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CONSTANT MEAN CURVATURE HYPERSURFACES IN SPHERES

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Abstract. In this paper, we first summarise the progress for the famous Chern conjecture, and then we consider n-dimensional closed hypersurfaces with constant mean curvature H in the unit sphere $\mathbb{S}^{n+1}$ with $n \leq 8$ and generalise the result of Cheng et al. (Q. M. Cheng, Y. J. He and H. Z. Li, Scalar curvature of hypersurfaces with constant mean curvature in a sphere, *Glasg. Math. J.* **51**(2) (2009), 413–423). In order to be precise, we prove that if $|H| \leq \varepsilon(n)$, then there exists a constant $\delta(n, H) > 0$, which depends only on n and H, such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S = S_0$ and M is isometric to the Clifford hypersurface, where $\varepsilon(n)$ is a sufficiently small constant depending on n.

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1. Introduction.

1.1. Minimal hypersurfaces in $\mathbb{S}^{n+1}$. In this section we will review some important results about the following famous conjecture. For more details, refer to Scherfner and Weiss' [**13**] excellent survey on this topic.

Chern conjecture. *Let S be the value of the squared norm of the second fundamental forms for n-dimensional closed minimal hypersurfaces in the unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature, then the set of S should be discrete.*

The above conjecture has been studied by many mathematicans after the work of Chern et al. [**7**]. In 1968, Simons [**14**] proved the following.

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THEOREM 1.1 ([**14**]). *Let M^n be an n-dimensional closed minimal hypersurface in a unit sphere $\mathbb{S}^{n+1}$, then*

$$\int_{M^n} S(S-n) \geq 0, \tag{1.1}$$

where S denotes the squared norm of the second fundamental form of M^n.

From Theorem 1.1, we immediately conclude that if $0 \leq S \leq n$, then either $S \equiv 0$ or $S \equiv n$. The latter case was independently characterised by Chern et al. [**7**] and Lawson [**9**].

THEOREM 1.2 (**7**, **9**). *The Clifford tori are the only closed minimal hypersurfaces in $\mathbb{S}^{n+1}$ with $S = n$.*

According to the example given by Cartan [**3**], the next value of S may be $2n$. Peng and Terng [**11**] considered the next value of S and proved the following theorem.

THEOREM 1.3 ([**11**]). *Let $M^n(n \geq 3)$ be a closed minimal hypersurface in $\mathbb{S}^{n+1}$ with $S =$ constant. If $S > n$, then*

$$S > n + \frac{1}{12n}.$$

Moreover, for the case of $n = 3$, Peng–Terng obtained the following sharp result.

THEOREM 1.4 ([**11**]). *Let M^3 be a closed minimal hypersurface in $\mathbb{S}^4$ with $S =$ constant. If $S > 3$, then $S \geq 6$. Moreover, $S = 6$ is assumed in the examples of Cartan* [**3**] *and Hsiang* [**8**].

Chang [**5**] proved the following classification theorem for $n = 3$. Thus, the Chern conjecture [7] is right for $n = 3$.

THEOREM 1.5 (Chang's Classification Theorem [**5**]). *A closed minimally immersed hypersurface with constant scalar curvature in $\mathbb{S}^4$ is either an equatorial 3-sphere, a product of sphere, or a Cartan's minimal hypersurface.*

In particular, S can only be 0, 3, 6.

For the closed hypersurface M^3 of $\mathbb{S}^4$ with constant mean curvature and scalar curvature, Almeida–Brito [**1**] and Chang [**4**] proved that M^3 is isoparametric. There is another recent result by Almeida et al. [**2**], which is as follows:

THEOREM 1.6 ([**2**]). *Let H, K and R be the mean curvature, the Gauss–Kronecker curvature and scalar curvature of M^3. If two out of these three functions are constant, then either* (1) *M^3 is an isoparametric hypersurface of $\mathbb{S}^4$, or* (2) $H = K \equiv 0$.

However, if M^3 is not closed, then we have the following conjecture, which is still open.

BRYANT CONJECTURE. *A piece of a minimally immersed hypersurface of constant scalar curvature in $\mathbb{S}^4$ is isoparametric.*

For the case $n = 4$, Lusala et al. [**10**] proved the following:

THEOREM 1.7 ([**10**]). *A closed minimal Willmore hypersurface M^4 of $\mathbb{S}^5$ with non-negative constant scalar curvature must be isoparametric.*

For general n, Cheng and Yang improved Theorem 1.3 as follows (please see [**17–19**] for details).

