup $N = N(a)$ of $G$ such that $[a, x] e N \forall x \in G$ but $a \notin N$.

REMARK. If $G$ satisfies (**) and $a(\neq 1)$ is an element of $G$, we can define $N_a = \{ \cap N | N < G, a \notin N \}$ and $[a, x] e N \forall x \in G$ then $N_a < G, a \notin N_a$ and $[a, x] e N_a$. $N_a$ is the unique smallest normal subgroup of $G$ with these properties.

As in the case of (*) we have:

LEMMA 3. (i) If $S$ is a subgroup of the group $G$, and if $G$ satisfies (**), then $S$ satisfies (**).
(ii) If \( K \) is a normal subgroup of the group \( G \), and if \( G \) satisfies (**), then given an element \( a \neq 1 \) in \( K \) there exists a normal subgroup \( N \) of \( G \) such that \( N < K \), \( a \notin N \) but \([a, x] \in N \ \forall x \in G\).

It is clear that (***) implies (*). For if \( a, b \neq 1 \) are elements of the group \( G \), then if \( a \neq 1 \) we can find a normal subgroup \( N \) of \( G \) such that \( a \notin N \) but \([a, x] \in N \) for all \( x \in G \). Thus \([a, b] \in N \) but not both \( a \) and \( b \) are in \( N \). If \( a = 1, b \neq 1 \) and we interchange the roles of \( a \) and \( b \).

**PROPOSITION 3.** If \( G \) is a Z-group, then \( G \) satisfies (**). In particular, if \( G \) is nilpotent, \( G \) satisfies (**).

**PROOF.** Let \( \Sigma \) be a central system for \( G \). Let \( a \neq 1 \) be an element of \( G \) and define
\[
N = \{ \cap K | K \in \Sigma, \ a \in K \} \\
N = \{ \cup L | L \in \Sigma, \ a \notin L \}
\]
Then \( N < N \) is a jump in \( \Sigma \); hence \( N/N \leq Z(G/N) \) and this implies that \([a, x] \in N \ \forall x \in G\).

**PROPOSITION 4.** Let \( G \) be a group and assume that for each element \( a \neq 1 \) a normal subgroup \( N_a \) can be chosen so that \( a \nsubseteq N_a \) but \([a, x] \in N_a \ \forall x \in G \) and that in addition these subgroups are linearly ordered. Then \( G \) is a Z-group.

The proof of this proposition is quite similar to the proof of Proposition 2 and will be omitted.

**THEOREM 4.** If the group \( G \) satisfies (***) locally, then \( G \) satisfies (**).

**PROOF.** Let \( \Sigma \) consist of all finitely generated subgroups of \( G \). For \( H \) in \( \Sigma \) and \( a \neq 1 \) in \( H \) let \( N_a(H) \) be a fixed normal subgroup of \( H \) such that \( a \nsubseteq N_a(H) \) but \([a, x] \in N_a(H) \ \forall x \in H \).

For \( a \neq 1 \) in \( G \) and \( S \) a finite subset of \( G \) containing \( a \), define \( K_a(S) = \{ \cap N_a(H) | H \in \Sigma, S \subseteq H \} \). Let \( K_a = \{ \cup K_a(S) | S \) a finite subset of \( G \) containing \( a \} \). It is easy to verify that \( K_a \) is a normal subgroup of \( G \) such that \( a \nsubseteq K_a \) but \([a, x] \in K_a \ \forall x \in G \).

**LEMMA 4.** Let \( G \) be a group which satisfies (***) and \( Z \) a subgroup of the centre of \( G \). Then \( G/Z \) satisfies (**).

**PROOF.** Let \( a \in G \setminus Z \) and let \( N_a \) be the minimal normal subgroup of \( G \) such that \( a \nsubseteq N_a \) but \([a, x] \in N_a, \ \forall x \in G \). Then \( ZN_a/Z < G/Z \) and \([Za, Zx] \in ZN_a/Z, \ \forall x \in G \). We have to verify that \( Z \notin ZN_a/Z \).

So suppose that \( a \in ZN_a \). Then we can write: \( a = zc \), where \( z \in Z \) and \( c \in N_a \).

