Canad. Math. Bull. Vol. **37** (3), 1994 pp. 310–314

ON LEVEL 0 AFFINE LIE MODULES

D. J. BRITTEN AND F. W. LEMIRE

ABSTRACT. It is proven that the dimensions of the homogeneous summands of a nontrivial **Z** graded module for an infinite dimensional Heisenberg algebra on which a central element acts as nonzero scalar are unbounded. This result is then applied to show that the central elements of an affine Lie algebra act trivially on any indecomposable diagonalizable module whose weight spaces are of bounded dimension.

Let $\mathbb{C}[t^{-1}, t]$ be the algebra of Laurent polynomials and let $\mathcal{G}$ be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$ of rank n. The non-twisted affine Lie algebras can be realized as

$$\hat{L}(\mathcal{G}) = (\mathcal{G} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where, after fixing a non-degenerate symmetric invariant bilinear C-valued form $(\cdot \mid \cdot)$ on $\mathcal{G}$, the multiplication is given by

$$\begin{aligned}[x \otimes t^k + \nu c + \mu d, y \otimes t^{k_1} + \nu_1 c + \mu_1 d] \\ = [x, y] \otimes t^{k+k_1} + \mu k_1 y \otimes t^{k_1} - \mu_1 k x \otimes t^k + k\delta_{k,-k_1}(x \mid y)c.\end{aligned}$$

The twisted affine algebras are described as being the fixed points of an automorphism on a non-twisted affine algebra. If σ is a diagram automorphism of $\mathcal{G}$ of order k and ϵ is a primitive k-th root of unity then σ extends to $\hat{L}(\mathcal{G})$ by $\sigma(c) = c$, $\sigma(d) = d$, $\sigma(x \otimes t^e) = \epsilon^e \sigma(x) \otimes t^e$, and linearity. The twisted affine algebra $\hat{L}^\sigma(\mathcal{G})$ is the subalgebra of $\hat{L}(\mathcal{G})$ of fixed points of σ. Denote the affine algebra under consideration as L and so L could be either twisted or non-twisted. Let H denote the Cartan subalgebra of L. Since every diagram automorphism have possible orders 1, 2 or 3 and each leaves some root vector fixed, the corresponding element h in the Cartan subalgebra of $\mathcal{G}$ is fixed and we find the Heisenberg subalgebra

$$\mathcal{H} = \oplus \sum_{0 \neq k \in \mathbf{Z}} \mathbb{C}(h \otimes t^{6k}) \oplus \mathbb{C}c.$$

in L. In the non-twisted case, we take σ to be the identity automorphism so that our algebra $\mathcal{H}$ is defined in all cases. We normalize the bilinear form so that $(h \mid h) = \frac{1}{6}$ and set $\hat{t} = t^6$. Then $[(h \otimes \hat{t}^k), (h \otimes \hat{t}^{-k})] = kc$.

Modules are called *diagonalizable* provided they admit a weight space decomposition $\mathcal{V} = \sum_{\mu \in \mathcal{H}^*} \mathcal{V}_\mu$ relative to H. In this paper we assume that all modules are diagonalizable with finite dimensional weight spaces. A nonzero module is said to be *torsion free*

The work of the authors was supported in part by the Natural Science and Engineering Research Council of Canada under grants A8471 and A7742 respectively.

Received by the editors August 4, 1992; revised February 5, 1993.

AMS subject classification: 17B67.

provided that each real root vector x acts injectively on all weight spaces (see [1]). A module is said to be *pointed* provided it is simple and it has at least one 1-dimensional weight space. Clearly, all the weight spaces of pointed torsion free modules are 1-dimensional. These modules are, in some sense, at the opposite end of the spectrum from integrable modules studied in [2], [3] and [4] which require that real root vectors act as locally nilpotent operators. For each simple integrable module $\mathcal{V}$ of a non-twisted affine Lie algebra, there exists an integer $\ell(\mathcal{V})$, called the *level* of $\mathcal{V}$, such that the central element c acts as the scalar $\ell(\mathcal{V})$, *i.e.* $cv = \ell(\mathcal{V})v$ for all $v \in \mathcal{V}$. If the level of a simple integrable module $\mathcal{V}$ is nonzero then $\mathcal{V}$ is either a highest weight module belonging to a dominant integral weight or its dual. If the level of $\mathcal{V}$ is zero then $\mathcal{V}$ is a so called *loop module*. It turns out that simple torsion free modules also have the property that c acts as the scalar 0. Since the natures of integrable modules and of torsion free modules are quite different, we seek a common property of these modules which forces $\ell(\mathcal{V}) = 0$. In this brief note, we prove:

MAIN THEOREM. *If $\mathcal{V}$ is a diagonalizable module for a simple affine algebra over $\mathbb{C}$ such that the dimensions of the weight spaces are bounded, and the central element c acts as a scalar $\tilde{c}$ then $\tilde{c} = 0$.*

Let $\mathcal{W}$ denote the $\mathcal{H}$-submodule of $\mathcal{V}$ generated by some weight space of $\mathcal{V}$. Clearly $\mathcal{W}$ is a $\mathbb{Z}$ graded $\mathcal{H}$ module where $h \otimes t^k$ acts as an operator of degree k and c acts as a nonzero scalar, say $\tilde{c}$. Since the graded summands of $\mathcal{W}$ are contained in distinct weight spaces of $\mathcal{V}$ the main theorem is a direct consequence of the following result on representations of Heisenberg algebras.

THEOREM. *Let $\mathcal{H}$ be a Heisenberg algebra with basis $\{a_k \mid k \in \mathbb{Z}\setminus\{0\}\}\cup\{c\}$ with multiplication*

$$[a_i, a_{-j}] = i\delta_{ij}c \quad [a_i, c] = 0$$

Let $\mathcal{W} = \sum_{i\in\mathbb{Z}} \mathcal{W}_i$ be a nontrivial $\mathbb{Z}$ graded $\mathcal{H}$ module on which a_k acts as an operator of degree k (i.e. $a_k\mathcal{W}_i \subset \mathcal{W}_{i+k}$ for all i) and c acts as a nonzero scalar $\tilde{c}$. Then for all $N \in \mathbb{Z}$ there exists $i \in \mathbb{Z}$ such that $\dim \mathcal{W}_i \geq N$.

Before tackling the proof of this theorem we introduce some notation and make some preliminary remarks.

First, without loss of generality, we may assume that each summand $\mathcal{W}_i$ is finite dimensional. We freely identify the element $a_i \in \mathcal{H}$ with their images as linear operators on $\mathcal{W}$. Then, for each $k \in \mathbb{Z}$, we define $T_k = a_k a_{-k}$. It is clear that $\{T_k\}$ is a set of commuting operators on $\mathcal{W}$ of degree 0 and hence for each $i \in \mathbb{Z}$ there exists a nonzero vector $w_i \in \mathcal{W}_i$ which is a common eigenvector for all T_k. Fix some $i \in \mathbb{Z}$ and let λ_k denote the eigenvalue of T_k corresponding to w_i. Then set

$$\mathcal{W}_i^{(k)} = \{w \in \mathcal{W}_i \mid T_k w = \lambda_k w\}.$$

We observe that for $k \neq 0$ at most one of the eigenvalues λ_k and λ_{-k} can be zero since $\lambda_k = \lambda_{-k} + k\tilde{c}$. Let

$$W_i = \bigcap_{k\in\mathbb{Z}} \mathcal{W}_i^{(k)}.$$

Clearly $W_i \neq \{0\}$ since $0 \neq w_i \in W_i$. By the finite dimensionality of $\mathcal{W}_i$ it follows that there is some K such that

$$W_i = \bigcap_{k \in \mathbf{Z}} \mathcal{W}_i^{(k)} = \bigcap_{k=-K}^{K} \mathcal{W}_i^{(k)}.$$

Fix an integer K with this property for the remainder of this paper. With these remarks in hand we now prove the theorem through a sequence of lemmas.

LEMMA 1. *Let $a = a_{k_1} a_{k_2} \cdots a_{k_m}$ with $k_1 = -\sum_{i=2}^{m} k_i$, $m > 2$ and $k_2, \ldots, k_m$ all integers of the same sign having absolute value greater than K. Then $aW_i = 0$.*

PROOF. Set $b = T_{k_1}$. Since for $-K \leq k \leq K$ we have $[a, T_k] = 0$ it follows that $aW_i \subseteq \mathcal{W}_i^{(k)}$ for all $-K \leq k \leq K$ and hence $aW_i \subseteq W_i$. Therefore, $[a,b]w = a(bw) - b(aw) = a(\lambda_{k_1} w) - \lambda_{k_1}(aw) = 0$ for all $w \in W_i$ and hence $[a,b]W_i = (0)$. However $[a,b] = k_1 ca$ and, since $\tilde{c} \neq 0$, we have that $aW_i = (0)$ as required. ∎