THEOREM 1.8 ([**17–19**]). *Let $M^n(n > 3)$ be a closed minimal hypersurface in $\mathbb{S}^{n+1}$ with $S =$ constant. If $S > n$, then*

$$S > n + \frac{n}{3}.$$

By using the method of Cheng–Yang and by carefully estimation, Suh–Yang [**15**] improved Cheng–Yang's result as follows.

THEOREM 1.9 ([**15**]). *Let $M^n(n > 3)$ be a closed minimal hypersurface in $\mathbb{S}^{n+1}$ with $S =$ constant. If $S > n$, then*

$$S > n + \frac{3n}{7}.$$

Until now, we still have the following open problem.

Open problem. Let $M^n(n > 3)$ be a closed minimal hypersurface in $\mathbb{S}^{n+1}$ with $S =$ constant. If $S > n$, then $S \geq 2n$?

If we do not add the condition that M^n has constant scalar curvature, then the following is obtained.

THEOREM 1.10 ([**12**]). *Let M be a closed minimally immersed hypersuface in $\mathbb{S}^{n+1}$, $n \leq 5$ and S* the square of the length of the second fundamental form of M. *Then there exists $\delta(n) > 0$ such that if $n \leq S(x) < n + \delta(n)$, then $S(x) \equiv n$, hence M is the Clifford torus.*

In [**16**], Wei and Xu improved the above theorem from $n \leq 5$ to $n \leq 7$, and then Zhang [**20**] extented it to $n \leq 8$.

1.2. Constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$. For constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$, we will prove the following theorem, which is corresponding to Theorem 1.10.

THEOREM 1.11. *Let M^n be an n-dimensional ($n \leq 8$) closed hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}(1)$ and S be the length of the second fundamental form of M^n. Then there exist positive constants $\varepsilon(n)$ depending only on n and $\delta(n, H)$ depending only on n and H such that if*

$$|H| \leq \varepsilon(n) \quad \textit{and} \quad S_0 \leq S \leq S_0 + \delta(n, H),$$

where

$$S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

then $S \equiv S_0$ and M^n is isometric to the Clifford hypersurface. To be precise, M^n is isometric to the Clifford torus $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ if $H = 0$; M^n is isometric to the Clifford hypersurface $\mathbb{S}^1(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbb{S}^{n-1}(\frac{\lambda}{\sqrt{1+\lambda^2}})$ if $H \neq 0$.

REMARK 1.12. In [**6**], Cheng et al. proved the above theorem for $n \leq 7$.

2. Basic formulas for closed hypersurfaces with constant mean curvature H in $\mathbb{S}^{n+1}(1)$. In this section we recall basic formulas for closed hypersurfaces with constant mean curvature H in $\mathbb{S}^{n+1}(1)$, which can be found in [**6**].

Let M^n be an n-dimensional closed hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}(1)$. Then we choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ such that $\{e_1, e_2, \ldots, e_n\}$ is tangent to M^n when restricted to M^n. Let h_{ij} and H denote the second fundamental form and mean curvature, respectively. Denote

$$S = \sum_{i,j} h_{ij}^2, \quad H = \frac{1}{n}\sum_i h_{ii}, \quad f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}, \quad f_4 = \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li}.$$

Denote by h_{ijk} and h_{ijkl} components of the first and second covariant derivatives of the second fundamental form, respectively. For an arbitrary fixed point $p \in M$, we take orthonormal frames such that $h_{ij} = \lambda_i\delta_{ij}$ at p, for all i, j. Then at this point p, we have

$$S = \sum_{i=1}^n \lambda_i^2, \quad H = \frac{1}{n}\sum_{i=1}^n \lambda_i, \quad f_3 = \sum_{i=1}^n \lambda_i^3, \quad f_4 = \sum_{i=1}^n \lambda_i^4.$$

We define A, B by

$$A = \sum_{i,j,k} h_{ijk}^2\lambda_i^2 \quad \text{and} \quad B = \sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_j.$$

Then by some computation, we have the following formulas:

$$\frac{1}{2}\Delta S = S(n-S) - n^2H^2 + nHf_3 + \sum_{i,j,k} h_{ijk}^2, \tag{2.1}$$

$$\frac{1}{2}\Delta \sum_{i,j,k} h_{ijk}^2 = (2n+3-S)\sum_{i,j,k} h_{ijk}^2 - 3(A-2B) - \frac{3}{2}|\nabla S|^2 + 3nH\sum_{i,j,k} h_{ijk}^2\lambda_i + \sum_{i,j,k,l} h_{ijkl}^2, \tag{2.2}$$

$$\Delta f_3 = 3(n-S)f_3 + 3nHf_4 - 3nHS + 6\sum_{i,j,k} h_{ijk}^2\lambda_i, \tag{2.3}$$

$$\int_M A - 2B = \int_M Sf_4 - f_3^2 - S^2 + nHf_3 - \frac{|\nabla S|^2}{4}. \tag{2.4}$$

