6. THE THEORETICAL BASIS OF THE FISSION THEORY OF BINARY STARS

By R. A. Lyttleton

I wish to say something today regarding the theoretical basis of the fission theory of binary stars. For many years it has been thought to be one of the established conclusions of astronomy that at least the close binaries have resulted from the fissional break-up of single massive stars, and theories of stellar evolution have regarded this as one of the settled issues that must be incorporated in a general theory accounting for all types of stars.

In discussing the basis of the fission theory it will not be my concern so much to consider actual stars, but only the theoretical stars as postulated in the dynamical formulation of the fission theory.

Several tacit assumptions as to the former condition of a binary star are implicit. For example, that they have evolved from single massive stars, that this single star has come to possess large angular momentum, that this star has somehow proceeded to a state of rotational instability. (The whole question of the origin of individual stars is ignored in the fission theory.)

In order to render the problem tractable it is necessary to assume uniform density of the star, however much this assumption may fail to hold. Jeans himself believed that the central regions of his stars, comprising the main part of the mass, would closely conform to this assumption.

Stability. In following the evolution of rotating systems, it is necessary to distinguish between two kinds of stability, usually termed *ordinary* and *secular*. For non-rotating systems the two are the same.

A rod standing upright on its end is simply *unstable*. Given a slight push the rod falls away from the position regardless of the presence of friction. The energy contributed by a disturbance, however slight, is sufficient to rid the time integral of its infinity and the rod reaches any given angular displacement *in a finite time*.

Similarly a non-rotating or slowly rotating top standing upright on its point is unstable. But a sufficiently rapidly rotating top can, as we know, stand upright on its point, and *in the absence of friction* will do so indefinitely and moreover if slightly disturbed will only oscillate about the upright position. The top is said to be *ordinarily stable*.

But if account is taken of friction, then however small this is, dissipation will set in, and the top will gradually spiral away from the upright position and precess with ever increasing inclination of its axis to the vertical. The top is said to be *secularly unstable*.

Notice that if there is no friction, the path of the top is reversed if all velocities (spin round the axis as well) are reversed. This is because the equations of motion do not contain terms involving the velocities, so if we write $-t$ for t, the equations are unaltered and we get the same motion described in the reverse direction; e.g. if we reverse the direction of motion of a planet round the Sun, the orbit is described the other way; but if we reverse the velocity of a meteor through the atmosphere it will not describe the reversed path. This complete reversibility is a characteristic property of motion away from a position of ordinary instability, and indeed of any motion not involving friction. But where *secular instability* is concerned the motion is *not reversible*. If we have a top that has spiralled some distance from the upright position, and reverse its direction of motion *and of spin*, it will *not* proceed to climb back again to the upright position, but will go on precessing in an ever widening angle. Dissipation is involved and cannot be reversed. This distinction between ordinary stability (no friction) and secular stability has constantly to be borne in mind in considering the evolution of rotating systems.

A dynamical system that is secularly stable is necessarily ordinarily stable; e.g. the conical pendulum.

A dynamical system that is secularly unstable may nevertheless be ordinarily stable, e.g. top spinning fast enough, or it may be ordinarily unstable, e.g. top spinning slowly enough.

This means that if an evolving system reaches a point at which it becomes secularly unstable, its ordinary stability may continue to hold, or may cease simultaneously (it cannot cease *before*). Jeans fell into error on just this point in thinking that secular instability always occurs *before* ordinary instability can set in, and hence that ordinary stability could not be of any importance in the fission problem.

Finally, a system that is ordinarily unstable is necessarily secularly unstable, e.g. inverted pendulum or top, with negligible spin.

Evolution of a rotating liquid mass. Evolution with gradually increasing density and constant angular momentum can readily be shown to be completely equivalent, as to the series of figures, to evolution with constant density and gradually increasing angular momentum. It is easier to think in terms of the latter as the mass then preserves its volume unchanged.

With zero angular momentum the system begins as a uniform sphere, and its angular velocity is zero. As angular momentum is added, the angular velocity begins by increasing and the mass flattens to a spheroid, the eccentricity of section increasing with the angular momentum.

