ON THE MAXIMUM MODULUS AND THE MEAN MODULUS OF AN ENTIRE FUNCTION

A.R. Reddy

Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function, but not a polynomial. As usual let,

$$M(r) = \max_{|z|=r} |f(z)|, \quad M_\delta(r) = \{\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta\}^{1/\delta}, \; 0 < \delta < \infty,$$

(1)

$$m(r) = \sum_0^\infty |a_n| r^n, \quad \mu(r) = \max_{n \geq 1} |a_n| r^n = |a_\nu| r^\nu,$$

$\nu = \nu(r)$, $\nu(r)$ is a monotonic increasing function of r. By Parseval's theorem, we have

(2) $$M_2(r) = \{\sum_0^\infty |a_n|^2 r^{2n}\}^{1/2}.$$

In this note we prove two theorems, of which Theorem 1 includes a result of Brinkmier [1, Satz (8)]. The proof of this theorem is not only shorter than the one used by Brinkmier, but also very elementary. By combining Theorems 1 and 2 we can have the well-known result of Valiron [2, Theorem 12]. In this case too the present proof is more elementary than the one used by Valiron.

THEOREM 1. *If* $f(z)$ *is an entire function of finite order* ρ *then the inequality*

$$m(r) \leq M_2(r)\, r^{\frac{1}{2}\rho+\varepsilon_1}$$

is satisfied for $r > r_{\varepsilon_1}$, ε_1 *being arbitrarily small.*

Proof. Let us choose $0 < r < R$, $\nu = \nu(R)$. By definition

$$m(r) = \sum_0^\infty |a_n| r^n = \sum_0^{\nu-1} |a_n| r^n + \sum_\nu^\infty |a_n| r^n .$$

Now by the Cauchy-Schwarz inequality,

$$m(r) < \sqrt{\nu} \left(\sum_0^{\nu-1} |a_n|^2 r^{2n} \right)^{1/2} + \sum_\nu^\infty |a_n| r^n .$$

Using here the definitions of $M_2(r)$ and $\mu(r)$,

$$m(r) \le \sqrt{\nu}\, M_2(r) + \mu(r) \sum_\nu^\infty \frac{|a_n| r^n}{|a_\nu| r^\nu} ,$$

$$m(r) \le \sqrt{\nu}\, M_2(r) + \mu(r) \sum_\nu^\infty \frac{|a_n| R^n}{|a_\nu| R^n} \left(\frac{r}{R}\right)^{n-\nu} ,$$

$$m(r) \le \sqrt{\nu(R)}\, M_2(r) + \mu(r) \; (R/R - r) ,$$

$$m(r) \le M_2(r) \, \{\sqrt{\nu(R)} + R/R - r\} , \quad \text{because } \mu(r) \le M_2(r) .$$

Now by choosing $R = 2r$, and also by using the fact that

$$\limsup_{r \to \infty} \frac{\log \nu(r)}{\log r} = \rho \quad [2, \text{ page } 30] ,$$

we have finally

$$m(r) \le M_2(r) \, r^{\frac{1}{2}\rho + \varepsilon_1} .$$

Hence the theorem is proved.

https://doi.org/10.4153/CMB-1969-115-0 Published online by Cambridge University Press

THEOREM 2. If $f(z)$ is an entire function of finite order ρ, then the inequality $M(r) \leq \mu(r)\, r^{\frac{1}{2}\rho+\varepsilon_2}$ is satisfied for $r > r_{\varepsilon_2}$, ε_2 being arbitrarily small.

Proof. Let us choose $0 < r < R$, $\nu = \nu(R)$.

Now by definition

$$\{M_2(r)\}^2 = \sum_0^\infty |a_n|^2 r^{2n} = \sum_0^{\nu-1} |a_n|^2 r^{2n} + \sum_\nu^\infty |a_n|^2 r^{2n},$$

$$\leq \nu\{\mu(r)\}^2 + \{\mu(r)\}^2 \sum_\nu^\infty \frac{|a_n|^2 R^{2n}}{|a_\nu|^2 R^{2\nu}} \left[\left(\frac{r}{R}\right)^2\right]^{(n-\nu)},$$

$$\leq \nu\{\mu(r)\}^2 + \{\mu(r)\}^2 \sum_\nu^\infty \left[\left(\frac{r}{R}\right)^2\right]^{n-\nu}$$

$$\{M_2(r)\}^2 \leq \nu\{\mu(r)\}^2 + \{\mu(r)\}^2 R^2/R^2 - r^2 .$$

By choosing here $R = 2r$ and using the fact [2, page 30] that

$$\limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho ,$$

we finally have the required result, i.e.

$$M_2(r) \leq \mu(r)\, \{\sqrt{\nu(R) + R^2/R^2 - r^2}\} \leq \mu(r)\, r^{\frac{1}{2}\rho+\varepsilon_2} .$$

Remark. By combining Theorems 1 and 2 we have $m(r) \leq \mu(r)\, r^{\rho+\varepsilon_3}$.

https://doi.org/10.4153/CMB-1969-115-0 Published online by Cambridge University Press

REFERENCES

1. H. Brinkmier, Über das mass der Bestimmutheit des wachstums einer ganzen transzendenten function durch die absoluten Betrage der Koeffizienten ihrer Potenzreihe. Math. Ann. 96 (1927) 108-118.

2. G. Valiron, Theory of integral functions. (Chelsea Publishing Co. 1949.)

University of Alberta
Edmonton

https://doi.org/10.4153/CMB-1969-115-0 Published online by Cambridge University Press