3. Proof of the theorem.

LEMMA 3.1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be real numbers. Denote*

$$\sum_{i=1}^{n} \lambda_i = nH, \quad \sum_{i=1}^{n} \lambda_i^2 = S,$$
$$\sum_{i=1}^{n} \lambda_i^3 = f_3, \quad \sum_{i=1}^{n} \lambda_i^4 = f_4.$$

Let a_{ij} $(1 \leq i \leq n, 1 \leq j \leq n)$ be real numbers satisfying

$$\sum_{j=1}^{n} a_{ij} = \frac{1}{2}\left[(n-S)\lambda_i + nH\lambda_i^2 - nH\right], \quad a_{ij} = a_{ji}.$$

Then

$$\sum_{i=1}^{n} a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2 \geq \frac{3}{2(n+4)}\left[(n-S)^2 S + n^2H^2 f_4 - n^3H^2 + 2(n-S)nHf_3\right].$$

Proof. Consider the function

$$f(a_{ij}, \alpha_i, \beta_{ij}) = \sum_{i=1}^{n} a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2 + \sum_{i\neq j} \beta_{ij}(a_{ij} - a_{ji})$$
$$+ \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} a_{ij} - \frac{1}{2}\left[(n-S)\lambda_i - nH\lambda_i^2 + nH\right] \right).$$

Then by direct computation, we can obtain

$$\frac{\partial f}{\partial a_{ii}} = 2a_{ii} + \alpha_i \quad (1 \leq i \leq n),$$
$$\frac{\partial f}{\partial a_{ij}} = 6a_{ij} + \alpha_i + \beta_{ij} - \beta_{ji} \quad (i \neq j).$$

According to the method of Lagrange multipliers, we solve the equations

$$\frac{\partial f}{\partial a_{ii}} = 0, \quad \frac{\partial f}{\partial a_{ij}} = 0,$$

and obtain

$$\alpha_i = -\frac{6}{n+4}\left[(n-S)\lambda_i + nH\lambda_i^2 - nH\right],$$
$$a_{ii} = -\frac{\alpha_i}{2}, \quad a_{ij} = -\frac{\alpha_i + \alpha_j}{12} \quad (i \neq j).$$

Put the expressions of a_{ii} and a_{ij} into $\sum_{i=1}^{n} a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2$, and we get the minimum. Hence, we finished the proof of the lemma. □

LEMMA 3.2. *Let M^n be a closed hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}$. Then*

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}\left(Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3\right) + \frac{3}{2(n+4)}\left[(n-S)^2S + n^2H^2f_4 - n^3H^2 + 2(n-S)nHf_3\right]. \tag{3.1}$$

Proof. Since

$$\begin{aligned}\sum_{i,j,k,l}(h_{ijkl}-h_{ijlk})^2 &= \sum_{i,j,k,l}\left(\sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}\right)^2 \\ &= \sum_{i,j,k,l}(\lambda_i-\lambda_j)^2R_{ijkl}^2 \\ &= 2\sum_{i,j}(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2 \\ &= 4\left(Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3\right),\end{aligned} \tag{3.2}$$

we have

$$\begin{aligned}\sum_{i,j,k,l} h_{ijkl}^2 &= \frac{1}{4}\sum_{i,j,k,l}(h_{ijkl}-h_{ijlk})^2 + \frac{1}{4}\sum_{i,j,k,l}(h_{ijkl}+h_{ijlk})^2 \\ &= \frac{1}{4}\sum_{i,j,k,l}(h_{ijkl}-h_{ijlk})^2 + \frac{1}{16}\sum_{i,j,k,l}(h_{ijkl}+h_{ijlk}-h_{klij}-h_{klji})^2 + \sum_{i,j,k,l}u_{ijkl}^2 \\ &= \frac{1}{4}\sum_{i,j,k,l}(h_{ijkl}-h_{ijlk})^2 + \frac{1}{16}\sum_{i,j,k,l}(h_{ijkl}-h_{klij})^2 + \frac{1}{16}\sum_{i,j,k,l}(h_{ijlk}-h_{klji})^2 + \sum_{i,j,k,l}u_{ijkl}^2 \\ &= \frac{3}{8}\sum_{i,j,k,l}(h_{ijkl}-h_{ijlk})^2 + \sum_{i,j,k,l}u_{ijkl}^2 \\ &= \sum_{i,j,k,l}u_{ijkl}^2 + \frac{3}{2}\left(Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3\right),\end{aligned} \tag{3.3}$$

where $u_{ijkl} = \frac{1}{4}(h_{ijkl}+h_{jkli}+h_{klij}+h_{lijk})$.