To start with, the spheroidal forms (the Maclaurin spheroids) are thoroughly stable for all displacements, i.e. both secularly and ordinarily stable. This holds till the form is reached for which $e = 0{\cdot}8127$. Its dimensions are:

e	a/r	c/r	$\omega^2/2\pi G\rho$	$H/G^{\frac{1}{2}}M^{\frac{3}{2}}r^{\frac{1}{2}}$	V/GM^2r^{-1}	T/GM^2r^{-1}
0·8127	1·1973	0·6976	0·1868	0·3035	−0·5850	0·0805

The investigation of *all* possible displacements is made by means of ellipsoidal harmonic analysis. It is found that the harmonic through which instability first enters is a certain second-order harmonic which when added at the surface of the spheroid transforms it into an ellipsoid with three unequal axes (the beginning of the Jacobi series).

Spheroidal forms exist beyond this but are secularly unstable. Nevertheless for a time ($e < 0{\cdot}9529$) they remain ordinarily stable, but after this they are both secularly and ordinarily unstable.

What does this secular instability mean? The system can (conceptually) be set up in any spheroidal form. But under the slightest disturbance (for e slightly greater than 0·8127, say) oscillations would occur causing it to depart more and more from the spheroidal form. Their amplitude would increase at a rate depending on the amount of friction. Also dissipation is involved causing a loss of energy and this cannot go on indefinitely; the system accordingly seeks some new form not involving further dissipation. In this case the new form is a Jacobi ellipsoid. For equal values of H the Jacobi ellipsoid has less energy than the corresponding spheroid. In passing over to the Jacobi form, the excess energy is dissipated by friction, and eventually a steady state (the Jacobi ellipsoid) is reached.

For $H > 0{\cdot}3035$ there exist equilibrium forms with three unequal axes. Physically the system cannot ever get into a state represented by any of the secularly unstable spheroids.

Once the system has got on to the Jacobi series, further evolution with increasing angular momentum now takes place along this series. It begins (at a point coinciding with the last stable spheroid) as a thoroughly stable configuration. The *angular velocity* now *diminishes* as the angular momentum increases, so that if break-up should eventually occur the pieces will be endowed with slow rotation and so can revert to stable spheroids at once without serious change of angular velocity. The (increasing) angular momentum is now carried in the figure which elongates considerably.

The Jacobi ellipsoids remain stable in all respects until that figure J is reached given by

(a)	(b)	(c)	$\omega^2/2\pi G\rho$	$H/G^{\frac{1}{2}}M^{\frac{3}{2}}r^{\frac{1}{2}}$	V/GM^2r^{-1}	T/GM^2r^{-1}
1·8858	0·8150	0·6507	0·1420	0·3896	−0·552	0·0894

The instability (secular) is found to enter for a certain third-order harmonic displacement, which when added to the last ellipsoid transforms it into the so-called 'pear-shaped' figure. The Jacobi series is crossed by a new series of configurations.

What happens next? Is this new series stable or unstable? This in effect was the great problem attacked by Poincaré, Darwin, Liapounov and Jeans.

If the new series should turn out to be stable, further evolution with increasing angular momentum would proceed along it (at least for a time).

The criterion for determining whether the pear is stable or not can be shown to depend on whether the angular momentum of the pear-shaped members near the last Jacobi (stable) form is greater than or less than that of *J*. If it is greater than that of *J*, then the pear-shaped series would (to begin with, anyway) be stable. If it were less, then the pears are unstable. So the question resolves itself into discovering whether the representative curve of the pear series turns up or down near *J*

Darwin maintained that he had proved it turned up, and hence that the series was *stable*. Liapounov said he had proved it turned down, and hence that the series was *unstable*.

Two-dimensional problem. With a view to helping resolve this difficulty, Jeans took up the corresponding two-dimensional problem of a rotating gravitating cylinder. He eventually announced that he had proved that the two-dimensional series was stable, and this appeared to confirm Darwin's result.

If Jeans had been correct in this conclusion, it would have been legitimate to regard the system as evolving next along the pear-shaped series. This would involve a deepening of the furrow, suggesting an ultimate division into two parts. This was the course of development envisaged by Darwin and Jeans.

But even had their conclusion as to stability been correct, this description omits to answer the relevant question whether any other series of configurations may branch off the pear-shaped series. It would be hard indeed to imagine how this question might even have been tackled had it occurred to Darwin and Jeans. This then was the position in 1902.

Evidently Jeans gradually came to entertain some deep-rooted suspicions about the whole theory, and ten years later he recommenced work on the three-dimensional problem, starting in an entirely fresh way. His final conclusion this time was that the initial members of the piriform series had *less* angular momentum than the last stable Jacobi form and were therefore *unstable*. Moreover, a re-scrutiny of his work on the two-dimensional problem disclosed a simple numerical error that had reversed the sign of the crucial term and which when corrected indicated that this series was also *unstable*. Both results were now in agreement with Liapounov's conclusion.

This altered the whole position because now *the system cannot evolve along the pear-shaped series.*

Jeans, however, seems to have been anxious to retain the fission theory whatever conclusions his analysis might lead to, and in face of the new result he made several mistakes which culminated in his retaining the fission hypothesis unchanged from what it had been when he considered the pear to be stable. First, he still retained his picture of the system evolving along the pear-shaped series and continued to use his invalid two-dimensional calculations unchanged. But this course of evolution assumes the angular momentum to go on increasing, whereas what is required is to find the course of development with a given angular momentum from an unstable configuration on the Jacobi series. There is now no question of the system being urged on by further increases of density, here represented by increase of angular momentum but simply of a system reaching and slightly passing a point of bifurcation and then undergoing further *motion with constant angular momentum.*

As it is, the initial members of the piriform series have *less* angular momentum than the last Jacobi form and so the system could never evolve along this series. Second, Jeans believed that the question of ordinary stability did not arise. This would have been valid if the system were secularly stable, for this would imply that it is also ordinarily stable, but if a system—here the last Jacobi ellipsoid—is secularly unstable, it may or may not be ordinarily stable. This becomes a further question that has to be investigated afresh by setting up the equations of small motion of the system.

What then happens when the system reaches the last stable Jacobi ellipsoid and passes just beyond?

If the Jacobi ellipsoid were ordinarily stable, although secularly unstable, it is conceivable that the system might depart only slowly from the Jacobi series under the effect of internal friction. (For example, the lunar orbit in the Earth-Moon system, taking account of the angular momentum of the planet, is secularly unstable but ordinarily stable, and departure is very slow because friction is very small.) To complete the discussion of the system it is therefore necessary to discover whether the Jacobi series remains ordinarily stable when its secular stability ceases, or whether it becomes ordinarily unstable at the same stage, as it might well do. In fact, Cartan has shown comparatively simply that ordinary stability disappears *at the same time* as the secular stability for third-order harmonic displacements.

Thus beyond the last stable Jacobi form there are no equilibrium forms that possess either kind of stability. The system has now reached a state similar to that of a nearly vertical top with insufficient spin for its stability. and it is no more stable than a stick balanced on end. Given the slightest disturbance it will move away from the position and attain a *finite displacement from it in a finite time*, but the direction of fall can only be predicted if the initial disturbance is known. In the present case the motion must also be a rapid one and there is no possibility of the system evolving slowly.

As the system is *ordinarily unstable*, whatever motion it undergoes must of necessity be *strictly reversible*. That is, if all velocities are reversed, the system would undergo the reversed motion describing the same path in the opposite direction. It follows from this that fission into a binary star is not possible. For if in a binary system every velocity is reversed, the system simply describes the orbit the other way and does not proceed to coalesce into a single star. Accordingly it follows from the ordinary instability of the Jacobi series (beyond the critical member) that fission into two pieces in closed orbital motion is not possible. *The process does not work.*

What then is the final result of the instability? Certainly it is not motion along the pear-shaped series. Nor can division into two masses in finite orbital motion round each other take place.

But somehow the system has got to find its way to another steady state, since dissipation cannot go on indefinitely.

Nor can the system remain a single mass since there is no equilibrium figure for it to move to. Some kind of break-up must occur in order to rid the mass of its embarrassment of angular momentum.

If we assume break-up into two main pieces, then we can show that the system can get out of all its difficulties and satisfy the various requirements we have indicated provided that *these pieces separate to infinity in hyperbolic orbits*.

If the velocity of separation were not hyperbolic, although the pieces might at first begin to separate, their attraction would soon reverse this, and they would eventually re-unite. This could only be temporary, since no loss of angular momentum could take place. Also it would physically involve dissipation, loss of heat, and therefore some increase of density, which here would mean an increase in angular momentum with density remaining constant. So the system would be more unstable than before and a further break-up of greater violence would ultimately ensue. Obviously the only end to this would be if the velocity of separation was enough to enable the masses to separate altogether in (almost) hyperbolic orbits. The excess angular momentum can then be stored in the orbital motion. Each of the two component pieces can revert to a form appropriate to the MacLaurin series, without undergoing any serious change of angular velocity (since the angular velocity *diminishes* as the Jacobi series is described). Evolution along the Jacobi series gradually slows up the rotation and so prepares as it were, to produce stable components.

The final steady state ultimately reached is therefore that of two stable masses separating from each other with constant velocity. No further dissipation need take place. Also the motion is strictly reversible, as required.

Mass-ratio. By considering the energy balance and the angular momentum balance as between the last stable ellipsoid and the double system it may be shown that the mass-ratio cannot be less than about 7·1. That is, only a small component need or can be ejected. Equality of masses of the two pieces is quite impossible. (This by the way is yet another argument against the fission theory, since if we assume stable binary orbits the least mass-ratio possible is about 3:1, and so close binaries of equal masses, which are common enough, could not be explained by fission. If loss of mass by radiation is invoked to equalize the masses, the orbit evolves and the system is no longer a close one.)

Given every factor in its favour, the theoretical considerations on which the fission hypothesis has been based are found in fact to be against the process on every count. This conclusion ignores the question whether actual stars are sufficiently closely representable by the assumption of uniform density. Of course we are now pretty certain that they depart very much from this. It ignores the criticism that evolution by gradually increasing density has no real valid basis in the theory of the structure and development of ordinary stars, and it ignores the question whether stars are in fact created as single massive stars already endowed with large angular momentum.

To sum up, *Jeans's main errors* in his theory of fission were as follows:

(1) In believing that secular instability invariably sets in before ordinary instability, and hence that any question of ordinary instability was irrelevant.

(2) In misunderstanding his own result that the system was secularly unstable, and reaching exactly the same conclusions as he had reached when he believed he had proved it was secularly stable.

(3) In supposing that evolution would take place along the pear-shaped series. This would require angular momentum to be subtracted from the system, whereas once the gradual increase of angular momentum has brought the system to an unstable state the subsequent motion is with constant angular momentum. The parameter defining the series is not to be regarded as a dynamical co-ordinate.

(4) In supposing circular orbits could result. (He actually states at one point that the masses must be projected away from each other, but then goes on to invoke collisions to round up the orbits. This in any case is ruled out by the fact that a stable binary system has a finite separation between its components.)

(5) In supposing that equal components could result. The simplest considerations of energy and angular momentum show this to be impossible.

Discussion du rapport de LYTTLETON

Martynov demande quel changement subissent les conclusions de Lyttleton pour les étoiles possédant une concentration de matière.

Lyttleton pense que la situation est pire et la fission impossible.

7. THE CHEMICAL COMPOSITION OF THE STARS AND ITS RELATION TO STELLAR EVOLUTION

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On the one hand the rate at which stellar evolution progresses will depend on the chemical composition of the star. On the other, as a by-product of the nuclear processes involved in the energy production, the chemical composition of the reacting material will change. Stellar models can be used to provide information as to the composition of the reacting zones, stellar spectroscopy will provide the composition of the atmospheres,

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