Now let \( N_c \) be a normal subgroup of \( G \) such that \( c \nsubseteq N_c \), but \([c, x] \in N_c, \ \forall x \in G \). Then \( N_c \cap N_a < G \) and \( N_c \cap N_a < N_a \) since \( c \nsubseteq N_a \) but \( c \in N_a \). Clearly \( a \nsubseteq N_c \cap N_a \) since \( a \nsubseteq N_c \). For any element \( x \in G \), we have:
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\[ [a, x] = [zc, x] = [z, x]c[c, x] = [c, x] \text{ since } z \text{ is a central element.} \]

Hence \([a, x] \in N_a \cap N_c\), and the minimality of \(N_a\) is contradicted. Thus \(a \notin ZN_a\) so that \(Za \notin ZN_a\).

**Theorem 5.** If the group \(G\) satisfies (**) and if \(C\) is the hypercentre of \(G\), then \(G/C\) satisfies (**) for \(C\).

**Proof.** We define the ascending central chain of \(G\) by \(Z_o = E, Z_1 = Z(G), \ldots, Z_{\beta+1}/Z_{\beta} = Z(G/Z_{\beta})\) and \(Z_{\alpha} = \cup Z_{\beta}|\beta < \alpha\) for \(\alpha\) a limit ordinal. Then there is an ordinal \(v\) such that \(Z_v = Z_{v+1}\). \(C = Z_v\) is the hypercentre of \(G\).

We prove by transfinite induction that each \(G/Z_a\) satisfies (**) for \(\alpha\) a limit ordinal. Clearly \(G/Z_o\) satisfies (**) for \(\alpha = 0\).

**Case 1.** \(\alpha = \beta + 1\) and \(G/Z_{\beta}\) satisfies (**) for \(\beta < \alpha\). Then \(G/Z_{\alpha} \cong G/Z_{\beta}/Z_{\beta+1}/Z_{\beta} = G/Z_{\beta}\) satisfies (**) by Lemma 4.

**Case 2.** \(\alpha\) is a limit ordinal, and \(G/Z_{\beta}\) satisfies (**) for \(\beta < \alpha\). Thus if \(a \in G \setminus Z_{\beta}\), there exists \(U/Z_{\beta} \leq G/Z_{\beta}\) such that \(a \notin U\) but \([a, x] \in U\) for all \(x \in G\). Hence \(Z_{\beta} \leq U \triangleleft G\), \(a \notin U\) but \([a, x] \in U\) for all \(x \in G\). Let \(V_{\beta}(a) = \{U/Z_{\beta} \leq U \triangleleft G, a \notin U, [a, x] \in U \forall x \in G\}\).

Then \(Z_{\beta} \leq V_{\beta}(a) \supset G, a \notin V_{\beta}(a)\) and \([a, x] \in V_{\beta}(a) \forall x \in G\), and \(V_{\beta}(a)\) is the unique minimal subgroup of \(G\) with these properties.

We verify that if \(\beta \leq \gamma < \alpha\) and \(a \in G \setminus Z_{\gamma}\), then \(V_{\beta}(a) \leq V_{\gamma}(a)\). For \(Z_{\beta} \leq Z_{\gamma} \leq V_{\gamma}(a), a \notin V_{\gamma}(a)\) and \([a, x] \in V_{\gamma}(a) \forall x \in G\). Hence by the minimality of \(V_{\gamma}(a), V_{\beta}(a) \leq V_{\gamma}(a)\).

Now let \(a \in G \setminus Z_{\alpha}\). Then \(a \in G \setminus Z_{\beta}\) for all \(\beta < \alpha\). Define \(V(a) = \{U/V_{\beta}(a)|\beta < \alpha\}\). Since the \(V_{\beta}(a)\) are linearly ordered and normal, \(V(a)\) is a normal subgroup of \(G\); \(a \notin V(a)\) but \([a, x] \in V(a) \forall x \in G\). Also, \(Z_{\beta} \leq V_{\alpha}(a)\) for \(\beta < \alpha \Rightarrow Z_{\alpha} = \cup Z_{\beta} \leq \cup V_{\beta}(a) = V(a)\).

Hence \(G/Z_{\alpha}\) satisfies (**) in this case also.

**Lemma 5.** If the group \(G(\neq E)\) satisfies (**) and the minimum condition for normal subgroups then for \(E < H \triangleleft G\), \(H \cap Z(G) \neq E\).

**Proof.** Let \(N\) be a minimal normal subgroup of \(G\) contained in \(H\). If \(N \triangleleft Z(G)\), there are elements \(a \in N\) and \(x \in G\) such that \([a, x] \neq 1\). By (**) and Lemma 3 (ii) we can find a normal subgroup \(N_a\) of \(G\) such that \(N_a < N, a \notin N_a\) and \([a, y] \in N_a \forall y \in G\). Hence \(1 \neq [a, x] \in N_a\) so that \(E < N_a < N\) contrary to the minimality of \(N\). Thus \(N \leq H \triangleleft Z(G)\).
THEOREM 6. If the group $G$ satisfies (***) and the minimum condition for normal subgroups, then $G$ is a ZA-group.

PROOF. Let $C$ be the hypercentre of $G$. If $C \neq G$, $G/C$ satisfies (***) by Theorem 5. Hence since $G/C$ satisfies the minimum condition for normal subgroups, $Z(G/C) \neq E$ (provided $G \neq C$) by Lemma 5. But this is impossible. Hence $G = C$ and $G$ is a ZA-group.

COROLLARY 4. If $G$ is a finitely generated group satisfying (***) and the minimum condition for normal subgroups, then $G$ is finite and nilpotent.

PROOF. By Theorem 2, $G$ is finite and by Theorem 6, $G$ is a ZA-group. Hence $G$ is finite and nilpotent.

Finally we recall two further conditions which may be imposed on groups:

DEFINITION 7. The group $G$ is an $N$-group if the normalizer condition holds in $G$, i.e. if every proper subgroup of $G$ is distinct from its normalizer.

DEFINITION 8. A group $G$ is an $\bar{N}$-group if in every subgroup $B$ of $G$ every maximal subgroup $A$ is normal.

THEOREM 7. Let $G$ be a group satisfying the minimum condition for subgroups. Then the following are equivalent:

(1) $G$ is a ZA-group.
(2) $G$ is an $N$-group.
(3) $G$ is an $\bar{N}$-group.
(4) $G$ satisfies (**). 
(5) $G$ is locally finite and locally nilpotent.

PROOF. It is well-known that (1) implies (2) (see e.g. Kurosh p. 215 and p. 219). A group $G$ is an $N$-group if and only if through each subgroup of $G$ there passes an ascending normal series, while $G$ is an $\bar{N}$-group if for every subgroup of $G$ there is some normal system passing through it (see Kurosh pp. 220—21). Hence (3) follows from (2).

Now assume that $G$ is an $\bar{N}$-group which does not satisfy (**), and let $H$ be a minimal subgroup of $G$ which does not satisfy (**). By Theorem 4, $H$ is finitely generated. Let $M$ be a maximal subgroup of $H$. Then $M$ is normal in $H$, and hence of finite index. Thus $M$ is a finitely generated subgroup of $G$ which satisfies (**). By Corollary 4, $M$ is finite. But this implies that $H$ is finite and a finite $\bar{N}$-group is nilpotent (see Kurosh p. 216). Hence by Proposition 3, $H$ satisfies (**) — contrary to the choice of $H$. Therefore, (3) implies (4).

(5) follows from (4) by Corollary 4. Finally if $G$ satisfies (5), it is a $Z$-group and hence a ZA-group since it satisfies the minimum condition for subgroups.
REMARK. It should be noted that the (equivalent) conditions in Theorem 7 do not imply that \( G \) is nilpotent. For example, let \( A \) be a group of type \( p^\infty \) and let \( B \) be cyclic of order \( p \). Then \( G = A \wr B \) (the wreath product of \( A \) and \( B \)) is solvable and satisfies the minimum condition. Any finitely generated subgroup \( H \) of \( G \) is solvable and satisfies the minimum condition. Hence \( H \) is a finite \( p \)-group and so nilpotent. Therefore, \( G \) is locally nilpotent. But \( G \) is not nilpotent since \( A \) is not bounded (see Baumslag [3, § 3]).

References


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