From Ricci identities, we obtain that

$$\sum_{j=1}^{n} u_{iijj} = \frac{1}{2}\left[(n-S)\lambda_i + nH\lambda_i^2 - nH\right], \tag{3.4}$$

then by setting $a_{ij} = u_{iijj}$, we have

$$\begin{aligned}\sum_{i,j,k,l}u_{ijkl}^2 &\geq \sum_i a_{ii}^2 + 3\sum_{i,j}a_{ij}^2 \\ &\geq \frac{3}{2(n+4)}\left[(n-S)^2S + n^2H^2f_4 - n^3H^2 + 2(n-S)nHf_3\right].\end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), we obtain (3.1). □

LEMMA 3.3. *Let M^n be an n-dimensional ($n \leq 8$) closed hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}(1)$. Then there exists a positive constant $\delta(n) < \min\{\frac{1}{2}, \frac{3}{n}\}$ depending only on n such that*

$$3(A-2B) \leq (2+\delta(n))(S-nH^2)\sum_{i,j,k} h_{ijk}^2 + 3H^2\sum_{i,j,k} h_{ijk}^2 - 6H\sum_{i,j,k} h_{ijk}^2\lambda_i. \tag{3.6}$$

Proof. Let

$$\mu_i = \lambda_i - H, \quad 1 \leq i \leq n, \tag{3.7}$$

then

$$\begin{aligned}
3(A-2B) &= 3\left(\sum_{i,j,k} h_{ijk}^2\lambda_i^2 - 2\sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_j\right)\\
&= 3\left(\sum_{i,j,k} h_{ijk}^2(\mu_i+H)^2 - 2\sum_{i,j,k} h_{ijk}^2(\mu_i+H)(\mu_j+H)\right)\\
&= 3\left(\sum_{i,j,k} h_{ijk}^2\mu_i^2 - 2\sum_{i,j,k} h_{ijk}^2\mu_i\mu_j\right) + 3H^2\sum_{i,j,k} h_{ijk}^2 - 6H\sum_{i,j,k} h_{ijk}^2\lambda_i\\
&\leq (2+\delta(n))\sum_i \mu_i^2\sum_{i,j,k} h_{ijk}^2 + 3H^2\sum_{i,j,k} h_{ijk}^2 - 6H\sum_{i,j,k} h_{ijk}^2\lambda_i\\
&= (2+\delta(n))(S-nH^2)\sum_{i,j,k} h_{ijk}^2 + 3H^2\sum_{i,j,k} h_{ijk}^2 - 6H\sum_{i,j,k} h_{ijk}^2\lambda_i,
\end{aligned} \tag{3.8}$$

where the above inequality follows from Lemma 3.4 in [**20**], as $\sum_i \mu_i = 0$. □

THEOREM 3.4. *Let M^n be an n-dimensional ($n \leq 8$) closed hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}(1)$ and S be the length of the second fundamental form of M^n. Then there exist positive constants $\varepsilon(n)$ depending only on n, and $\delta(n, H)$ depending only on n and H such that if*

$$|H| \leq \varepsilon(n) \quad \text{and} \quad S_0 \leq S \leq S_0 + \delta(n, H),$$

where

$$S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

then $S \equiv S_0$ and M^n is isometric to the Clifford hypersurface. To be precise, M^n is isometric to the Clifford torus $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ if $H = 0$; M^n is isometric to the Clifford hypersurface $\mathbb{S}^1(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbb{S}^{n-1}(\frac{\lambda}{\sqrt{1+\lambda^2}})$ if $H \neq 0$.

