BOUNDS FOR SOLUTIONS OF A SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS ON DOMAINS WITH BERGMAN-SILOV BOUNDARY

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1. Introduction.

1. The method of integral operators has been used by Bergman and others **(4; 6; 7; 10; 12)** to obtain many properties of solutions of linear partial differential equations. In the case of equations in two variables with entire coefficients various integral operators have been introduced which transform holomorphic functions of one complex variable into solutions of the equation. This approach has been extended to differential equations in more variables and systems of differential equations. Recently Bergman **(6; 4)** obtained an integral operator transforming certain combinations of holomorphic functions of two complex variables into functions of four real variables which are the real parts of solutions of the system

$$(1)\qquad \frac{\partial^2\psi}{\partial z_1\,\partial z_1^*} = F_1(z_1, z_1^*)\psi, \qquad \frac{\partial^2\psi}{\partial z_2\,\partial z_2^*} = F_2(z_2, z_2^*)\psi,$$

where z_1, z_1^*, z_2, z_2^* are independent complex variables and the functions F_j $(j = 1, 2)$ are entire functions of the indicated variables. (In general, j takes the values 1 and 2. Note that if the variables x_1, y_1, x_2, y_2 are introduced in the usual manner by writing $z_j = x_j + iy_j$, $z_j^* = x_j - iy_j$ and if the new variables are restricted to real values, z_j^* coincides with the conjugate $\bar{z}_j$ of z_j).

Bergman showed that there exist four functions $T_j(z_j, z_j^*, \zeta_j)$ and $P_j(z_j, z_j^*, \zeta_j)$ which are entire functions of the indicated variables such that every real solution of (1), regular at the origin, can be represented in a neighbourhood of the origin in the form

$$(2)\qquad \psi(z_1, \bar{z}_1, z_2, \bar{z}_2) = \mathrm{Re}[\psi'(z_1, \bar{z}_1, z_2, \bar{z}_2) + \psi''(z_1, \bar{z}_1, z_2, \bar{z}_2)],$$

where

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$$\text{(2a)}\quad \psi'(z_1, \bar{z}_1, z_2, \bar{z}_2) = g_1(z_1, z_2) + \int_{\zeta_1=0}^{z_1} T_1(z_1, \bar{z}_1, \zeta_1) g_1(\zeta_1, z_2) d\zeta_1 + \int_{\zeta_2=0}^{z_2} T_2(z_2, \bar{z}_2, \zeta_2) g_1(z_1, \zeta_2) d\zeta_2 + \int_{\zeta_1=0}^{z_1} \int_{\zeta_2=0}^{z_2} \prod_j T_j(z_j, \bar{z}_j, \zeta_j) g_1(\zeta_1, \zeta_2) d\zeta_1\, d\zeta_2,$$

$$\text{(2b)}\quad \psi''(z_1, \bar{z}_1, z_2, \bar{z}_2) = g_2(z_1, \bar{z}_2) + \int_{\zeta_1=0}^{z_1} P_1(z_1, \bar{z}_1, \zeta_1) g_2(\zeta_1, \bar{z}_2) d\zeta_1 + \int_{\zeta_2=0}^{\bar{z}_2} P_2(z_2, \bar{z}_2, \zeta_2) g_2(z_1, \zeta_2) d\zeta_2 + \int_{\zeta_1=0}^{z_1} \int_{\zeta_2=0}^{z_2} \prod_j P_j(z_j, \bar{z}_j, \zeta_j) g_2(\zeta_1, \zeta_2) d\zeta_1\, d\zeta_2$$

and g_1 and g_2 are arbitrary functions of z_1, z_2 and z_1, $\bar{z}_2$ respectively, holomorphic in a neighbourhood of the origin.