LEMMA 2. *Let $p_1, \ldots, p_q$ be distinct integers of the same sign and $p = \sum p_i$. Then*

$$T_{k(K+1)} a_{p_1(K+1)} \cdots a_{p_q(K+1)} W_i = \zeta_k a_{p_1(K+1)} \cdots a_{p_q(K+1)} W_i \subseteq \mathcal{W}_{i+p(K+1)}$$

where $\zeta_k = \lambda_{k(K+1)}$ if $k \neq \pm p_i$ for any p_i and $\zeta_k = \lambda_{k(K+1)} \mp p_i(K+1)\tilde{c}$, if $k = \pm p_i$.

PROOF. The containment in $\mathcal{W}_{i+p(K+1)}$ is due to the fact that $a_{p_i(K+1)}$ is an operator of degree $p_i(K+1)$. If $k \neq \pm p_i$ for any p_i, the result follows from the commutativity of the operators. If $k = \pm p_i$ for some p_i the result follows from $[a_{-p_i(K+1)}, a_{p_i(K+1)}] = -p_i(K+1)c$. ∎

LEMMA 3. *Either $\lambda_{k(K+1)} = 0$ for all $k > 1$ or $\lambda_{-k(K+1)} = 0$ for all $k > 1$.*

PROOF. Since

$$a_{K+1} a_{K+1} a_{-(K+1)} a_{-(K+1)} = T_{K+1}^2 + (K+1)cT_{K+1}$$

it follows that

$$a_{K+1} a_{K+1} a_{-(K+1)} a_{-(K+1)} w = \lambda_{K+1}\big(\lambda_{K+1} + (K+1)\tilde{c}\big) w$$

for all $w \in W_i$. Similarly,

$$a_{-(K+1)} a_{-(K+1)} a_{K+1} a_{K+1} w = \lambda_{-(K+1)}\big(\lambda_{-(K+1)} - (K+1)\tilde{c}\big) w$$

for all $w \in W_i$. Set $a = a_{2(K+1)} a_{-(K+1)} a_{-(K+1)}$ and $b = a_{-2(K+1)} a_{K+1} a_{K+1}$. By Lemma 1, for all $w \in W_i$ we have $aw = 0$ and $bw = 0$ and hence $baw = 0$ which implies that

$$(*) \qquad \lambda_{-2(K+1)} \lambda_{K+1}\big(\lambda_{K+1} + (K+1)\tilde{c}\big) = 0$$

while $abw = 0$ yields

$$(**) \qquad \lambda_{2(K+1)} \lambda_{-(K+1)}\big(\lambda_{-(K+1)} - (K+1)\tilde{c}\big) = 0.$$

https://doi.org/10.4153/CMB-1994-047-9 Published online by Cambridge University Press

Suppose that $\lambda_{K+1} = 0$. Then, as observed earlier, this implies that $\lambda_{-(K+1)} = -(K+1)\tilde{c}$. By $(**)$ we have $2\lambda_{2(K+1)}(K+1)^2\tilde{c}^2 = 0$ and hence $\lambda_{2(K+1)} = 0$. On the other hand, if $\lambda_{K+1} \neq 0$, then by $(*)$ either $\lambda_{-2(K+1)} = 0$ or $\lambda_{K+1} = -(K+1)\tilde{c}$. If $\lambda_{K+1} = -(K+1)\tilde{c}$ then substituting into $(**)$ we conclude that $\lambda_{2(K+1)} = 0$. Therefore in any case either $\lambda_{2(K+1)} = 0$ or $\lambda_{-2(K+1)} = 0$. Since not both $\lambda_{2(K+1)} = 0$ and $\lambda_{-2(K+1)} = 0$ can be valid, we may assume without loss of generality that $\lambda_{-2(K+1)} = 0$ and $\lambda_{2(K+1)} \neq 0$. The assumption that $\lambda_{2(K+1)} \neq 0$ implies that $\lambda_{K+1} \neq 0$. Now for $m > 2$ set

$$a_m = a_{m(K+1)}a_{-(m-1)(K+1)}a_{-(K+1)} \quad \text{and} \quad b_m = a_{-m(K+1)}a_{(m-1)(K+1)}a_{K+1}.$$

By Lemma 1, $b_m a_m W_i = (0)$ and so $\lambda_{-m(K+1)}\lambda_{(m-1)(K+1)}\lambda_{K+1} = 0$, and by an inductive argument $\lambda_{(m-1)(K+1)}\lambda_{(K+1)} \neq 0$. Therefore, $\lambda_{-m(K+1)} = 0$ for all $m > 1$. ∎