Proof. By using (2.1), (2.2) and (2.3) and direct computations, we have

$$\int_M \sum_{i,j,k} h_{ijk}^2 = \int_M -S(n-S) + n^2H^2 - nHf_3, \tag{3.9}$$

$$-\frac{1}{2}\int_M |\nabla S|^2 = \int_M S^2(n-S) - n^2H^2S + nHf_3S + S\sum_{i,j,k} h_{ijk}^2, \tag{3.10}$$

$$\int_M -3\sum_{i,j,k} h_{ijk}^2\lambda_i = \frac{3}{2}\int_M (n-S)f_3 + nHf_4 - nHS, \tag{3.11}$$

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 = \int_M (S-2n-3)\sum_{i,j,k} h_{ijk}^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2 - 3nH\sum_{i,j,k} h_{ijk}^2\lambda_i. \tag{3.12}$$

Applying (3.1) to (3.12), we can obtain

$$\begin{aligned}\int_M (S-2n-3)\sum_{i,j,k} h_{ijk}^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2 - 3nH\sum_{i,j,k} h_{ijk}^2\lambda_i \\ \geq \int_M \frac{3}{2}\left(Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3\right) \\ + \int_M \frac{3}{2(n+4)}\left[(n-S)^2S + n^2H^2f_4 - n^3H^2 + 2(n-S)nHf_3\right]. \end{aligned}\tag{3.13}$$

Combining (2.4), (3.9), (3.10) and (3.11) we get

$$\begin{aligned}\int_M -\left(\frac{5}{4}S - \frac{n}{4} + \frac{3}{2}\right)|\nabla h|^2 + \frac{3}{2}(A-2B) - \left(3 - \frac{3}{n+4}\right)nH\sum_{i,j,k} h_{ijk}^2\lambda_i \\ + \int_M \frac{9n+30}{4(n+4)}\left[-S^2(n-S) + n^2H^2S - n|\nabla h|^2 - nHSf_3\right] \geq 0.\end{aligned}$$

By (3.6), we have

$$\begin{aligned}\int_M -\left(\frac{5}{4}S - \frac{n}{4} + \frac{3}{2}\right)|\nabla h|^2 - \left(3n + 3 - \frac{3n}{n+4}\right)H\sum_{i,j,k} h_{ijk}^2\lambda_i \\ + \int_M \frac{9n+30}{4(n+4)}S\left[-S(n-S) + n^2H^2 - nHf_3\right] - \frac{9n+30}{4(n+4)}n|\nabla h|^2 \\ + \int_M \frac{2+\delta}{2}(S-nH^2)|\nabla h|^2 + \frac{3}{2}H^2|\nabla h|^2 \geq 0.\end{aligned}\tag{3.14}$$

Since $S \geq S_0$, then by direct computation or [**6**], it is not difficult to get $-S(n-S) + n^2H^2 - nHf_3 \geq 0$, and, moreover, if $S_0 \leq S \leq S_0 + \delta(n,H)$, the inequality (3.14) implies

$$\begin{aligned}\int_M -\left(\frac{5}{4}S - \frac{n}{4} + \frac{3}{2}\right)|\nabla h|^2 - \left(3n + 3 - \frac{3n}{n+4}\right)H\sum_{i,j,k} h_{ijk}^2\lambda_i \\ + \frac{9n+30}{4(n+4)}(S_0 + \delta(n,H) - n)|\nabla h|^2 + \frac{2+\delta}{2}(S-nH^2)|\nabla h|^2 + \frac{3}{2}H^2|\nabla h|^2 \geq 0.\end{aligned}\tag{3.15}$$

Because $\sum_{i,j,k} h_{ijk}^2 \lambda_i \leq \sqrt{S}|\nabla h|^2$, this means

$$\int_M -\left(\frac{5}{4}S - \frac{n}{4} + \frac{3}{2}\right)|\nabla h|^2 + \left(3n + 3 - \frac{3n}{n+4}\right) H\sqrt{S}|\nabla h|^2 + \frac{9n+30}{4(n+4)}(S_0 + \delta(n,H) - n)|\nabla h|^2 + \frac{2+\delta}{2}(S - nH^2)|\nabla h|^2 + \frac{3}{2}H^2|\nabla h|^2 \geq 0. \tag{3.16}$$

By the definition of S_0, and noting that $\delta < \frac{3}{n}$ and $|H| \leq \varepsilon(n)$ if $\varepsilon(n)$ is small enough, we can choose $\delta(n, H) > 0$ such that

$$-\left(\frac{5}{4}S - \frac{n}{4} + \frac{3}{2}\right) + \left(3n + 3 - \frac{3n}{n+4}\right) H\sqrt{S} + \frac{9n+30}{4(n+4)}(S_0 + \delta(n,H) - n) + \frac{2+\delta}{2}(S - nH^2) + \frac{3}{2}H^2 < 0.$$

According to (3.16) and the above inequality, we conclude that $|\nabla h| \equiv 0$. Hence, all of the above inequalities are equalities. Therefore, $-S(n - S) + n^2H^2 - nHf_3 = 0$, which implies $S = S_0$ and M is isometric to the Clifford hypersurface. Thus, the proof is completed. $\square$

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