In this paper we assume that g_j are defined on a domain lying in the space C^2 of two complex variables whose boundary consists of a finite number of segments of analytic hypersurfaces. The intersections of these hypersurfaces form a 2-dimensional manifold called the Bergman–Silov boundary of the domain on which a function holomorphic on the domain takes the maximum of its absolute value. The closed domain consists of the interior $\mathfrak{M}^4$, the Bergman–Silov boundary $\mathfrak{D}^2$, and the complementary part $\mathfrak{b}^3$ of the 3-dimensional boundary $\mathfrak{m}^3$. (The superscript indicates the dimension of the set.) We investigate what properties of the solution on $\mathfrak{b}^3$ can be used to obtain bounds for the solution on $\mathfrak{M}^4$. In §§2 and 3 bounds for the solution in a set $\mathfrak{N}^4 \subset \mathfrak{M}^4$, where $z = (z_1, z_2) \in \mathfrak{N}^4$ implies that a 2-dimensional set $\mathfrak{S}^2(z)$ lies in $\mathfrak{M}^4$ (see (3)), are obtained by means of the Schottky inequality for holomorphic functions of one complex variable. In §2 it is assumed that through every point ζ of $\mathfrak{S}^2(z)$ there passes an analytic surface $\mathfrak{A}^2(\zeta)$ which intersects the boundary of $\mathfrak{M}^4$ in a set lying on one analytic hypersurface only. In §3 this is extended to the case that $\mathfrak{A}^2(\zeta)$ meets $\mathfrak{m}^3$ in a Jordan curve which cuts the Bergman–Silov boundary in a finite number of points if the functions g_j in (2) are bounded in a neighbourhood of the Bergman–Silov boundary lying on $\mathfrak{m}^3$. For other possible bounds for holomorphic functions of two complex variables, see **(9; 13)**.

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2. *Geometry of the problem.* Let $\mathfrak{M}^4$ be a domain in C^2 with boundary $\mathfrak{m}^3$ and $0 \in \mathfrak{M}^4$, which possesses a distinguished piece of boundary $\mathfrak{D}^2$ in the sense of Bergman–Silov boundary. $\mathfrak{D}^2$ is constructed as follows **(3; 5)**:

$$\mathfrak{m}^3 = \bigcup_{k=1}^{n} \bar{\mathfrak{i}}_k^3,$$

where $\bar{\mathfrak{i}}_k^3$ is a closed segment of analytic hypersurface and n is finite:

$$\bar{\mathfrak{i}}_k^3 = \bigcup_{0 \leqslant \lambda_k \leqslant 2\pi} \bar{\mathfrak{F}}_k^2(\lambda_k),$$

$\bar{\mathfrak{F}}_k^2(\lambda_k)$ being a segment of analytic surface given by

$$\bar{\mathfrak{F}}_k^2(\lambda_k) = [z | z_j = h_{kj}(Z_k, \lambda_k), j = 1, 2, |Z_k| \leqslant 1],$$

$H_k = (h_{k1}, h_{k2})$ a 1 to 1 continuous map of $D_k^3 = [|Z_k| \leqslant 1] \times [0 \leqslant \lambda_k \leqslant 2\pi]$ onto $\bar{\mathfrak{i}}_k^3$, each h_{kj} being a continuously differentiable function on D_k^3 and holomorphic on $|Z_k| < 1$ for each λ_k in $[0, 2\pi]$. Since D_k^3 is compact, the set $\bar{\mathfrak{i}}_k^3$ is compact and H_k is a homeomorphism. Hence $\mathfrak{m}^3$ is compact and since $0 \in \mathfrak{M}^4$ and $\mathfrak{M}^4$ is connected, $\mathfrak{M}^4$ is bounded. For fixed λ_k let $A_k(\lambda_k)$ be the point $(h_{k1}(0, \lambda_k), h_{k2}(0, \lambda_k))$ corresponding to $Z_k = 0$ and call

$$l_k^1 = \cup A_k(\lambda_k) \quad (0 \leqslant \lambda_k \leqslant 2\pi)$$

the axis of $\mathfrak{i}_k^3$. The representation of $\mathfrak{i}_k^3$ given here is said to be normalized with respect to the axis l_k^1 (**3**, p. 186). On $\mathfrak{m}^3$ there are two kinds of points, namely those that belong to one $\bar{\mathfrak{i}}_k^3$ only and those that belong to the intersection of two or more $\bar{\mathfrak{i}}_k^3$'s. Bergman has shown that every point of the boundary curve $\mathfrak{i}_k^1(\lambda_k)$ of $\bar{\mathfrak{F}}_k^2(\lambda_k)$ must belong to the intersection of two or more $\bar{\mathfrak{i}}_k^3$'s (**2**). Thus

$$\mathfrak{i}_k^1(\lambda_k) = \bigcup_{s=1}^{n} \mathfrak{i}_{ks}^1(\lambda_k), \qquad \mathfrak{i}_{ks}^1(\lambda_k) = \mathfrak{i}_k^1(\lambda_k) \cap \bar{\mathfrak{i}}_s^3 \qquad (s \neq k).$$

Set

$$\mathfrak{G}_{ks}^2 = \bigcup_{0 \leqslant \lambda_k \leqslant 2\pi} \mathfrak{i}_{ks}^1(\lambda_k) = \bigcup_{0 \leqslant \lambda_s \leqslant 2\pi} \mathfrak{i}_{ks}^1(\lambda_s),$$

and

$$\mathfrak{D}^2 = \bigcup_{k=1}^{n} \bigcup_{s=1}^{n} \mathfrak{G}_{ks}^2 \qquad (s \neq k)$$

is the Bergman–Silov boundary of $\mathfrak{M}^4$.

If we assume for every s in $(0, s_0]$ with s_0 sufficiently small that the sets

$$[z | z_j = h_{kj}(Z_j, \lambda_k - is), Z_k \in B_k^2(\lambda_k, s), k = 1, \ldots, n]$$

form the boundary of a domain $\mathfrak{M}_s$ with $\bar{\mathfrak{M}}_s \subset \mathfrak{M}^4$, where $B_k^2(\lambda_k, s)$ are simply connected domains which for $s = 0$ become the unit disk $|Z_k| < 1$, and for each λ_k, $h_{kj}(Z_k, \lambda_k - is)$ are continuous in Z_k and s on $|Z_k| \leqslant 1$, $0 \leqslant s \leqslant s_0$, then it follows from Cauchy and Morera's theorems that $f(z_1, z_2)$, holomorphic in $\mathfrak{M}^4$ and continuous on $\bar{\mathfrak{M}}^4$, implies $f[h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)]$ holomorphic on $|Z_k| < 1$ for every $\lambda_k \in [0, 2\pi]$ (**3**, p. 188).

Let $\mathfrak{C}_j^1(z_j)$ be a curve in the z_j-plane connecting 0 to z_j whose points ζ_j are such that $|\zeta_j| \leqslant |z_j|$. Set

$$\mathfrak{S}^2(z) = \mathfrak{C}_1^1(z_1) \times \mathfrak{C}_2^1(z_2) = [\zeta = (\zeta_1, \zeta_2) | \zeta_j \in \mathfrak{C}_j^1(z_j)]. \tag{3}$$

Then for each $z \in \mathfrak{M}^4$ for which $\mathfrak{S}^2(z) \subset \mathfrak{M}^4$, bounds can be obtained for the functions g_j in (2). Let $\mathfrak{N}^4$ be the largest subset of $\mathfrak{M}^4$ such that $z \in \mathfrak{N}^4$ implies that $\mathfrak{S}^2(z) \subset \mathfrak{M}^4$. (Notice that for any bicylinder or complete Reinhardt circular domain with center at the origin $\mathfrak{N}^4 = \mathfrak{M}^4$.)

Let $\mathfrak{A}_0^2 = [w|\, w_j = f_j(t)]$ be an analytic surface through the point z, that is, f_j are holomorphic functions of the complex variables t, chosen so that the boundary $\mathfrak{a}^1$ of the set $\mathfrak{A}^2 = \mathfrak{A}_0^2 \cap \mathfrak{M}^4$ lies on $\mathfrak{m}^3$ and the inverse image of $\bar{\mathfrak{A}}^2$ under the mapping $F = (f_1, f_2)$ is a compact set in the t-plane. (The boundary of $\mathfrak{A}^2$ could lie partly in $\mathfrak{M}^4$.) Similarly assume that through every point ζ of $\mathfrak{S}^2(z)$, there is an analytic surface $\mathfrak{A}_0^2(\zeta)$ with the same properties as $\mathfrak{A}_0^2$.

The representation (2b) is valid for ψ'' only if the domain $\mathfrak{M}^4$ is symmetric with respect to $x_1\, y_1\, x_2$-space; that is, $(z_1, z_2) \in \mathfrak{M}^4$ implies $(z_1, \bar{z}_2) \in \mathfrak{M}^4$, and we may take the curve joining 0 to $\bar{z}_2$ in the z_2-plane as the reflection of $\mathfrak{C}_2^1(z_2)$ with respect to the x_2-axis. Also the functions $g_1(z_1, 0)$ and $g_2(0, z_2)$ are assumed to be holomorphic on $\mathfrak{M}^4$ and continuous on $\bar{\mathfrak{M}}^4$.

2. Bounds for solutions of system (1.1) on analytic surfaces $\mathfrak{A}_0^2$ which meet the boundary hypersurfaces of $\mathfrak{M}^4$ along sets lying in one segment $\mathfrak{i}_k^3$.

1. If the curve $\mathfrak{a}^1$ lies entirely in one segment $\mathfrak{i}_k^3$, then there exists an $r < 1$ such that $|Z_k| \leqslant r$ for all points on $\mathfrak{a}^1$. Otherwise there is a sequence $P^{(n)} \in \mathfrak{a}^1$ such that the corresponding coordinate $Z_k^{(n)} \to Z_k^0$ and $|Z_k^0| = 1$. Let $\lambda_k^{(n)}$ be the corresponding value of λ_k for $P^{(n)}$. There exists a convergent subsequence of $\lambda_k^{(n)}$ converging to $\lambda_k^0 \in [0, 2\pi]$ and the corresponding subsequence of $Z_k^{(n)}$ converges to Z_k^0. Reletter these subsequences as $(Z_k^{(n)}, \lambda_k^{(n)})$. By continuity of h_{kj}, the corresponding coordinate of $P^{(n)}$ converges to $h_{kj}(Z_k^0, \lambda_k^0)$ with $|Z_k^0| = 1$, but the point P^0 with these coordinates lies on the boundary of $\mathfrak{i}_k^3$ since $H_k = (h_{k1}, h_{k2})$ is a homeomorphism. Since P^0 is a limit point of the closed set $\mathfrak{a}^1$, $P^0 \in \mathfrak{a}^1$, which is a contradiction. Thus such an $r < 1$ exists. Let

$$\text{(1)} \qquad \mathfrak{t}_k^3 = [z|\, z_j = h_{kj}(Z_k, \lambda_k),\ |Z_k| < r].$$

and say that $\mathfrak{t}_k^3$ has a representation normalized with respect to the axis l_k^1 and in this representation is of radius r. We also assume that the boundary $\mathfrak{a}^1(\zeta)$ of $\mathfrak{A}^2(\zeta) = \mathfrak{A}_0^2(\zeta) \cap \mathfrak{M}^4$ lies in $\mathfrak{t}_k^3$ for each $\zeta \in \mathfrak{S}^2(z)$.

Since T_j and P_j in (1.2) are entire functions of z_j, z_j^*, ζ_j, there exist functions $\tilde{T}_j$, $\tilde{P}_j$ depending on $|z_j|$, $|z_j^*|$, and $\mathfrak{M}^4$ such that

$$\text{(2)} \qquad |T_j(z_j, z_j^*, \zeta_j)| \leqslant \tilde{T}_j(|z_j|, |z_j^*|), \qquad |P_j(z_j, z_j^*, \zeta_j)| \leqslant \tilde{P}_j(|z_j|, |z_j^*|)$$

on $\mathfrak{M}^4$ for $|\zeta_j| \leqslant |z_j|$.

2. We now obtain a bound for solutions ψ of (1.1) in terms of the bounds (2), $g_0 = |g_1(0, 0)|$ and various quantities connected with the boundary segment $\mathfrak{i}_k^3$.

THEOREM 2.1. (a) *Let $\mathfrak{M}^4$ be a domain with a Bergman-Silov boundary surface satisfying the hypotheses of* §1.2 *and symmetric with respect to $x_1\, y_1\, x_2$-space.*

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(b) *Let* ψ *be a solution of* (1.1) *with the representation* (1.2), *where* $g_1, g_2, g_1(z_1, 0)$, *and* $g_1(0, z_2)$ *are holomorphic on* $\mathfrak{M}^4$ *and continuous on* $\bar{\mathfrak{M}}^4$ *and* $g_1(0, 0)$ *is real, and such that* (i) $\psi_1(z_1, z_2) = \psi(z_1, 0, z_2, 0)$ *and* $\psi_2(z_1, z_2) = \psi(z_1, 0, 0, z_2)$ *omit the values* $e_{1j}(\lambda_k)$, $e_{2j}(\lambda_k)$ *resepctively on the lamina* $\mathfrak{J}_k^2(\lambda_k)$ *where*

$$(3) \qquad \begin{array}{l} |e_{\nu 1}(\lambda_k)| + |e_{\nu 2}(\lambda_k)| \leqslant E_{k\nu} < \infty \\ |e_{\nu 1}(\lambda_k) - e_{\nu 2}(\lambda_k)| \geqslant F_{k\nu} > 0 \end{array} \qquad (\nu = 1, 2),$$

$E_{k\nu}$, $F_{k\nu}$ *constants depending only on k and* ν; (ii) *on the axis* l_k^1 *of* $\mathfrak{i}_k^3$, ψ' *and* ψ'' *are bounded by* $A_{kj}(l_k^1)$ $(j = 1, 2)$ *respectively.* (c) *Let* $\mathfrak{A}^2 = \mathfrak{A}_0^2 \cap \mathfrak{M}^4$ *be a segment of analytic surface whose boundary* $\mathfrak{a}^1$ *lies in the segment* $\mathfrak{t}_k^3$ *of* $\mathfrak{i}_k^3$ *of radius r when the representation of* $\mathfrak{t}_k^3$ *is normalized with respect to the axis* l_k^1 *and similarly for the boundary* $\mathfrak{a}^1(\zeta)$ *of* $\mathfrak{A}^2(\zeta)$ *for all* $\zeta \in \mathfrak{S}^2(z)$ (see (1.3)).

Then for any $z \in \mathfrak{N}^4$

$$(4) \quad |\psi(z_1, \bar{z}_1, z_2, \bar{z}_2)| \leqslant \prod_{j=1}^{2} [1 + \tilde{T}_j(|z_j|)|z_j|] C_{kl}[g_0, \mathrm{r}, E_{kl}, F_{k1}, A_{k1}(l_k^1)] + \prod_{j=1}^{2} [1 + \tilde{P}_j(|z_j|)|z_j|] C_{k2}[g_0, r, E_{k1}, E_{k2}, F_{k1}, F_{k2}, A_{k1}(l_k^1), A_{k2}(l_k^1)],$$

where C_{kj} *are constants depending only on the indicated quantities.*

Proof. Continue x_j, y_j, $z_j = x_j + iy_j$ to complex values. Using the bounds (2) for T_j and P_j we need bounds on $\mathfrak{M}^4$ for the functions g_1 and g_2.

By (**4**, formula (16))

$$(5) \qquad g_1(z_1, z_2) + \bar{g}_1(0, 0) = 2\psi_1(z_1, z_2).$$

Setting $z_1 = z_2 = 0$ in (5) gives, since $g_1(0, 0)$ is real, $g_1(0, 0) = \psi_1(0, 0)$. From (5) and the hypothesis of the theorem, ψ_1 is holomorphic on $\mathfrak{M}^4$ and continuous on $\bar{\mathfrak{M}}^4$. Hence by the second paragraph of §1.2 the function

$$(6) \qquad \Psi_{k1}(Z_k, \lambda_k) = \psi_1[h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)]$$

is holomorphic on $|Z_k| < 1$ for each $\lambda_k \in [0, 2\pi]$ and omits there the values $e_{1j}(\lambda_k)$. Then $\Psi_{k1}^* = (\Psi_{k1} - e_{11})(e_{12} - e_{11})^{-1}$ is holomorphic on $|Z_k| < 1$ and omits there the values 0 and 1 so that Ahlfors' form of Schottky's theorem (**1**) gives for $|Z_k| \leqslant r$

$$|\Psi_{k1}^*(Z_k, \lambda_k)| < \exp\frac{1 + r}{1 - r}(7 + \log^+|\Psi_{k1}^*(0, \lambda_k)|).$$

By (6) and (ii) of the theorem, $|\Psi_{k1}(0, \lambda_k)| \leqslant A_{k1}(l_k^1)$, which gives a bound for $\Psi_{k1}^*(0, \lambda_k)$. Thus, using (3) for $|Z_k| \leqslant r$,

$$(7) \quad |\psi_1[h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)]| = |\Psi_{k1}(Z_k, \lambda_k)| \leqslant E_{k1}\left\{1 + \exp\frac{1 + r}{1 - r}(7 + \log^+(A_{k1}(l_k^1) + E_{k1})/F_{k1})\right\} \equiv B_k(r, E_{k1}, F_{k1}, A_{k1}(l_k^1)).$$

Since the boundary $\mathfrak{a}^1$ of $\mathfrak{A}^2$ lies in the segment $\mathfrak{t}_k{}^3$ of $\mathfrak{i}_k{}^3$ with $|Z_k| < r$ and the domain of the mapping F in the t-plane is compact when F is restricted to $\bar{\mathfrak{A}}^2$, $\psi_1[f_1(t), f_2(t)]$ is an analytic function of t for $z \in \mathfrak{A}^2$ and continuous on a compact set. Hence by the maximum modulus theorem, $|\psi_1|$ takes its maximum on the boundary of the set in the t-plane which corresponds to $\mathfrak{a}^1$ under the holomorphic transformation F (**8**, p. 86). Thus

$$(8) \qquad |g_1(z_1, z_2)| \leqslant g_0 + 2B_k(r, E_{k1}, F_{k1}, A_{k1}(l_k{}^1)) \equiv C_{k1}[g_0, r, E_{k1}, F_{k1}, A_{k1}(l_k{}^1)].$$

Similarly by the hypotheses on $\mathfrak{A}^2(\zeta)$ and $\mathfrak{a}^1(\zeta)$, $g_1(\zeta_1, \zeta_2)$ satisfies inequality (8) for $\zeta \in \mathfrak{S}^2(z)$. Thus ψ' in (1.2) is bounded for all $z \in \mathfrak{N}^4$ and $z^* \in \bar{\mathfrak{M}}^4$.

To get bounds for ψ'' in (1.2) we need bounds for the functions g_2. By (**4**, formula (17))

$$(9) \qquad \psi_2(z_1, z_2{}^*) = \tfrac{1}{2}[g_2(z_1, z_2{}^*) + g_1(z_1, 0) + \bar{g}_1(0, z_2{}^*)].$$

By hypothesis (b), ψ_2 is holomorphic on $\mathfrak{M}^4$ and continuous on $\bar{\mathfrak{M}}^4$. Thus as for ψ_1 the function $\psi_2[h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)]$ is holomorphic on $|Z_k| < 1$ and bounded in absolute value on $|Z_k| \leqslant r$ for $\lambda_k \in [0, 2\pi]$ by $B_k(r, E_{k2}, F_{k2}, A_{k2}(l_k{}^1))$ (see (7)). Since by hypothesis any point $(z_1, z_2{}^*)$ of $\mathfrak{N}^4$ lies in the analytic segment $\mathfrak{A}^2$, similarly as for ψ_1, the function $\psi_2(z_1, z_2{}^*)$ has the same bound. Since also $(z_1, 0) \in \mathfrak{S}^2(z)$, $g_1(z_1, 0)$ satisfies the bound (8) and similarly for $g_1(0, z_2{}^*)$. Thus

$$(10) \qquad |g_2(z_1, z_2{}^*)| \leqslant 2B_k(r, E_{k2}, F_{k2}, A_{k2}(l_k{}^1)) + 2C_{k1}[g_0, r, E_{k1}, F_{k1}, A_{k1}(l_k{}^1)] \equiv C_{k2}[g_0, r, E_{k1}, E_{k2}, F_{k1}, F_{k2}, A_{k1}(l_k{}^1), A_{k2}(l_k{}^1)].$$

Similarly $g_2(\zeta_1, \zeta_2)$ satisfies (10) for all $\zeta \in \mathfrak{S}^2(z)$. Thus by inequalities (2), (8), and (10), on setting $z_j{}^* = \bar{z}_j$, $\tilde{P}_j(|z_j|, |z_j|) = \tilde{P}_j(|z_j|)$ and similarly for T_j, we obtain (4) as a bound for ψ.

3. Bounds for solutions of (1.1) if the analytic surface $\mathfrak{A}_0{}^2$ meets $\mathfrak{m}^3$ in a closed curve lying on more than one segment $\mathfrak{i}_k{}^3$. Suppose that the analytic surface $\mathfrak{A}_0{}^2$ meets $\mathfrak{m}^3$ in a Jordan curve $\mathfrak{a}^1$ and the Bergman–Silov boundary $\mathfrak{D}^2$ in a finite number of points; also there exists a number r_k, $0 < r_k < 1$ such that $\mathfrak{a}^1$ crosses the set

$$(1) \qquad \mathfrak{t}_k{}^2 = [z|\, z_j = h_{kj}(Z_k, \lambda_k),\ |Z_k| = r_k,\ 0 \leqslant \lambda_k \leqslant 2\pi]$$

$\subset \mathfrak{i}_k{}^3$ at most a finite number of times, although a piece of $\mathfrak{a}^1$ may lie on $\mathfrak{t}_k{}^2$. The curve $\mathfrak{a}^1(\zeta)$ for $\zeta \in \mathfrak{S}^2(z)$ is assumed to have similar properties. Then

THEOREM 3.1. *In addition to hypotheses* (a) *and* (b) *of Theorem* 2.1, (ci) *the analytic surface* $\mathfrak{A}_0{}^2$ *meets* $\mathfrak{m}^3$ *in a Jordan curve* $\mathfrak{a}^1$ *which intersects the Bergman–Silov boundary* $\mathfrak{D}^2$ *in a finite number of points and crosses the set* $\mathfrak{t}_k{}^2$ *given by* (1) *at most a finite number of times; similarly for the curve* $\mathfrak{a}^1(\zeta)$ *for* $\zeta \in \mathfrak{S}^2(z)$;

(cii) *the functions* ψ_j *are bounded on that part of* $\bar{\mathfrak{i}}_k^3$ *such that* $\mathfrak{A}_0^4(z) \cap \mathfrak{i}_k^3 \neq \emptyset$, *where*

$$\mathfrak{A}_0^4(z) = \bigcup_{\zeta \in \mathfrak{S}^2(z)} \mathfrak{A}_0^2(z), \qquad z \in \mathfrak{N}^4$$

and $r_k \leqslant |Z_k| \leqslant 1$ $(k = 1, \ldots, n)$. *Then for all* $z \in \mathfrak{N}^4$

$$(2) \quad |\psi(z_1, \bar{z}_1, z_2, \bar{z}_2)| \leqslant \max_{\{k\}} \mathbf{B}_{k1}[g_0, r_k, E_{k1}, F_{k1}, A_{k1}(l_k^1), D_{k1}] \prod_{j=1}^{2} [1 + \tilde{T}_j(|z_j|)|z_j|] + \max_{\{k\}} \mathbf{B}_{k2}[g_0, r_k, E_{k1}, E_{k2}, F_{k1}, F_{k2}, A_{k1}(l_k^1), A_{k2}(l_k^1), D_{k1}, D_{k2}] \times \prod_{j=1}^{2} [1 + \tilde{P}_j(|z_j|)|z_j|],$$

where $\mathbf{B}_{kj}$ *are constants depending only on the indicated quantities.*

Proof. For points on $\mathfrak{i}_k^3$ for which $|Z_k| \leqslant r_k$, g_1 has the bound (2.8) and for points on $\mathfrak{i}_k^3$ for which $r_k \leqslant |Z_k| \leqslant 1$

$$(3) \qquad |g[h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)]| \leqslant g_0 + 2D_{k1},$$

where $|\psi_1| \leqslant D_{k1}$ for all λ_k and Z_k given in (cii) of the theorem.

The Jordan curve $\mathfrak{a}^1$ has a representation $\mathfrak{a}^1 = [z| \, z_j = f_j(e^{i\phi}), 0 \leqslant \phi < 2\pi]$, f_j continuous functions of $e^{i\phi}$ and $\mathfrak{a}^1$ a 1 to 1 map of $[0, 2\pi)$. Thus

$$\mathfrak{A}^2 = [z| \, z_j = f_j(t), |t| < 1].$$

By (ci) $\mathfrak{a}^1$ meets $\mathfrak{D}^2$ at points corresponding to ϕ_ν $(\nu = 1, \ldots, q)$ say, $0 \leqslant \phi_1 < \phi_2 < \ldots < \phi_q < 2\pi$, $q < \infty$. Let $\mathfrak{a}_\nu^1 \subset \mathfrak{a}^1$ correspond to $\phi_\nu < \phi < \phi_{\nu+1}$ $(\nu = 1, \ldots, q-1)$. Then $\mathfrak{a}_\nu^1$ lies entirely in one segment $\mathfrak{i}_k^3$ of $\mathfrak{m}^3$, and the points P_ν, $P_{\nu+1}$ on $\bar{\mathfrak{a}}_\nu^1$ corresponding to ϕ_ν, $\phi_{\nu+1}$ respectively lie on the boundary of $\bar{\mathfrak{i}}_k^3$ and correspond to values of Z_k with $|Z_k| = 1$. Also $\mathfrak{a}_\nu^1$ crosses $\mathfrak{t}_k^2$ a finite number of times, say at points $Q_1, Q_2, \ldots, Q_p$. Since $\mathfrak{a}^1$ is a Jordan curve, to each Q_i corresponds a distinct $\phi^{(i)}$ with the possible exception of $\phi^{(i)} = 0$. Now for all $\phi \in (\phi^{(i)}, \phi^{(i+1)})$ such that the corresponding piece of $\mathfrak{a}^1$ does not lie on $\mathfrak{t}_k^2$, either $|Z_k| > r_k$ or $|Z_k| < r_k$ but not both. This can be seen as follows. Since $H_k = (h_{k1}, h_{k2})$ is a homeomorphism and hence 1 to 1, $\mathfrak{t}_k^2$ subdivides $\bar{\mathfrak{i}}_k^3$ into two disjoint sets $\mathfrak{t}_{k1}^3$ with $|Z_k| < \mathrm{r}_k$ and $\mathfrak{t}_{k2}^3$ with $|Z_k| > r_k$. Also $\mathfrak{t}_{kj}^3$ is connected since $H_k^{-1}(\mathfrak{t}_{kj}^3)$ is connected, but $H_k^{-1}(\mathfrak{t}_{k1}^3 \cup \mathfrak{t}_{k2}^3)$ is not connected so that $\mathfrak{t}_{k1}^3 \cup \mathfrak{t}_{k2}^3$ is not connected. Now the set $\mathfrak{a}_{\nu i}^1 = F[(\phi^{(i)}, \phi^{(i+1)})]$ is connected since f_j are continuous so that $\mathfrak{a}_{\nu i}^1$ cannot intersect both $\mathfrak{t}_{k1}^3$ and $\mathfrak{t}_{k2}^3$. Hence $(\phi_\nu, \phi_{\nu+1})$ is further subdivided into a finite number of intervals in each of which only one of $|Z_k| > r_k$, $|Z_k| \leqslant r_k$ holds:

$$\phi_\nu < \phi_\nu^{(1)} < \ldots < \phi_\nu^{(p)} < \phi_{\nu+1}.$$

Let $t = e^{i\phi}$, $\phi \in (\phi_\nu^{(i)}, \phi_\nu^{(i+1)})$, which either corresponds to Z_k with $|Z_k| \leqslant r_k$ or with $r_k < |Z_k| \leqslant 1$. For intervals of the first type, $g_1[f_1(e^{i\phi}), f_2(e^{i\phi})]$ has the

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bound (2.8) and for intervals of the second the bound (3). Since g_1 is a holomorphic function of t on $|t| < 1$ for $z \in \mathfrak{A}^2$ and continuous on $|t| \leqslant 1$, the Poisson integral for the unit disk may be used and gives

$$|g_1(z_1, z_2)| \leqslant \int_0^{2\pi} |g_1[f_1(e^{i\phi}), f_2(e^{i\phi})]P(e^{i\phi}, \zeta)|d\phi = \int_{I_1} + \int_{I_2},$$

where I_1 is the sum of a finite number of integrals whose points correspond to $|Z_k| \leqslant r_k$ $(k = 1, \ldots, n)$ and I_2 is similar with $|Z_k| > r_k$. Thus from the bounds for g_1 and well-known properties of the Poisson kernel we deduce that

$$|g_1(z_1, z_2)| \leqslant \max_k \mathbf{B}_{k1}[g_0, r_k, E_{k1}, F_{k1}, A_{k1}(l_k^{\,1}), D_{k1}].$$

Since $\mathfrak{a}^1(\zeta)$ is also a Jordan curve for $\zeta \in \mathfrak{S}^2(z)$, $g_1(\zeta_1, \zeta_2)$ has the same bound for such ζ.

As in §2, $\psi_2[h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)]$ is bounded by B_k for all points on $i_k^{\,3}$ with $|Z_k| \leqslant r_k$. As in the case of g_1 for those intervals with $|Z_k| \leqslant r_k$ for some k, $\psi_2[f_1(e^{i\phi}), f_2(e^{i\phi})]$ has the same bound B_k and for intervals with $r_k < |Z_k| \leqslant 1$ by (cii) a bound D_{k2}. Since ψ_2 is holomorphic in (z_1, z_2^*) on $\mathfrak{A}^2$ and continuous on $\bar{\mathfrak{M}}^4$, $\psi_2[f_1(t), f_2(t)]$ is holomorphic in t on $|t| < 1$ and continuous on $|t| \leqslant 1$. Thus from these bounds for ψ_2 and the bound for g_1 we obtain from (2.9), by using the Poisson integral formula, that

$$|g_2(z_1, z_2^*)| \leqslant \max_k \mathbf{B}_{k2}[g_0, r_k, E_{k1}, E_{k2}, F_{k1}, F_{k2}, A_{k1}(l_k^{\,1}), A_{k2}(l_k^{\,1}), D_{k1}, D_{k2}]$$

and the bound is valid for $g_2(\zeta_1, \zeta_2)$ if $\zeta \in \mathfrak{S}^2(z)$. Thus we obtain a bound for $\psi''(z_1, z_1^*, z_2, z_2^*)$, and replacing z_j^* by $\bar{z}_j$, (2) follows for all $(z_1, z_2) \in \mathfrak{M}^4$.

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