The proof of the theorem involves the concept of a partition of a positive integer $n = n_1 + \cdots + n_p$ into p distinct parts. Let $\mathcal{P}(n)$ be the number of such partitions. These numbers have generating function

$$\prod_{i \in \mathbb{Z}^+} (1 + x^i).$$

It is easy to see that 10 is the first integer such that $\mathcal{P}(n) \geq n$. An easy constructive induction argument proves that $\mathcal{P}(n) \geq n$ is true for all $n \geq 10$.

PROOF OF THEOREM. Assume, contrary to what we want to prove, that there exists an integer N which is an upper bound on the dimensions of the summands $\mathcal{W}_i$. Let $M = \max\{10, N+1\}$ and let $\{M_{1i}, \ldots, M_{p_i i}\}$ for $1 \leq i \leq \mathcal{P}(n)$ be the i-th partition of M having distinct parts. The strategy is to show that the dimension of $\mathcal{W}_{i+M(K+1)}$ is greater than N providing the desired contradiction.

The two cases given to us by Lemma 3 are similar and so we treat the case of $\lambda_{k(K+1)} = 0$ for all $k > 1$. We observe that $(0) \neq a_{-2M_{1i}(K+1)} \cdots a_{-2M_{p_i i}(K+1)} W_i$ since $\lambda_{-k(K+1)} \neq 0$ for all $k > 1$. By Lemma 2,

$$\begin{aligned} T_{k(K+1)}a_{2M_{1i}(K+1)} \cdots a_{2M_{p_i i}(K+1)} W_i &= \zeta_{ki} a_{2M_{1i}(K+1)} \cdots a_{2M_{p_i i}(K+1)} W_i \\ &\subseteq \mathcal{W}_{i+M(K+1)} \end{aligned}$$

where $\zeta_{ki} = \lambda_{k(K+1)}$ if $k \neq \pm 2M_{ji}$ for any M_{ji} and $\zeta_{ki} = \lambda_{k(K+1)} \mp 2M_{ji}(K+1)\tilde{c}$, if $k = \pm 2M_{ji}$.

If $k \neq 2M_{ji}$ for all M_{ji} then $\zeta_{ki} = 0$ and if $k = 2M_{ji}$ then $\zeta_{ki} = -2M_{ji}(K+1)\tilde{c}$. Let $0 \neq v_i \in a_{2M_{1i}(K+1)} \cdots a_{2M_{p_i i}(K+1)} W_i$. Then v_i is a common eigenvector for all the operators T_k. We claim that $\{v_i \mid 1 \leq i \leq \mathcal{P}(M)\}$ is a linearly independent set. Suppose that $\sum_{i=1}^m d_i v_i = 0$ with $d_m \neq 0$. Set $T'_j = T_{2M_{jm}(K+1)}$ and observe that

$$0 = T'_1 \cdots T'_m \Big(\sum d_i v_i\Big) = (-1)^{p_m} 2^{p_m} (K+1)^{p_m} \Big(\prod_{j=1}^{p_m} M_{jm}\Big) d_m \tilde{c} v_m.$$

This contradicts $d_m \neq 0$ and proves the claim. Now observe that by our choice of M the number of elements in the set $\{v_i \mid 1 \leq i \leq \mathcal{P}(M)\}$ is greater than N. This contradiction completes the proof. ∎

ACKNOWLEDGEMENT. The authors would like to thank the referee for suggestions which improved the presentation of the paper.

REFERENCES

1. D. J. Britten, F. W. Lemire and F. Zorzitto, *Pointed torsion free modules for affine Lie algebras*, Comm. Alg. **18**(1990), 3307–3321.
2. Vyjayanthi Chari,*Integrable representations of affine Lie-algebras*, J. Algebra **133**(1988), 438–464.
3. Vyjayanthi Chari and Andrew Pressley, *Integrable representations of twisted affine Lie-algebras*, Invent. Math. **85**(1986), 317–335.
4. ———, *Integrable representations of Kac-Moody algebras: Results and open problems*, Infinite Dimensional Lie Algebras and Groups, Advanced series in Mathematical Physics, (ed. Victor Kac), World Scientific, 1988.
5. Victor G. Kac, *Infinite dimensional Lie algebras*, Cambridge Press, 1988.

Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario

Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario