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Abstract

We consider the Prym map from the space of double coverings of a curve of genus g with r branch points to the moduli space of abelian varieties. We prove that $\mathcal{P}: \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1+r/2}$ is generically injective if

$$r > 6 \text{ and } g \geqslant 2, \quad r = 6 \text{ and } g \geqslant 3, \quad r = 4 \text{ and } g \geqslant 5 \quad \text{or} \quad r = 2 \text{ and } g \geqslant 6.$$

We also show that a very general Prym variety of dimension at least 4 is not isogenous to a Jacobian.

1. Introduction

Let C be a complex projective curve of genus $g \geqslant 1$, and let $\pi : D \to C$ be a two-sheeted covering of C ramified at r points. The *Prym variety* P of π is the identity component of the kernel of the norm homomorphism

$$N : J(D) \to J(C).$$

Note that if $r \neq 0$, then ker N is connected (see, for example, [Mum74, § 3, Lemma] and [Kan04, § 1, Lemma 1.1]). The Prym variety P of π is an abelian variety of dimension $g - 1 + r/2$, and the divisor $\Theta_P := \Theta_{J(D)} \cap P$ gives a polarization of type

$$\delta = (1, \ldots, 1, \underbrace{2, \ldots, 2}_{g}) \tag{1}$$

(see, for instance, [Kan04, § 1, Lemma 1.1]).

In this paper we deal with the generic Torelli theorem for Prym varieties of ramified coverings. The infinitesimal Torelli theorem stated in [NR95] (see also Proposition 2.2) and the étale case suggest that the result should hold when the dimension of the space of coverings is strictly smaller than the dimension of the moduli space of abelian varieties.

Let C be a curve, η a line bundle on C and R a multiplicity-free divisor in the linear system $|\eta^2|$. Following [Mum74], we can provide the coherent $\mathcal{O}_C$-module $\mathcal{O}_C \oplus \eta^{-1}$ with a natural $\mathcal{O}_C$-algebra structure depending on R. The natural projection

$$\pi : D := \mathrm{Spec}(\mathcal{O}_C \oplus \eta^{-1}) \to C = \mathrm{Spec}(\mathcal{O}_C)$$

is a ramified double covering with branch points in the support of R and, conversely, each double covering π comes in this form. Let $\mathcal{R}_{g,r}$ denote the scheme parametrizing triples (C, η, R) up to

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isomorphism; the *Prym map* is the morphism

$$\mathcal{P} : \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1+r/2}$$

which associates to (C, η, R) the Prym variety P of π. We call the closure of the set of Prym varieties in $\mathcal{A}^{\delta}_{g-1+r/2}$ the *Prym locus* and denote it by $\mathcal{P}^{\delta}_{g-1+r/2}$.

In the étale case, the Prym map is generically finite for $g \geqslant 6$ (Wirtinger's theorem; see [Bea77b]), and it is never injective [Don81, DS81, Nar92, Nar96]. Kanev [Kan82] and Friedman and Smith [FS82] proved independently that the Prym map has generically degree 1 for $g \geqslant 7$. This is the so-called *generic Torelli theorem for Prym varieties*. It is known (see [GL85, LS96, NR95] and § 2.1) that the Prym map is generically finite (onto its image) if and only if

$$\dim \mathcal{R}_{g,r} \leqslant \dim \mathcal{A}^{\delta}_{g-1+r/2}.$$

In [NR95], Nagaraj and Ramanan proved that if

$$r = 4, \quad g \geqslant 4 \quad \text{and} \quad h^0(\eta^2) = 1,$$

then the triple (C, η, R) can be recovered from the Prym variety $\mathcal{P}(C, \eta, R)$. Furthermore, the Prym map

$$\mathcal{P} : \mathcal{R}_{3,4} \to \mathcal{A}^{\delta}_{4}$$

is a dominant morphism of degree 3 (see [NR95, Theorem 9.14] and [BCV95, Theorem 5.11]). We shall prove the following.

Theorem 1.1 (Generic Torelli theorem). *If one of the conditions*

$$r > 6 \text{ and } g \geqslant 2, \quad r = 6 \text{ and } g \geqslant 3, \quad r = 4 \text{ and } g \geqslant 4, \quad r = 2 \text{ and } g \geqslant 6$$

holds, then the Prym map is generically injective.

This shows that our expectation is essentially satisfied. In the bi-elliptic case ($r \geqslant 6$ and $g = 1$) the generic injectivity of the Prym map has recently been proved by the first author and Naranjo (see [MN11]) by using the techniques we develop in this paper together with a construction of Del Centina and Recillas (see [DR89]). Thus there are only two cases left ($r = 2$ and $g = 5$, $r = 6$ and $g = 2$) that will be the object of future studies.

The proof consists of two parts. In the spirit of [CG80, GT84], the first part is based on the *infinitesimal variation of Hodge structures* approach to Torelli problems (the Prym étale case is described in [SV02]). We prove that, for a general point, the inclusion

$$d\mathcal{P}(T_{(C,\eta,R)}) \subset T_{\mathcal{P}(C,\eta,R)}, \tag{2}$$

i.e. the position of the image of the differential as a subspace of the tangent space of $\mathcal{A}^{\delta}_{g-1+r/2}$ at $\mathcal{P}(C, \eta, R)$, determines the semi-canonical model C_η of C, that is, the image of the curve C through the embedding associated to $\omega_C \otimes \eta$. Since C_η identifies, up to isomorphism, the pair (C, η), this proves the theorem when $h^0(\eta^2) = 1$, i.e. when $g \geqslant r$. As in the étale case (cf. [Deb89]), the kernel of the co-differential of the Prym map at (C, η, R) is the space of quadrics vanishing on C_η. We show that we can recover C_η in the intersection of the quadrics. It turns out that the more interesting and difficult case is where $g = 6$ and $r = 2$. In this case, the intersection of the quadrics consists of, scheme-theoretically, the curve C_η and five projective lines corresponding to the five morphisms of degree 4 from C to $\mathbb{P}^1$.

In the second part, using degeneration methods (see [Bea77a, FS86] for the étale case), we prove the theorem for $g < r$. We fix a general curve C and show that the Prym map is generically injective on the space of double coverings of C. To do this, we extend the Prym map to some admissible coverings of C (see, for example, Figure 1 on page 1157) and obtain a proper morphism to a suitable compactification of $\mathcal{A}^{\delta}_{g-1+r/2}$. Then we prove that the infinitesimal Torelli theorem holds also at the boundary, and we give a lemma that allows us to compute the degree of the Prym map once we know the behaviour at the boundary.

We then consider two applications. By Theorem 1.1, a general Prym variety P arises from a unique covering $\pi : D \to C$. Thus there is a canonical choice of divisor Θ_P inducing the polarization. In analogy with the hyperelliptic case of Andreotti's proof of the Torelli theorem (see [And58]), we prove that the Gauss map of Θ_P in P allows us to determine the branch locus of π.

The second application is the following.

Theorem 1.2. *A very general Prym variety of dimension at least* 4 *is not isogenous to a Jacobian variety.*

We recall that a Prym variety is said to be very general when it is outside a countable union of proper subvarieties of the Prym locus. The argument is again based on degeneration techniques and is similar to the ones used in [BP89, NP94]. However, the comparison of extension classes requires a more sophisticated geometric analysis. In particular, we use the fact that a very general Prym variety of a ramified covering is simple (see [BP02, Pir88]).

In [Mar] it is proved that on a very general Jacobian variety of dimension $n \geqslant 4$ there are no curves of genus g with $n < g < 2n-2$ and, by using Theorem 1.2, that there are strong obstructions to the existence of curves of genus $2n-2$. For example, there are no curves of genus 6 on a very general Jacobian variety of dimension 4. This was our original motivation.

1.1 Plan of the paper

In §§ 2 and 3 we prove Theorem 1.1. In § 2.1 we prove an infinitesimal version of the Torelli theorem and determine when the Prym map is finite. In § 2.2 we describe the intersection of the quadrics vanishing on the semi-canonical model C_η of C. In particular, we show that C_η is the only non-hyperplane curve in the intersection. This concludes the proof of the theorem when $g \geqslant r$.

In § 3.1 we fix a curve C and show that it is possible to define a rational map

$$\mathcal{S} : \Upsilon \dashrightarrow \bar{\mathcal{A}}^{\delta}_{g-1+r/2},$$

from the space of admissible coverings of C to a suitable compactification of the moduli space of abelian varieties, which coincides with the Prym map on the space of smooth coverings. In § 3.2 we blow up $\mathcal{S}$ in its indeterminacy locus to get a proper map. Then we give a result on the cardinality of the general fibre of a proper morphism and apply it to our case in order to show that $\mathcal{S}$ is generically injective (if $g > 2$).

In § 4.1 we describe the Gauss map of a Prym variety, while in § 4.2 we prove Theorem 1.2.

1.2 Notation and preliminaries

(i) We work over the field $\mathbb{C}$ of complex numbers. Each time we have a family of objects parametrized by a scheme X (respectively, by a subset $Y \subset X$), we say that the *general* element of the family has a certain property $\mathfrak{p}$ if $\mathfrak{p}$ holds on a dense Zariski-open subset of X (respectively, of Y). Moreover, we say that a *very general* element of X (respectively, of Y) has the property $\mathfrak{p}$ if $\mathfrak{p}$ holds on a dense subset that is the complement of a union of countably many proper subvarieties of X (respectively, of Y).

(ii) Given an effective line bundle L on C, we will often need to consider the natural map

$$m : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^2)$$

and its dual

$$m^* : H^1(C, \omega_C \otimes L^{-2}) \to \mathrm{Sym}^2 H^1(C, \omega_C \otimes L^{-1}).$$

Let $f : C \to \mathbb{P}^N$ be the projective morphism associated to the linear system $|L|$. We recall (cf. [GL85, Laz89]) that the kernel of m can be identified with the space of homogeneous quadratic polynomials vanishing on $f(C)$. Furthermore, m^* can be identified (up to multiplication by a non-zero scalar) with the map

$$\mathrm{Ext}^1(L, \omega_C \otimes L^{-1}) \to \mathrm{Hom}(H^0(C, L), H^1(C, \omega_C \otimes L^{-1}))$$

which sends an extension of L by $\omega_C \otimes L^{-1}$ to the connecting homomorphism it determines.

(iii) Let P be the Prym variety of a ramified covering $\pi : D \to C$, and let $i : D \to D$ be the involution induced on the curve D by π. The *Abel–Prym map* of π is the morphism

$$\begin{aligned} a : D &\to J(D) \\ x &\mapsto [x - i(x)]. \end{aligned}$$

The image of a is a curve D' contained in P, called the *Abel–Prym curve*. If D is hyperelliptic, a has degree 2; otherwise it is birational.

(iv) Given an abelian variety A, its *Kummer variety* $\mathcal{K}(A)$ is the quotient of A by the involution that maps x to $-x$. We denote by $\mathcal{K}^0(A)$ the Kummer variety of $\mathrm{Pic}^0(A)$.

(v) A *semi-abelian variety* G of rank n is an extension $0 \to T \to G \to B \to 0$ of an abelian variety B by an algebraic torus $T = \prod^n \mathbb{G}_m$ (see [Cha85, ch. II, § 2] and [FC90]).

(vi) Let D be a projective curve having only nodes (ordinary double points) as singularities. The *generalized Jacobian variety* of D is defined as the semi-abelian variety $\mathrm{Pic}^0(D)$. If D is obtained from a non-singular curve C by identifying $p, q \in C$, the semi-abelian variety $J(D)$ is the extension of $J(C)$ by $\mathbb{G}_m$ determined by $\pm[p - q] \in \mathcal{K}^0(J(C))$ (see [Ser88]).

(vii) Given a smooth curve C of genus greater than 1, we denote by Γ_C the image of the difference map

$$\begin{aligned} C \times C \to J(C) &\xrightarrow{\sim} \mathrm{Pic}^0(J(C)) \\ (p, q) &\mapsto [a - b]. \end{aligned} \tag{3}$$

The image Γ'_C of Γ_C through the projection $\sigma_C : \mathrm{Pic}^0(J(C)) \to \mathcal{K}^0(J(C))$ is a surface, in the Kummer variety, that parametrizes generalized Jacobian varieties of rank 1 with compact part $J(C)$. Given an integer $n \in \mathbb{N}$, we denote by $n\Gamma_C$ the image of Γ_C under multiplication by n and by $n\Gamma'_C$ the projection of $n\Gamma_C$ in $\mathcal{K}^0(J(C))$.

2. A first Torelli-type theorem

2.1 Finiteness of the Prym map

In the present subsection our aim is to determine when the Prym map is finite. Here we state a simple lemma that will be used throughout the paper.

Lemma 2.1. *Let (C, η, R) be a general point of $\mathcal{R}_{g,r}$.*

(i) *If $r = 6$ and $g = 3$, then $h^0(\eta) = 1$.*

(ii) *Suppose that one of the following conditions holds:*

  (a) *$r = 2$ and $g \geqslant 5$;*
  (b) *$r = 4$ and $g \geqslant 3$;*
  (c) *$r = 6$ and $g \geqslant 4$.*

  Then η is not effective.

(iii) *Suppose that one of the following conditions holds:*

  (a) *$r = 2$ and $g \geqslant 4$;*
  (b) *$r = 4$ and $g \geqslant 5$.*

  Then, given any point $p \in C$, the line bundle $\eta \otimes \mathcal{O}_C(p)$ is not effective.

(iv) *If $r = 2$ and $g \geqslant 6$, then given any two points $p, q \in C$, the line bundle $\eta \otimes \mathcal{O}_C(p + q)$ is not effective.*

Proof. Statements (i) and (ii) are trivial. We prove (iii)(a), the other cases being analogous. Assume, by way of contradiction, that for each η there exists a point $p \in C$ such that $\eta \otimes \mathcal{O}_C(p) \simeq \mathcal{O}_C(a_1 + a_2)$ for some $a_1, a_2 \in C$. Then $R + 2p \equiv 2a_1 + 2a_2$ and so C has a two-dimensional family of g^1_4 of type $|2a_1 + 2a_2|$. Since the moduli space of coverings of degree 4 of $\mathbb{P}^1$ with two ramification points over the same branch point has dimension $2g + 2$, a general curve of genus $g \geqslant 4$ has, at most, a one-dimensional family of g^1_4 of this type, and we get a contradiction. □

We now give conditions under which the Prym map is a local embedding. We recall that the tangent space of $\mathcal{R}_{g,r}$ at (C, η, R) can be identified with the space of first-order deformations of the r-pointed curve C, i.e. with $H^1(C, T_C(-R))$ (see [HM98, ch. 3, § B]). Following the computation in [Bea77b, Proposition 7.5] (see also [LO11, Proposition 4.1] and [NR95, Proposition 3.1] for the ramified case), we can identify the co-differential of the Prym map at (C, η, R) with the natural map

$$d\mathcal{P}^* : \mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2 \otimes \mathcal{O}_C(R)).$$

By [GL85, Theorem 1] (see also [LS96, Theorem 1.1]), to prove the surjectivity of $d\mathcal{P}^*$ it is sufficient to show that $\omega_C \otimes \eta$ is very ample and that the following inequality holds:

$$\deg(\omega_C \otimes \eta) \geqslant 2g + 1 - \mathrm{Cliff}(C).$$

By a straightforward computation we get the following proposition (cf. [LO11, § 5]).

Proposition 2.2. *Let $(C, \eta, R) \in \mathcal{R}_{g,r}$, and assume that one of the following conditions holds:*

(i) *$r \geqslant 6$ and $g \geqslant 1$;*

(ii) *$r = 4$, C is not hyperelliptic and $h^0(\eta) = 0$;*

(iii) *$r = 2$, $\mathrm{Cliff}(C) > 1$ and $h^0(\eta \otimes \mathcal{O}_C(p)) = 0$ for each $p \in C$.*

Then the differential of the Prym map at (C, η, R) is injective.

In view of Lemma 2.1, we can restate Proposition 2.2 in a more compact form.

Corollary 2.3. *The Prym map is generically finite (onto its image) if and only if $r \geqslant 6$ and $g \geqslant 1$, or $r = 4$ and $g \geqslant 3$, or $r = 2$ and $g \geqslant 5$.*

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Proof. Lemma 2.1 proves that the statement of Proposition 2.2 is realized for a general point (C, η, R). Thus the *if* part is proved.

For the *only if* direction, we notice that if $r = 4$ and $g < 3$, or $r = 2$ and $g < 5$, then $\dim \mathcal{R}_{g,r} > \mathcal{A}^{\delta}_{g-1+r/2}$. □

2.2 Quadrics vanishing on the semi-canonical model

We start with the following definition.

Definition 2.4. Let (C, η, R) be a point of $\mathcal{R}_{g,r}$. The projective map

$$f_\eta : C \to \mathbb{P}H^0(C, \omega_C \otimes \eta)^*$$

defined by the linear system $|\omega_C \otimes \eta|$ is called the *semi-canonical map*, and $C_\eta := f_\eta(C)$ is the *semi-canonical model* of C (cf. [Deb89]).

In this subsection we consider a point $(C, \eta, R) \in \mathcal{R}_{g,r}$ that satisfies the hypotheses of Proposition 2.2. In particular, $\omega_C \otimes \eta$ is very ample and f_η is an embedding. We recall that the co-differential of the Prym map at (C, η, R) is the surjective map

$$d\mathcal{P}^* : \mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2 \otimes \mathcal{O}_C(R)).$$

As we have remarked in (ii) of § 1.2, $I_2(C) := \ker d\mathcal{P}^*$ is the space of homogeneous quadratic polynomials vanishing on C_η; thus the image of $d\mathcal{P}$ in $T_{\mathcal{P}(C,\eta,R)}$ determines the projective space $\mathbb{P}I_2(C_\eta)$ of the quadrics containing C_η (see (2) in § 1). The aim is to recover the curve C_η in the intersection of the quadrics. We remark that, by a dimensional count, in the cases

$$r = 6 \text{ and } g = 2, \quad r = 4 \text{ and } g = 4, \quad r = 2 \text{ and } g = 5$$

this is not possible.

We recall that there is a natural bijective correspondence between the points of $\mathbb{P}H^0(C, \omega_C \otimes \eta)^*$ lying in the intersection of the quadrics of $\mathbb{P}I_2(C_\eta)$ and the extensions (up to isomorphism and multiplication by a scalar)

$$0 \to \eta^{-1} \to F \to \omega_C \otimes \eta \to 0$$

with connecting homomorphism $\delta : H^0(C, \omega_C \otimes \eta) \to H^1(C, \eta^{-1})$ of rank 1 (see [LS96, Lemma 1.2]). Given $p \in \mathbb{P}H^0(C, \omega_C \otimes \eta)^*$ in the intersection of the quadrics, the image of the corresponding δ in $H^1(C, \eta^{-1}) = H^0(C, \omega_C \otimes \eta)^*$ is the one-dimensional vector space defining p.

In the proof of Theorem 2.8 we will classify all extensions of the previous type. In order to do this, we will need the following technical results on the number of points of order 2 lying on the theta divisor of a Jacobian. For a more exhaustive analysis of this topic we refer to [Mar11].

Proposition 2.5. *Let C be a general curve of genus g, then the following hold.*

(i) *Let Θ be a symmetric theta divisor in $J(C)$. For each $a \in A$ there exists a point of order 2 not contained in $t_a^*\Theta$.*

(ii) *Let $M_1, \ldots, M_N$ be a finite number of line bundles of degree $d \leqslant g-1$ on C. Given an integer $k \leqslant g-1-d$, if η is a general line bundle of degree k such that $h^0(\eta^2) > 0$, then*

$$h^0(\eta \otimes M_i) = 0 \quad \text{for } i = 1, \ldots, N.$$

Proof. Statement (i) can be proved by reducing $J(C)$ to the product of elliptic curves. In this case, $t_a^*\Theta$ contains at most $2^{2g}-3^g$ points of order 2.

We will prove statement (ii) for $N=1$. The general case follows immediately.

Let $\Lambda := \{\eta \in \mathrm{Pic}^k(C) : h^0(\eta^2)>0\}$; we prove that $\Lambda' := \{\eta \in \Lambda : h^0(M_1 \otimes \eta) > 0\}$ is a proper subset. Specifically, we claim that given an effective $L \in \mathrm{Pic}^{2k}(C)$, there exists a line bundle $\eta \in \Lambda \backslash \Lambda'$ such that $\eta^2 \simeq L$. Setting $n := g-1-k-d$ and considering the line bundle $M_1^2 \otimes L \otimes \mathcal{O}_C(p)^{2n}$ where p is an arbitrary point of C, we can assume $M_1 \simeq \mathcal{O}_C$ and $k=g-1$. Then the result follows from (i). □

Remark 2.6. Statement (i) of Proposition 2.5 can be further improved. Specifically, let A be a principally polarized abelian variety of dimension g and let Θ be its symmetric theta divisor. For each $a \in A$ there are at most $2^{2g}-2^g$ points of order 2 lying on $t_a^*\Theta$ (see [Mar11, Proposition 2.21]). If Θ is irreducible and $t_a^*\Theta$ is not symmetric with respect to the origin, the statement also holds for $2^{2g}-(g+1)2^g$ points. One might expect the right bound to be $2^{2g}-3^g$, as in the product of elliptic curves.

In the proof of Theorem 2.8 we will also use the following lemma (see, for instance, [Bea83, Lemma X.7]).

LEMMA 2.7. *Let C be a smooth projective curve and let F be a vector bundle of rank 2 on C. If*

$$2h^0(F)-3 > h^0(\det F), \tag{4}$$

then there exists a line bundle $L \subset F$ such that $h^0(L) \geqslant 2$ and F/L is a line bundle.

THEOREM 2.8. *Let $(C,\eta,R) \in \mathcal{R}_{g,r}$ and let $\mathbb{P}I_2(C_\eta)$ be the space of quadrics vanishing on the semi-canonical model C_η of C.*

(i) *Suppose that one of the following holds:*

- *$r \geqslant 8$ and $g \geqslant 1$;*
- *$r=6$, C is not hyperelliptic and $h^0(\eta)=0$;*
- *$r=4$, $\mathrm{Cliff}(C)>1$ and $h^0(\eta \otimes \mathcal{O}_C(p))=0$ for each $p \in C$;*
- *$r=2$, $\mathrm{Cliff}(C)>2$ and $h^0(\eta \otimes \mathcal{O}_C(p+q))=0$ for any $p,q \in C$.*

Then the intersection of the quadrics of $\mathbb{P}I_2(C_\eta)$ is, set-theoretically, C_η.

(ii) *If $r=6$, $g=3$, C is not hyperelliptic and $h^0(\eta)=1$, then the intersection of the quadrics of $\mathbb{P}I_2(C_\eta)$ consists of the curve C_η and a line (note that C_η is not a plane curve).*

(iii) *If $r=2$, $g=6$ and (C,η,R) is a general point of $\mathcal{R}_{g,r}$, then the intersection of the quadrics of $\mathbb{P}I_2(C_\eta)$ consists of the curve C_η and five projective lines.*

Proof. In view of the previous discussion, we want to classify the extensions of type

$$0 \to \eta^{-1} \to F \to \omega_C \otimes \eta \to 0 \tag{5}$$

with connecting homomorphism of rank 1. Since, by hypothesis, $g \geqslant 7-r$, we can apply Lemma 2.7: F has a sub-line bundle L such that $h^0(L) \geqslant 2$ and F/L is a line bundle. Fixing such a sub-line bundle, the situation is summarized in the following diagram, in which (5) appears as

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the horizontal sequence.

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & L \arrow[d] \arrow[dr, dashed, "\tau"] & & \\ 0 \arrow[r] & \eta^{-1} \arrow[r] & F \arrow[r] \arrow[d] & \omega_C \otimes \eta \arrow[r] & 0 \\ & & \omega_C \otimes L^{-1} \arrow[d] & & \\ & & 0 & & \end{tikzcd} \tag{6}$$

We note that, since the map τ in diagram (6) is not zero, we must have

$$h^0(\omega_C \otimes \eta \otimes L^{-1}) > 0. \tag{7}$$

If L is not special, $h^0(L) = h^0(F) = h^0(\omega_C \otimes \eta) - 1$ and so $L = \omega_C \otimes \eta \otimes \mathcal{O}_C(-p)$ for some $p \in C$. By diagram (6) we get

$$h^0(L) + h^0(\omega_C \otimes L^{-1}) \geqslant h^0(F),$$
$$h^0(L) - h^0(\omega_C \otimes L^{-1}) = \deg L + 1 - g.$$

Thus, when L is special, we obtain

$$3 - \frac{r}{2} \geqslant \mathrm{Cliff}(L) \geqslant 0. \tag{8}$$

Proof of (i). In the hypotheses of (i), equations (7) and (8) are never simultaneously satisfied. Thus L is not special and C_η is, set-theoretically, the intersection of quadrics.

Proof of (ii). By (8) and (7), we get $L = \omega_C$ and $F = \omega_C \oplus \mathcal{O}_C$. Furthermore, by hypothesis, there is only one non-zero morphism $\omega_C \to \omega_C \otimes \eta$. Therefore there is a two-dimensional family of extensions of type

$$0 \to \eta^{-1} \to \omega_C \oplus \mathcal{O}_C \to \omega_C \otimes \eta \to 0$$

corresponding to the points of the line determined by

$$H^0(\omega_C) \subset H^0(\omega_C \otimes \eta).$$

Proof of (iii). The proof is long and we will divide it into four steps. First, recall that on a general curve C of genus 6 there are exactly five non-isomorphic line bundles $M_1, \ldots, M_5$ of degree 4 such that $h^0(M_i) = 2$ (see [ACGH85, ch. 5]).

Step I: *$L \simeq M_i$ for some i.*

By (8) we have $\mathrm{Cliff}(L) = 2$. Since C is a general curve of genus 6, either $\deg L = 4$ and $h^0(L) = 2$, or $\deg L = 6$ and $h^0(L) = 3$, or $\deg L = 8$ and $h^0(L) = 4$. In the last case, $L \simeq \omega_C \otimes \mathcal{O}_C(-p - q)$ for some $p, q \in C$, but this contradicts (iv) of Lemma 2.1. It follows that either $L \simeq M_i$ or $L \simeq \omega_C \otimes M_i^{-1}$ for some $i = 1, \ldots, 5$. By (ii) of Proposition 2.5,

$$h^0(M_i \otimes \eta) = 0 \quad \text{for } i = 1, \ldots, 5, \tag{9}$$

for a general η. Then there are no non-zero maps from $\omega_C \otimes M_i^{-1}$ to $\omega_C \otimes \eta$, and we can assume $L \simeq M_i$.

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Step II: *the vector bundle F is determined by $M_i \subset F$.*
Notice that

$$h^0(M_i) + h^0(\omega_C \otimes M_i^{-1}) = 5 = h^0(F).$$

It follows that the vertical exact sequence in diagram (6) (see Proposition 2.2), namely

$$0 \to M_i \to F \to \omega_C \otimes M_i^{-1} \to 0, \tag{10}$$

is exact on the global sections. Furthermore, by (9), it cannot split. The dimension of the space of extensions of this type is the dimension of the co-kernel of the map

$$m : \operatorname{Sym}^2 H^0(C, \omega_C \otimes M_i^{-1}) \to H^0(C, \omega_C^2 \otimes M_i^{-2}).$$

The image of C under the morphism associated to $|\omega_C \otimes M_i^{-1}|$ is a plane curve of degree 6. Therefore m is injective and $\dim H^0(C, \omega_C^2 \otimes M_i^{-2}) - \operatorname{rk} m = 1$.

Step III: *the intersection of the quadrics of $\mathbb{P}I_2(C_\eta)$ is, set-theoretically, contained in the union of C_η with five projective lines.*
Dualizing and tensoring (10) with $\omega_C \otimes \eta$, we get

$$0 \to M_i \otimes \eta \to F^* \otimes \omega_C \otimes \eta \to \omega_C \otimes M_i^{-1} \otimes \eta \to 0.$$

By (9), $h^1(C, M_i \otimes \eta) = 0$ and

$$h^0(C, F^* \otimes \omega_C \otimes \eta) = h^0(C, \omega_C \otimes M_i^{-1} \otimes \eta) = 7 + 1 - 6 + h^0(M_i \otimes \eta^{-1}) = 2.$$

It follows that there is a two-dimensional family of maps $F \to \omega_C \otimes \eta$.

Step IV: *the intersection of the quadrics of $\mathbb{P}I_2(C_\eta)$ coincides, as a scheme, with the union of C_η with five projective lines.*
We recall that

$$\dim I_2(C_\eta) = 4 \quad \text{and} \quad \dim \mathbb{P}H^0(C, \omega_C \otimes \eta) = 5.$$

Thus the intersection of the quadrics is complete and, consequently, Cohen–Macaulay [Har77, ch. II, Proposition 8.23]. It follows that it has degree 16 (see [Har77, ch. I, Theorem 7.7]) and has no embedded components [Mat80, ch. 6, § 16, Theorem 30]. Since the degree of C_η is 11, this concludes the proof. □

Remark 2.9. Cases (i) and (ii) of Theorem 2.8 can be also deduced from [Sai72, § 1, Proposition] and [LS96, Theorem 1.3].

COROLLARY 2.10. *Let (C, η, R) be a general point of $\mathcal{R}_{g,r}$. If one of the conditions*

$$r > 6 \text{ and } g \geqslant 1, \quad r = 6 \text{ and } g \geqslant 3, \quad r = 4 \text{ and } g \geqslant 5, \quad r = 2 \text{ and } g \geqslant 6$$

holds, then the image of the differential of the Prym map at (C, η, R) determines the pair (C, η).

Proof. Lemma 2.1 shows that the statement of Theorem 2.8 is realized for a general point (C, η, R). □

3. The generic Torelli theorem

3.1 The Prym map at the boundary

In the following, we assume C to be a general curve of genus $1 < g < r$, with $r \geqslant 6$. Set

$$\Upsilon := \{(\eta, R) \in \operatorname{Pic}^{r/2}(C) \times C_r : R \in |\eta^2|\} \tag{11}$$

and consider the partition

$$\Upsilon = \bigsqcup_{k=1}^{r} Y_k,$$

where

$$Y_k := \left\{ \left(\eta, \sum_i n_i y_i \right) \in \Upsilon : \sum_i (n_i - 1) = k - 1 \right\}. \tag{12}$$

We remark that Υ is an étale 2^{2g}-covering of the symmetric product C_r of C and, in particular, Y_1 is an étale covering of the open set of divisors with no multiple points. The rational map

$$\begin{aligned} \mathcal{T} : \Upsilon &\dashrightarrow \mathcal{R}_{g,r} \\ (\eta, R) &\mapsto (C, \eta, R) \end{aligned} \tag{13}$$

is clearly defined over Y_1, and for $g > 2$ it is an isomorphism there. If $g = 2$, the map $\mathcal{T}|_{Y_1}$ has degree 2; specifically, if $i : C \to C$ is the hyperelliptic involution, then

$$\mathcal{T}^{-1}(C, \eta, R) = \{(\eta, R), (i^*\eta, i(R))\}. \tag{14}$$

Let

$$\mathcal{P}' : \Upsilon \dashrightarrow \mathcal{A}^{\delta}_{g-1+r/2} \tag{15}$$

be the composition (regular over Y_1) of $\mathcal{T}$ with the Prym map. In this subsection we extend the rational map $\mathcal{P}'$.

Let (η, R) be a point of Y_k, and let $\varphi : \Delta \to \Upsilon$ be a non-constant map from the complex unit disk to Υ such that $\varphi(0) = (\eta, R)$ and $\varphi(\Delta \backslash \{0\}) \subset Y_1$. By pullback, up to a finite base change, we have a map of families of curves

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "\Gamma"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{16}$$

with the following properties:

- for $t \neq 0$, the curve $\mathcal{C}_t$ is isomorphic to C and $\Gamma_t : \mathcal{D}_t \to \mathcal{C}_t$ is the double covering of smooth projective curves corresponding to $(C, \varphi(t)) \in \mathcal{R}_{g,r}$;
- $\mathcal{D}_0 \to \mathcal{C}_0$ is a double admissible covering of semi-stable curves (see [HM82] and [HM98, ch. 3, § G]).

The family of coverings in (16) determines a family of semi-abelian varieties

$$\mathcal{P}/\Delta \tag{17}$$

where for each $t \in \Delta \backslash \{0\}$, $\mathcal{P}_t$ is the Prym variety of the double covering $\mathcal{D}_t \to \mathcal{C}_t$, while $\mathcal{P}_0$ is the kernel of the morphism of semi-abelian varieties

$$J(\mathcal{D}_0) \to J(\mathcal{C}_0).$$

Proposition 3.1. *Let $\varphi : \Delta \to \Upsilon$ be a non-constant map from the complex unit disk to Υ such that $\varphi(0) = (\eta, R) \in Y_k$ and $\varphi(\Delta \backslash \{0\}) \subset Y_1$. Consider the family of semi-abelian varieties $\mathcal{P}/\Delta$ defined as in (17).*

(i) *If $k = 2$, the semi-abelian variety $\mathcal{P}_0$ has rank 1 and its compact part is irreducible. Furthermore, it is uniquely determined by the point (η, R).*

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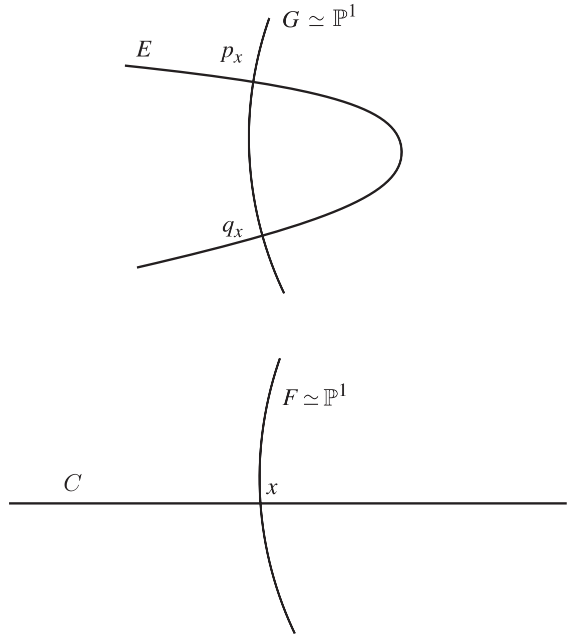

Figure 1. Admissible double covering ($k = 2$).

(ii) *If $k = 3$, $\mathcal{P}_0$ is either a trivial extension of rank 1, or a product of an abelian variety and an elliptic curve, or a semi-abelian variety of rank 2.*

(iii) *If $k > 3$, $\mathcal{P}_0$ is either a semi-abelian variety of rank 1 with reducible compact part or a semi-abelian variety of rank greater than 1.*

Proof. In the following, we want to describe the possible admissible coverings $\mathcal{D}_0 \to \mathcal{C}_0$. Roughly speaking, the covering is obtained as the limit of a smooth double covering of C when two or more branch points come together. If, for example, $R = R' + kx$, then k branch points on C collapse on the point x. In order to get the stable limit of the double covering, we have to attach a rational curve to C in x. More generally, if $(\eta, R) \in Y_k$, we have $\mathcal{C}_0 \simeq C \cup F$, where F is a union of (possibly singular and reducible) rational curves intersecting C in the points of the support of R of multiplicity greater than 1. The curve $\mathcal{D}_0$ is isomorphic to $E \cup G$, where E is a smooth double covering of C and G is a union of (possibly singular and reducible) curves mapping two to one on F (see [HM82, HM98]).

We want to describe the admissible covering $\mathcal{D}_0 \to \mathcal{C}_0$ and the semi-abelian variety $\mathcal{P}_0$ for $k = 2, 3$ and $k > 3$.

(i) If $k = 2$, then $R = R' + 2x$ (see Figure 1). The double covering $\pi : E \to C$ has $r - 2 > 0$ branch points. Since x is not a branch point for this covering, G intersects E in two different points p_x and q_x such that $\pi^{-1}(x) = \{p_x, q_x\}$. We can conclude that $G \simeq \mathbb{P}^1$ and $F \simeq \mathbb{P}^1$. The semi-abelian variety $\mathcal{P}_0$ has rank 1 and is the extension of the Prym variety P of π determined by $\pm[p_x - q_x] \in \mathcal{K}^0(P)$.

(ii) If $(\eta, R) \in Y_3$, then either $R = R' + 2x + 2y$ or $R = R' + 3x$. In the first case, the double covering $\pi : E \to C$ has $r - 4 > 0$ branch points and so $g(E) = 2g - 3 + r/2$. It follows that $\mathcal{P}_0$ is a semi-abelian variety of rank 2.

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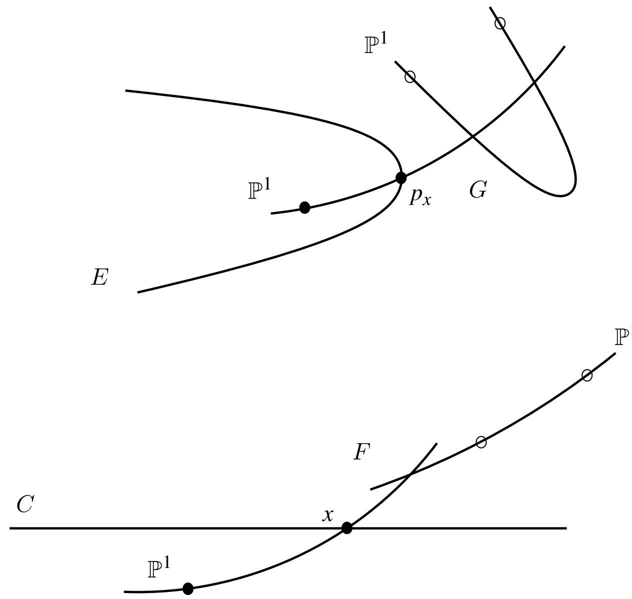

Figure 2. Admissible double covering ($k=3$).

In the second case, $\pi : E \to C$ has $r-2$ branch points and x is a branch point for this covering. Therefore G intersects E in the point $p_x = \pi^{-1}(x)$ and, consequently, the arithmetic genus of E is 1. There are two possibilities: E is either a rational nodal curve (see Figure 2) or a smooth elliptic curve. Then $\mathcal{P}_0$ either is a trivial extension of the Prym variety P of π by an algebraic torus of rank 1, or it is the product of P by an elliptic curve.

(iii) If $(\eta, R) \in Y_k$ and $k > 3$, the situation is more complicated and, as in the $k=3$ case, there are different possible limits. We observe only that E has genus strictly lower than $2g-2+r/2$. It follows that either the compact part of $\mathcal{P}_0$ is reducible or $\mathcal{P}_0$ is a semi-abelian variety of rank greater than 1. $\square$

Proposition 3.1 suggests that, in order to extend $\mathcal{P}'$ (see (15)), we need to compactify the moduli space of abelian varieties $\mathcal{A}^{\delta}_{g-1+r/2}$. We consider the normalized blowing up of the Satake compactification of $\mathcal{A}^{\delta}_{g-1+r/2}$ and denote it by $\bar{\mathcal{A}}^{\delta}_{g-1+r/2}$ (see [Gru09, Nam76a, Nam76b, Nam80]). It is a projective variety and its boundary points parametrize polarized semi-abelian varieties.

From Proposition 3.1 and its proof we have the following corollary.

Corollary 3.2. *It is possible to extend $\mathcal{P}'$ to a rational map*

$$\mathcal{S} : \Upsilon \dashrightarrow \bar{\mathcal{A}}^{\delta}_{g-1+r/2}$$

such that:

(i) *the indeterminacy locus of $\mathcal{S}$ is contained in $\bigsqcup_{k\geqslant 3} Y_k$;*

(ii) *$\mathcal{S}(\eta, R) = \mathcal{P}(C, \eta, R)$ for $(\eta, R) \in Y_1$;*

(iii) *given $z = (\eta' \otimes \mathcal{O}_C(x), R' + 2x)$, if $\pi : E \to C$ denotes the double covering associated to (C, η', R'), then $\mathcal{S}(z)$ is described by the following data:*

- *the compact Prym variety P of π;*
- *the class $\pm[p_x - q_x] \in \mathcal{K}(P)$, where $\pi^{-1}(x) = \{p_x, q_x\}$.*

The next section is devoted to proving the following proposition. This is the last step in completing the proof of Theorem 1.1.

Proposition 3.3. *Let C be a general curve of genus $1 < g < r$ and assume $r \geqslant 6$. The rational map*

$$\mathcal{S} : \Upsilon \dashrightarrow \bar{\mathcal{A}}^{\delta}_{g-1+r/2}$$

has generically degree 1 for $g > 2$ and degree 2 for $g = 2$.

3.2 Proof of Proposition 3.3

We want to apply to $\mathcal{S}$ an abstract lemma on the degree of a proper morphism.

Lemma 3.4. *Let $f : X \to Z$ be a generically finite, surjective, proper morphism of varieties over an algebraically closed field $\Bbbk$. Consider an integral, locally closed subset Y of X of codimension 1 that is not contained in the singular locus of X, and set $H := f(Y) \cap f(Y^c)$. Assume that:*

(i) *the codimension of the closure $\bar{H}$ of H in Z is at least 2;*

(ii) *the differential df is injective in a non-empty open set of Y;*

(iii) *there is a non-empty open set V of $f(Y)$ such that $f^{-1}(y)$ has cardinality n for each $y \in V$.*

Then there is a non-empty open set U of Z such that $f^{-1}(y)$ has cardinality $m \leqslant n$ for each $y \in U$.

Proof. We can prove the statement when Y is a closed subset of X. Specifically, given Y locally closed, conditions (ii) and (iii) clearly hold also for $\bar{Y}$. In order to prove (i), notice that

$$f(\bar{Y}) \cap f(\bar{Y}^c) \subset H \cup (f(\bar{Y} \backslash Y) \cap f(Y^c)) \subset H \cup f(\bar{Y} \backslash Y).$$

It follows that

$$\operatorname{codim}(\overline{f(\bar{Y}) \cap f(\bar{Y}^c)}) \geqslant \min\{\operatorname{codim} \bar{H}, \operatorname{codim} f(\bar{Y} \backslash Y)\} \geqslant 2.$$

Observe that, up to restriction, we can assume X to be smooth. We claim that we can also assume f to be a finite morphism. Let $W \subset Z$ be the maximal open subset of Z such that $f^{-1}(W) \to W$ is a proper quasi-finite morphism. By [Gro67, Theorem 8.11.1], $f|_{f^{-1}(W)}$ is a finite morphism. We show that $f(Y)$ is not contained in the complement of W and so, by (i), $Y \cap f^{-1}(W) \neq \emptyset$. This implies that all the hypotheses still hold when we replace X with $f^{-1}(W)$, Z with W and Y with $Y \cap f^{-1}(W)$. Assume, for contradiction, that $f(Y) \subset W^c$; then, by (ii), $f(Y)$ is an irreducible component of W^c. Let us consider the union of the closure of $W^c \backslash f(Y)$ and $f(Y) \backslash V$. The complement of this set is open and it has the same property as W, thus we get a contradiction.

Let $\widetilde{Z}$ be the normalization of Z. By the universal property of normalization, we can factorize f as

$$X \xrightarrow{g} \widetilde{Z} \xrightarrow{\pi} Z$$

where π and g are finite morphisms. Furthermore, there is an open set T of Z such that T^c has codimension at least 2 and $\widetilde{T} := \pi^{-1}(T)$ is smooth. This implies that

$$g : g^{-1}(\widetilde{T}) \to \widetilde{T}$$

is a finite flat morphism. We recall that $f(Y)$ has codimension 1 in Z; then $Y' := Y \cap g^{-1}(\widetilde{T})$ is a non-empty set. By (ii), Y' is not contained in the ramification locus of g and, by (i), we can

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assume that $g^{-1}(g(Y')) = Y'$. By (iii), we can conclude that the degree of g is at most n. Since π is birational, the statement is proved. □

Solving the indeterminacy locus (by a suitable blow up) of the rational map

$$\mathcal{S} : \Upsilon \dashrightarrow \bar{\mathcal{A}}^{\delta}_{g-1+r/2},$$

we obtain a proper map

$$\mathcal{Q} : \mathcal{B}(\Upsilon) \to \mathcal{Q}(\mathcal{B}(\Upsilon)) \subset \bar{\mathcal{A}}^{\delta}_{g-1+r/2}$$

where $\mathcal{B}(\Upsilon)$ has a projection

$$p : \mathcal{B}(\Upsilon) \to \Upsilon.$$

Set $X := \mathcal{B}(\Upsilon)$, $Z := \mathcal{Q}(\mathcal{B}(\Upsilon))$, $f := \mathcal{Q}$ and $Y := p^{-1}(Y_2)$, where we recall that

$$Y_2 := \left\{ \left(\eta, \sum_i y_i + 2x \right) \in \Upsilon \right\}.$$

By Corollary 2.3 the morphism $\mathcal{Q}$ is generically finite. Moreover, when $g = 2$, $\deg \mathcal{Q} \geqslant 2$ (see (14)). Proposition 3.3 will follow from a direct application of Lemma 3.4, once we have shown that conditions (i), (ii) and (iii) are fulfilled (with $n = 1$ when $g > 2$ and $n = 2$ when $g = 2$).

By Proposition 3.1, the points in $\mathcal{Q}(Y) \cap \mathcal{Q}(Y^c)$ parametrize semi-abelian varieties of rank 1 that are trivial extensions. On the other hand, by Corollary 3.2, associated to the general point of $\overline{\mathcal{Q}(Y)}$ is an irreducible semi-abelian variety of rank 1. Therefore $\overline{\mathcal{Q}(Y) \cap \mathcal{Q}(Y^c)}$ is a proper closed subset of $\overline{\mathcal{Q}(Y)}$ and condition (i) is satisfied. Statements (ii) and (iii) will follow, respectively, from Lemmas 3.5 and 3.7 (or Lemma 3.8 if $g = 2$) below. We recall that, by Corollary 3.2, the indeterminacy locus of $\mathcal{S}$ is contained in $\bigsqcup_{k \geqslant 3} Y_k$. Therefore $\mathcal{Q}|_Y$ coincides with $\mathcal{S}|_{Y_2}$.

In the following lemma, we adapt to our case the proof of the infinitesimal Torelli theorem for curves at the boundary (see, for instance, [Usu91]).

Lemma 3.5. *The differential of the map $\mathcal{S}$ at a general point $z \in Y_2$ is injective.*

Proof. Let z be a general point in Y_2, and write $z := (\eta, R)$, $R := R' + 2x$ and $\eta := \eta' \otimes \mathcal{O}_C(x)$. Moving $x \in C$, we define a one-dimensional subvariety of Y_2,

$$W := \{(\eta' \otimes \mathcal{O}_C(y), R' + 2y) : y \in C \backslash \operatorname{supp} R'\}.$$

Clearly, W is birational to C. The inclusions $W \subset Y_2 \subset \Upsilon$ induce a filtration

$$T''_z \subset T'_z \subset T_z, \tag{18}$$

where T_z is the tangent space of Υ at the point z, T'_z is the tangent space of Y_2 at z, and T''_z is the tangent space of W at z. We recall (see (11)) that Υ is an étale covering of the symmetric product C_r and so we can identify T_z with the tangent space of C_r at R. Note that, under this identification, T'_z is the tangent space of the diagonal of C_r passing through R and, consequently, we can identify it with the tangent space of C_{r-1} at the point $R' + x$. We have two exact sequences

$$0 \to T'_z \to T_z \to N \to 0,$$
$$0 \to T''_z \to T'_z \to N' \to 0,$$

where N' corresponds to the tangent space of C_{r-2} at the point R'.

The next step is to describe the tangent space at $\mathcal{S}(z)$. We can write the period matrix $\bar{M}$ corresponding to the point $\mathcal{S}(z)$ in the form

$$\bar{M} = \begin{pmatrix} 0 & w^t \\ w & M \end{pmatrix}$$

where M is the period matrix of $P := \mathcal{P}(C, \eta', R')$ and, with the notation of Corollary 3.2(iii), the vector w represents essentially the class of $\pm[p_x - q_x] \in \mathcal{K}(P)$. We consider now a filtration

$$T''_{\mathcal{S}(z)} \subset T'_{\mathcal{S}(z)} \subset T_{\mathcal{S}(z)},$$

analogous to that in (18), where $T'_{\mathcal{S}(z)}$ is the tangent space of $\bar{\mathcal{A}}_{g-1+r/2} \backslash \mathcal{A}_{g-1+r/2}$ at $\mathcal{S}(z)$, i.e. the space of infinitesimal deformations of the matrix $\bar{M}$ such that the first entry is always zero. We define $T''_{\mathcal{S}(z)}$ as the space of infinitesimal deformations of $\bar{M}$ such that M is constant, the first entry of $\bar{M}$ is zero and w varies. We again have two exact sequences

$$0 \to T'_{\mathcal{S}(z)} \to T_{\mathcal{S}(z)} \to N_{\mathcal{S}(z)} \to 0,$$
$$0 \to T''_{\mathcal{S}(z)} \to T'_{\mathcal{S}(z)} \to N'_{\mathcal{S}(z)} \to 0,$$

where $N_{\mathcal{S}(z)}$ is the normal space of $\mathcal{A}_{g-1+r/2}$ in $\bar{\mathcal{A}}_{g-1+r/2}$ and $N'_{\mathcal{S}(z)}$ is the space of infinitesimal deformations of the compact Prym variety P.

The differential $d\mathcal{S}$ preserves the filtration and defines the diagram

$$\begin{tikzcd}
0 \arrow[r] & T''_z \arrow[r] \arrow[d, "d\mathcal{S}|_{T''_z}"] & T'_z \arrow[r] \arrow[d, "d\mathcal{S}|_{T'_z}"] & N'_z \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & T''_{\mathcal{S}(z)} \arrow[r] & T'_{\mathcal{S}(z)} \arrow[r] & N'_{\mathcal{S}(z)} \arrow[r] & 0
\end{tikzcd}$$

where d is the map that makes the diagram commutative. We identify d with the differential of the Prym map $\mathcal{R}_{g,r-2} \to \mathcal{A}^{\delta}_{g-2+r/2}$ at a general point (C, η', R'). By Proposition 2.2, d is injective. The restriction

$$d\mathcal{S}|_{T''_z} : T''_z \to T''_{\mathcal{S}(z)}$$

is the differential of the map

$$\begin{aligned} C &\to \mathcal{K}(P) \\ x &\mapsto \pm[p_x - q_x] \end{aligned}$$

at the point x. Since the map is finite on the image and x is a general point of C, $d\mathcal{S}(T''_z) \neq 0$. We can conclude that $d\mathcal{S}|_{T'_z}$ is injective.

To complete the proof, let $0 \neq v \in T_z \backslash T'_z$; we claim that $d\mathcal{S}(v) \neq 0$. Since $\dim T_z = \dim T'_z + 1$, this implies $\ker d\mathcal{S} \subset T'_z$. Consider the complex unit disk Δ, a non-constant map $\varphi : \Delta \to \Upsilon$ such that $\varphi(0) = z$ and $d\varphi(d/dt) = v$, and the family of coverings (16) described in § 3.1. We define the map

$$\tau : \Delta \to \bar{\mathcal{M}}_{2g-2+r/2}$$

which sends t to the class of isomorphisms of $\mathcal{D}_t$. Let

$$T : \Delta \to \bar{\mathcal{A}}_{2g-1+r/2}$$

be the composition of τ with the Torelli map. Note that $J(\mathcal{D}_t)$ is isogenous to $J(C) \times \mathcal{P}_t$ for each $t \in \Delta \backslash \{0\}$. Thus, in order to prove that $d\mathcal{S}(v) \neq 0$, it is sufficient to show that $dT/dt|_{t=0} \neq 0$. Since $d\tau_0(v)$ is different from zero, this is a consequence of the infinitesimal Torelli theorem at the boundary [Usu91]. □

In order to compute the degree of $\mathcal{S}$ on Y_2, we need the following lemma.

Lemma 3.6. *Let $\pi : D \to C$ be a double covering of a curve of genus $g \geqslant 1$ with $r > 0$ branch points.*

(i) *If D is not hyperelliptic, then the Abel–Prym curve $D' \subset P$ determines the covering π.*

(ii) *Assume that $\pi : D \to C$ defines a general point in $\mathcal{R}_{g,r}$. The curve D is hyperelliptic (and in particular the Abel–Prym map has degree 1; see (iii) of § 1.2) if and only if $g = 1$ and $r = 2$.*

Proof. (i) Let us consider the natural projection $\sigma : P \to \mathcal{K}(P)$ and set $C' := \sigma(D')$. The morphism $\sigma : D' \to C'$ is a double covering. Passing to the normalization, we get $\pi : D \to C$.

(ii) If $g = 1$ and $r = 2$, then $g(D) = 2$ and D is hyperelliptic. To prove the converse, assume for contradiction that D is hyperelliptic, and denote by $j : D \to D$ the hyperelliptic involution and by $i : D \to D$ the involution associated to π. Since j commutes with i (see, e.g., [FK80, § III.8, Corollary 3]), it induces a non-trivial involution $j' : C \to C$ that is invariant on the branch divisor of π. If $g > 2$, we get a contradiction, since a generic curve has only trivial automorphisms. Otherwise, when $g = 2$ and $r \geqslant 2$ or $g = 1$ and $r \geqslant 4$, it is always possible to find an effective divisor of degree r that is not fixed by any involution. □

Lemma 3.7. *If $g > 2$,*

$$\mathcal{S} : Y_2 \to \mathcal{S}(Y_2) \subset \bar{\mathcal{A}}^{\delta}_{g-1+r/2}$$

is generically injective.

Proof. We recall that, by Corollary 3.2, a point

$$\mathcal{S}(\eta' \otimes \mathcal{O}_C(x), R' + 2x) \in \mathcal{S}(Y_2)$$

is determined by the Prym variety $P = \mathcal{P}(C, \eta', R')$ of dimension $(g - 1 + r/2) - 1$ and by the class $\pm[p_x - q_x] \in \mathcal{K}(P)$. Let P be a general Prym variety and let $V \subset \mathcal{S}(Y_2)$ be the set of points that parametrize semi-abelian varieties of rank 1 with compact part isomorphic to P. To prove the statement, it is sufficient to prove that $\mathcal{S}$ is generically injective on $W := \mathcal{S}^{-1}(V)$.

Let us consider the Prym map

$$\mathcal{P} : \mathcal{R}_{g,r-2} \to \mathcal{A}^{\delta}_{g-2+r/2}.$$

By Corollary 2.3, there are only finitely many points

$$\{(C, \eta^k, R^k) \in \mathcal{R}_{g,r-2}\}_{k=1,\ldots,n}$$

such that $P = \mathcal{P}(C, \eta^k, R^k)$. Denote by $\pi^k : D^k \to C$ the double covering of smooth curves associated to (C, η^k, R^k), and for $x \in C$ set $(\pi^k)^{-1}(x) = \{p^k_x, q^k_x\}$. We have

$$W = \bigcup_{k=1}^{n} W^k,$$

where

$$W^k := \{(\eta^k \otimes \mathcal{O}_C(x), R^k + 2x) \in \Upsilon : x \in C \backslash \operatorname{supp} R^k\}.$$

To prove that $\mathcal{S}|_W$ is generically injective, we show that, for each $i \neq j$, the curve $\mathcal{S}(W^i)$ intersects $\mathcal{S}(W^j)$ only in a finite number of points and that, for each i, the map $\mathcal{S}|_{W^i} : W^i \to \mathcal{S}(W^i)$ is generically injective.

We recall that

$$\mathcal{S}(\eta^i \otimes \mathcal{O}_C(x), R^i + 2x) = \mathcal{S}(\eta^j \otimes \mathcal{O}_C(y), R^j + 2y)$$

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if and only if

$$\pm[p_x^i - q_x^i] = \pm[p_y^j - q_y^j] \in \mathcal{K}(P).$$

It follows that $\overline{\mathcal{S}(W^i)} = \overline{\mathcal{S}(W^j)}$ if and only if the image of the Abel–Prym curve of π^i in $\mathcal{K}(P)$ coincides with that of π^j, i.e. (see Lemma 3.6) if and only if $i = j$. Furthermore, the map $\mathcal{S}|_{W^i}$ is generically injective (see Lemma 3.6(ii)). □

Lemma 3.8. *If $g = 2$,*

$$\mathcal{S} : Y_2 \to \mathcal{S}(Y_2) \subset \bar{\mathcal{A}}^{\delta}_{g-1+r/2}$$

has generically degree 2.

Proof. Let $i : C \to C$ be the hyperelliptic involution. With the same notation used in Lemma 3.7, we have

$$W = \bigcup_{k=1}^{n} W^k \cup \bigcup_{k=1}^{n} W^k_*,$$

where

$$W^k_* := \{(i^*\eta^k \otimes \mathcal{O}_C(x), i(R^k) + 2x) \in \Upsilon : x \in C \backslash \operatorname{supp} i(R^k)\}.$$

Furthermore, for each x,

$$\mathcal{S}(\eta^k \otimes \mathcal{O}_C(x), R^k + 2x) = \mathcal{S}(i^*\eta^k \otimes \mathcal{O}_C(x), i(R^k) + 2i(x)).$$

The rest of the proof is analogous to that of Lemma 3.7. □

3.3 End of the proof of Theorem 1.1

We are now ready to complete the proof of the generic Torelli theorem for Prym varieties of ramified coverings. We denote by

$$\mathcal{U}_C := \{(C, \eta, R) \in \mathcal{R}_{g,r}\} \subset \mathcal{R}_{g,r}$$

the moduli space of double coverings of a fixed curve C with r branch points.

Lemma 3.9. *Let C be a general curve of genus $1 < g < r$, with $r \geqslant 6$. Then the Prym map is generically injective on $\mathcal{U}_C$.*

Proof. We recall that on the open set $Y_1 \subset \Upsilon$ (see (12)), the map $\mathcal{S}$ coincides with the composition of $\mathcal{T} : \Upsilon \dashrightarrow \mathcal{R}_{g,r}$ (see (13)) with the Prym map. Furthermore, $\mathcal{T}$ is an isomorphism for $g > 2$ and a map of degree 2 for $g = 2$. Since $\mathcal{U}_C = \mathcal{T}(Y_1)$, the statement follows from Proposition 3.3. □

Proof of Theorem 1.1. If $r = 4$ and $g = 4$, the theorem is a direct consequence of [NR95, Theorem 7.7]. By Corollary 2.10 the theorem is proved for $g \geqslant r$. Let us assume $g < r$ and consider a general point $y \in \mathcal{P}(\mathcal{R}_{g,r})$. By Corollary 2.3,

$$\mathcal{P}^{-1}(y) = \{(C_i, \eta_i, R_i)\}_{i=1,\dots,n}.$$

Furthermore, since we can assume that y is smooth in the Prym locus, by Corollary 2.10 we have that $C_i = C_j$ for any i, j. Finally, by Lemma 3.9, $n = 1$. □

4. Applications

4.1 The Gauss map of a Prym variety

We recall the definition of the Gauss map. If M is a complex torus of dimension $n + 1$, then the tangent spaces $\{T_x M\}_{x \in M}$ are all naturally identified with $T_0 M \simeq \mathbb{C}^{n+1}$. Let X be an analytic

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subvariety of dimension $k+1$ of X, and denote by $X_{\rm ns}$ the smooth locus of X and by $\mathbb{G}(k,n)$ the Grassmannian of the k-planes in $\mathbb{P}^n$. The *Gauss map* of X in M is the map

$$\begin{aligned} X_{\rm ns} &\to \mathbb{G}(k,n) \\ x &\mapsto \mathbb{P}T_x(X) \subset \mathbb{P}T_x(M) = \mathbb{P}^n. \end{aligned}$$

Note that a Prym variety P is not principally polarized; in fact, if L is the polarization on P, then $h^0(P, \mathcal{O}_P(\Theta_P)) = 2^g$, where g is the genus of C. Thus, given P, there is no natural choice of divisor in the associated linear system. However, we have proved in Theorem 1.1 that a general Prym variety P arises from a unique covering $\pi : D \to C$, so in this case we may assign to P the divisor $\Theta_P := \Theta_{J(D)} \cap P$, which is well-defined up to translation. In analogy with the case of Jacobian varieties of hyperelliptic curves (see [And58]), the pair (P, Θ_P) allows us to determine the branch locus of the covering π. In fact, we have the following result.

PROPOSITION 4.1. *Let $P := \mathcal{P}(C, \eta, R)$ be the Prym variety of $\pi : D \to C$. If we identify the tangent space at any point of P with $H^0(C, \omega_C \otimes \eta)^*$, then the branch locus $\mathfrak{B}$ of the Gauss map*

$$(\Theta_P)_{\rm ns} \to \mathbb{P}H^0(C, \omega_C \otimes \eta)$$

of Θ_P in P is dual to the branch locus of $f_\eta \circ \pi$, where $f_\eta : C_\eta \to \mathbb{P}H^0(C, \omega_C \otimes \eta)^*$ is the semi-canonical map. In particular, if $\omega_C \otimes \eta$ is very ample, $\mathfrak{B}$ consists of r distinct hyperplanes.

Proof. Set $d := 2g - 2 + r/2 = g(D) - 1$ and define

$$\widehat{P} := \pi^{-1}(\omega_C \otimes \eta) \subset \mathrm{Pic}^d(D), \quad \Theta_{\widehat{P}} := \widetilde{\Theta} \cap \widehat{P},$$

where $\widetilde{\Theta}$ is the set of the effective line bundles of degree d on D. Owing to the isomorphism between $J(C)$ and $\mathrm{Pic}^d(C)$ that maps 0 to $\omega_C \otimes \eta$, we can identify $(\widehat{P}, \Theta_{\widehat{P}})$ with (P, Θ_P). Consider the vector spaces

$$H := H^0(D, \omega_D)^*, \quad H^+, \quad H^-$$

where H^+ and H^- are the eigenspaces of 1 and -1 for the involution induced on H by π. The map π^* identifies canonically H^+ and H^- with, respectively, $H^0(C, \omega_C)^*$ and $H^0(C, \omega_C \otimes \eta)^*$ (cf. [Ver01] for the étale case). Thus we have the commutative diagram

$$\begin{tikzcd} D \arrow[r, "f_{\omega_D}"] \arrow[d, "\pi"'] & \mathbb{P}H \arrow[d, dashed, "h^-"] \\ C \arrow[r, "f_\eta"] & \mathbb{P}H^- \end{tikzcd}$$

where f_{ω_D} is the canonical map of D and h^- is the projection of centre $\mathbb{P}H^+$. Notice that H can be identified with the tangent space of $\mathrm{Pic}^d(D)$ at any point and, consequently, H^- is the tangent space to $\widehat{P}$.

Denote by $\mathcal{G}$ the Gauss map of $\Theta_{\widehat{P}}$ in $\widehat{P}$ and by $\mathcal{G}'$ the Gauss map of $\widetilde{\Theta}$ in $\mathrm{Pic}^d(D)$. Given $\sum_{j=1}^d p_j \in (\Theta_{\widehat{P}})_{\rm ns}$, we have $\sum_{j=1}^d p_j \in \widetilde{\Theta}_{\rm ns}$, and $\mathcal{G}(\sum_{j=1}^d p_j)$ is simply the projection of $\mathcal{G}'(\sum_{j=1}^d p_j)$ (the hyperplane spanned by the points $\{f_{\omega_D}(p_j)\}_{j=1}^d$) under h^-. We can conclude that $\mathcal{G}(\sum_{j=1}^d p_j)$ is the hyperplane of $\mathbb{P}H^-$ which intersects the semi-canonical model C_η in the points $\{f_\eta \circ \pi(p_j) = h^- \circ f_{\omega_D}(p_j)\}_{j=1}^d$. $\square$

Remark 4.2. Observe that if $r \geqslant 6$, then $\omega_C \otimes \eta$ is very ample. When this condition is not satisfied, the description of the branch locus $\mathfrak{B}$ is trickier. If, for example, $r = 4$ and $g \geqslant 2$, there are two more possible situations.

(i) If $h^0(\eta) = 1$ and $\eta = p + q$ (respectively, $\eta = 2p$), the semi-canonical model C_η is the nodal curve obtained from C by identifying p and q (respectively, a cuspidal curve). The branch locus $\mathfrak{B}$ is made up of four (respectively, five) hyperplanes counted with multiplicities.

(ii) If $h^0(\eta) = 2$, C is hyperelliptic and the linear system $|\eta|$ defines the g^1_2. Then C_η is a rational normal curve and $\deg f_\eta = 2$. The branch locus $\mathfrak{B}$ is made up of $4g + 8$ hyperplanes counted with multiplicities: there are $2g + 2$ hyperplanes of multiplicity 2, which are dual to the branch points of f_η, and a hyperplane with multiplicity 4 (possibly not distinct from the previous ones), which is dual to the image of the branch points of π in C_η.

Note that in these cases also, it is always possible to recover the branch locus of π from $\mathfrak{B}$.

4.2 Proof of Theorem 1.2.

In this section we prove that a very general Prym variety of dimension at least 4 is not isogenous to a Jacobian variety. For some values of g and r the statement is a simple consequence of the fact that the Prym locus has dimension larger than the moduli space of Jacobian varieties (see Proposition 4.3(i)). In the other cases the moduli count shows that a general Prym variety is not isogenous to a Jacobian of a hyperelliptic curve (see Proposition 4.3(ii)). Using this fact together with degeneration techniques, as in the proof of Theorem 1.1, we can conclude the proof of Theorem 1.2.

Proposition 4.3. *Let P be a general Prym variety in the Prym locus $\mathcal{P}^\delta_{g-1+r/2}$.*

(i) *If*

$$r = 4 \text{ and } g \geqslant 3 \quad \text{or} \quad r = 2 \text{ and } g \geqslant 4,$$

then P is not isogenous to a Jacobian variety.

(ii) *If*

$$r \geqslant 6 \text{ and } g \geqslant 1 \quad \text{or} \quad r = 4 \text{ and } g = 2,$$

then P is not isogenous to a Jacobian of a hyperelliptic curve.

Remark 4.4. Notice that if $r = 6$ and $g = 1$ or $r = 4$ and $g = 2$, then P has dimension 3 and, consequently, is isogenous to a Jacobian.

Proof. We compare the dimension of the Prym locus $\mathcal{P}^\delta_{g-1+r/2}$ with that of the Jacobian locus (respectively, hyperelliptic locus). When the Prym map is generically finite (see Corollary 2.3), the result follows from a count of parameters. Hence we only need to consider the case of $r = 2$ and $g = 4$, or $r = 4$ and $g = 2$. We claim that in these cases the differential of the Prym map is generically surjective, which implies that

$$\dim \mathcal{P}^\delta_{g-1+r/2} = \dim \mathcal{A}^\delta_{g-1+r/2}.$$

To see this, we show that the co-differential is injective. We recall (see § 2.2) that the co-differential of the Prym map

$$d\mathcal{P}^* : \operatorname{Sym}^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2 \otimes \mathcal{O}_C(R))$$

is injective if and only if the semi-canonical model C_η of C is not contained in any quadric. If $r = 4$ and $g = 2$, the statement follows from the fact that C_η is a plane curve of degree 4. In the other case, it is a consequence of the following lemma. □

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Lemma 4.5. *Let C be a non-hyperelliptic curve of genus 4, and let $\eta \in \mathrm{Pic}^1(C)$ be a general line bundle of degree 1 on C such that $h^0(\eta^2) > 0$. Then the image C_η of the semi-canonical map*

$$f_\eta : C \to \mathbb{P}^3$$

does not lie on any quadric.

Proof. We recall that, since C is not hyperelliptic, $\omega_C \simeq L_1 \otimes L_2$ where $\deg L_i = 3$, $h^0(L_i) = 2$ and the line bundles may coincide [ACGH85, ch. 5]. As in Lemma 2.1, one can prove that the semi-canonical map is an embedding. Assume for contradiction that C_η lies on a quadric Q. If Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then by the adjunction formula, since $\deg C_\eta = 7$ we have that the bidegree (d_1, d_2) of C_η satisfies the relations

$$(d_1 - 1)(d_2 - 1) = 4,$$
$$d_1 + d_2 = 7.$$

Thus, either $d_1 = 2$ or $d_2 = 2$, that is, C is hyperelliptic, contradicting the hypothesis. If Q is a cone, let $g : C \to \mathbb{P}^2$ be the composition of f_η with the projection from the vertex of the cone. The image of g is a conic and so g factors as

$$C \to \mathbb{P}^1 \xrightarrow{|\mathcal{O}_{\mathbb{P}^1}(1)|} \mathbb{P}^2,$$

where the first morphism has degree $2 \leqslant d \leqslant 3$ and the second has degree 2. Since C is not hyperelliptic, $\deg g = 6$. It follows that g is the map associated to $|\omega_C \otimes \eta \otimes \mathcal{O}_C(-p)|$ or, equivalently, that $p \in C_\eta$ is the vertex of the cone. By the previous discussion we can conclude that

$$\omega_C \otimes \eta \otimes \mathcal{O}_C(-p) \simeq L_1^2$$

and so, since $\omega_C \simeq L_1 \otimes L_2$,

$$\eta \simeq L_1 \otimes L_2^{-1} \otimes \mathcal{O}_C(p).$$

If $L_1 \simeq L_2$, this implies that η is effective and we get a contradiction. Otherwise, η varies at most in a one-dimensional family. On the other hand, by hypothesis, η depends on two parameters. This yields a contradiction. □

Remark 4.6. We note that the argument of Proposition 4.3 shows that, in the étale case also, a general Prym variety of dimension greater than or equal to 4 is not isogenous to a Jacobian variety.

To complete the proof of Theorem 1.2, we need the following lemma concerning the difference surface (see (vii) in § 1.2).

Lemma 4.7. *Let C be a non-hyperelliptic curve and $n \in \mathbb{N}$ a non-zero integer. Then:*

(i) *$n\Gamma_C$ is birational to $C \times C$;*

(ii) *$n\Gamma'_C$ is birational to the symmetric product C_2 of C.*

In particular, Γ_C is birational to $C \times C$ and Γ'_C is birational to C_2.

Proof. Arguing as in [BP89, Lemma 3.1.1 and Proposition 3.2.1], we can assume $n = 1$. To prove (i), notice that if for the general point $(a, b) \in C \times C$ there exists $(c, d) \in C \times C$ such that $[a - b] = [c - d]$, then C is hyperelliptic. Statement (ii) follows from (i). □

Proof of Theorem 1.2. By Proposition 4.3, we have to consider only the cases where $r \geqslant 8$ and $g \geqslant 1$ and where $r = 6$ and $g \geqslant 2$. We assume, by way of contradiction, that a very general Prym

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variety is isogenous to a Jacobian. Then there exists a map of families

$$\mathcal{J}/\mathcal{U} \xrightarrow{G} \mathcal{P}/\mathcal{U}, \tag{19}$$

where $\mathcal{U}$ is a finite étale covering of a dense open subset of $\mathcal{R}_{g,r}$ and, for each $t \in \mathcal{U}$:

(i) $\mathcal{P}_t$ is a Prym variety of dimension $g-1+r/2$;

(ii) $\mathcal{J}_t$ is the Jacobian of a curve of genus $g-1+r/2$;

(iii) $G_t : \mathcal{J}_t \to \mathcal{P}_t$ is a surjective morphism of abelian varieties.

Step I: *limits of* G *in (19) at the boundary.*

We want to extend the map G to some point on the boundary of Υ (see (11)) associated to an admissible covering, as in Figure 1 on page 1157.

Let $\pi : E \to C$ be a very general double covering of a curve of genus g with $r-2$ branch points, and assume that π is determined by the triple (C, η', R'). For each non-branch point $x \in C$, we consider a family of admissible coverings

$$\mathcal{D}^x/\Delta \to \mathcal{C}^x/\Delta$$

obtained as in § 3.1 (see (16)) by choosing a unit disk centred at the point $(\eta' \otimes \mathcal{O}_C(x), R' + 2x)$. Let us restrict our initial map of families (19) to $\Delta \backslash \{0\}$. Changing base, if necessary, by completion, we obtain a map of families

$$\mathcal{J}^x/\Delta \to \mathcal{P}^x/\Delta.$$

The semi-abelian variety $\mathcal{P}_0^x$ is the kernel of the morphism $J(\mathcal{D}_0^x) \to J(\mathcal{C}_0^x)$, where $J(\mathcal{C}_0^x) = J(C)$, $J(\mathcal{D}_0^x) = J(E_x)$, and E_x is the singular curve obtained from E by identifying p_x and q_x, with $\pi^{-1}(x) = \{p_x, q_x\}$. We denote by P the compact part of $\mathcal{P}_0^x$. The semi-abelian variety $\mathcal{J}_0^x$ is a generalized Jacobian variety of a singular curve H_x, obtained from H by identifying two distinct points a_x and b_x. We denote by $\varphi : J(H) \to P$ the isogeny induced on the compact quotients.

Step II: *comparing the extension classes of* E *and* H.

By varying $x \in C$, we can perform different degenerations of the families in (19). Notice that the compact quotient P of $\mathcal{P}_0^x$ does not depend on x. It follows that the normalization H of H_x and the isogeny $\varphi : J(H) \to P$ are also independent of the chosen degeneration. Thus, for each non-branch point $x \in C$, there exist $a_x, b_x \in H$ such that

$$\varphi^*([p_x - q_x]) = n_x[a_x - b_x], \tag{20}$$

for some n_x different from zero.

Step III: *conclusion.*

Let us consider the diagram

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\sigma"] & \mathrm{Pic}^0(P) \arrow[r, "\varphi^*"] \arrow[d, "\sigma^\circ"] & \mathrm{Pic}^0(J(H)) \arrow[d, "\sigma_H^\circ"] \\ \mathcal{K}(P) \arrow[r, "\psi_{\mathcal{K}}"] & \mathcal{K}^0(P) \arrow[r, "\varphi_{\mathcal{K}}^*"] & \mathcal{K}^0(J(H)) \end{tikzcd}$$

where ψ is the isogeny induced by the polarization, the vertical arrows are the natural projections, and $\psi_{\mathcal{K}}$ and $\varphi^*_{\mathcal{K}}$ are the maps induced, respectively, by ψ and φ^* on the Kummer varieties.

Set

$$E'' = \psi(E') \subset \mathrm{Pic}^0(P),$$
$$C'' = \psi_{\mathcal{K}}(C') = \sigma^{\circ}(E'') \subset \mathcal{K}^0(P),$$

where E' is the Abel–Prym curve of π and C' is its projection in the Kummer variety. Denote by Γ_H the image of the difference map in $\mathrm{Pic}^0(J(H))$ (see (vii) in § 1.2) and by Γ'_H its projection in the Kummer variety. By (20), arguing as in [BP89, § 2], we find $n \neq 0$ such that $\varphi^*(E'') \subseteq n\Gamma_H$. It follows that

$$\varphi^*_{\mathcal{K}}(C'') \subseteq n\Gamma'_H.$$

Since H is not hyperelliptic (Proposition 4.3), by Proposition 4.7 we have that $n\Gamma'_H$ is birational to the symmetric product H_2 of H. By composition, we obtain a non-constant rational map

$$C' \xrightarrow{\varphi_{\mathcal{K}}} C'' \xrightarrow{\varphi^*_{\mathcal{K}}} \varphi^*_{\mathcal{K}}(C'') \hookrightarrow n\Gamma'_H \dashrightarrow H_2 \hookrightarrow J(H).$$

Observe that, since by hypothesis $r \geqslant 6$, we have $g < g - 2 + r/2 = \dim P = \dim J(H)$. Moreover, the geometric genus of C' is at most g. Thus we can conclude that $J(H)$ is not simple. On the other hand, P is very general and so, by [BP02], the Néron–Severi group $\mathrm{NS}(P)$ is isomorphic to $\mathbb{Z}$. It follows that P, and hence $J(H)$, is simple. Thus we get a contradiction. □

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References

And58 A. Andreotti, *On a theorem of Torelli*, Amer. J. Math. **80** (1958), 801–828.

ACGH85 E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves. Volume I*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 267 (Springer, New York, 1985).

BCV95 F. Bardelli, C. Ciliberto and A. Verra, *Curves of minimal genus on a general abelian variety*, Compositio Math. **96** (1995), 115–147.

BP89 F. Bardelli and G. P. Pirola, *Curves of genus g lying on a g-dimensional Jacobian variety*, Invent. Math. **95** (1989), 263–276.

Bea77a A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), 149–196.

Bea77b A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. Éc. Norm. Supér. (4) **10** (1977), 309–391.

Bea83 A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Lecture Note Series, vol. 68 (Cambridge University Press, Cambridge, 1983), translated from the French by R. Barlow, N. I. Shepherd-Barron and M. Reid.

BP02 I. Biswas and K. H. Paranjape, *The Hodge conjecture for general Prym varieties*, J. Algebraic Geom. **11** (2002), 33–39.

CG80 J. Carlson and P. Griffiths, *Infinitesimal variations of Hodge structure and the global Torelli problem*, in *Journées de géometrie algébrique d'Angers (juillet 1979): algebraic geometry Angers 1979* (Sijthoff & Noordhoff, Alphen aan den Rijn, 1980), 51–76.

Cha85 C.-L. Chai, *Compactification of Siegel moduli schemes*, London Mathematical Society Lecture Note Series, vol. 107 (Cambridge University Press, Cambridge, 1985).

Deb89 O. Debarre, *Sur le probleme de Torelli pour les varieties de Prym*, Amer. J. Math. **111** (1989), 111–134.

DR89 A. Del Centina and S. Recillas, *On a property of the Kummer variety and a relation between two moduli spaces of curves*, in *Algebraic geometry and complex analysis (Pátzcuaro, 1987)*, Lecture Notes in Mathematics, vol. 1414 (Springer, Berlin, 1989), 28–50.

Don81 R. Donagi, *The tetragonal construction*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 181–185.

DS81 R. Donagi and R. C. Smith, *The structure of the Prym map*, Acta Math. **146** (1981), 25–102.

FC90 G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 22 (Springer, Berlin, 1990), with an appendix by David Mumford.

FK80 H. M. Farkas and I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics, vol. 71 (Springer, New York, 1980).

FS82 R. Friedman and R. Smith, *The generic Torelli theorem for the Prym map*, Invent. Math. **67** (1982), 473–490.

FS86 R. Friedman and R. Smith, *Degenerations of Prym varieties and intersections of three quadrics*, Invent. Math. **85** (1986), 615–635.

GL85 M. Green and R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1985), 73–90.

GT84 P. Griffiths and L. Tu, *Infinitesimal variation of Hodge structure and the generic global Torelli theorem*, in *Topics in transcendental algebraic geometry (Princeton, NJ, 1981/1982)*, Annals of Mathematics Studies, vol. 106 (Princeton University Press, Princeton, NJ, 1984), 227–237.

Gro67 A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Publ. Math. Inst. Hautes Études Sci. **32** (1967), 5–361.

Gru09 S. Grushevsky, *Geometry of $\mathcal{A}_g$ and its compactifications*, in *Algebraic geometry—Seattle 2005. Part 1*, Proceedings of Symposia in Pure Mathematics, vol. 80 (American Mathematical Society, Providence, RI, 2009), 193–234.

HM98 J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187 (Springer, New York, 1998).

HM82 J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23–88, with an appendix by William Fulton.

Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, New York, 1977).

Kan82 V. I. Kanev, *A global Torelli theorem for Prym varieties at a general point*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 244–268.

Kan04 V. Kanev, *Hurwitz spaces of triple coverings of elliptic curves and moduli spaces of abelian threefolds*, Ann. Mat. Pura Appl. (4) **183** (2004), 333–374.

LO11 H. Lange and A. Ortega, *Prym varieties of cyclic coverings*, Geom. Dedicata **150** (2011), 391–403.

LS96 H. Lange and E. Sernesi, *Quadrics containing a Prym-canonical curve*, J. Algebraic Geom. **5** (1996), 387–399.

Laz89 R. Lazarsfeld, *A sampling of vector bundle techniques in the study of linear series*, in *Lectures on Riemann surfaces (Trieste, 1987)* (World Scientific, Teaneck, NJ, 1989), 500–559.

Mar V. Marcucci, *On the genus of curves in a Jacobian variety.* Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

Mar11 V. Marcucci, *Curves in Jacobian and Prym varieties.* PhD thesis, Università degli Studi di Pavia (2011).

https://doi.org/10.1112/S0010437X12000280 Published online by Cambridge University Press

MN11 V. Marcucci and J. C. Naranjo, *Prym varieties of double coverings of elliptic curves*, Preprint (2011), available at http://arxiv.org/abs/1111.3340.

Mat80 H. Matsumura, *Commutative algebra*, Mathematics Lecture Note Series, vol. 56, second edition (Benjamin Cummings, Reading, MA, 1980).

Mum74 D. Mumford, *Prym varieties. I*, in *Contributions to analysis (a collection of papers dedicated to Lipman Bers)* (Academic Press, New York, 1974), 325–350.

NR95 D. S. Nagaraj and S. Ramanan, *Polarisations of type* $(1, 2, \ldots, 2)$ *on abelian varieties*, Duke Math. J. **80** (1995), 157–194.

Nam76a Y. Namikawa, *A new compactification of the Siegel space and degeneration of Abelian varieties. I*, Math. Ann. **221** (1976), 97–141.

Nam76b Y. Namikawa, *A new compactification of the Siegel space and degeneration of Abelian varieties. II*, Math. Ann. **221** (1976), 201–241.

Nam80 Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Mathematics, vol. 812 (Springer, Berlin, 1980).

Nar92 J. C. Naranjo, *Prym varieties of bi-elliptic curves*, J. Reine Angew. Math. **424** (1992), 47–106.

Nar96 J. C. Naranjo, *The positive-dimensional fibres of the Prym map*, Pacific J. Math. **172** (1996), 223–226.

NP94 J. C. Naranjo and G. P. Pirola, *On the genus of curves in the generic Prym variety*, Indag. Math. (N.S.) **5** (1994), 101–105.

Pir88 G. P. Pirola, *Base number theorem for abelian varieties. An infinitesimal approach*, Math. Ann. **282** (1988), 361–368.

Sai72 B. Saint-Donat, *Sur les équations définissant une courbe algébrique*, C. R. Acad. Sci. Paris Sér. A–B **274** (1972), A324–A327.

Ser88 J.-P. Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics, vol. 117 (Springer, New York, 1988), translated from the French.

SV02 R. Smith and R. Varley, *The Prym Torelli problem: an update and a reformulation as a question in birational geometry*, in *Symposium in honor of C. H. Clemens (Salt Lake City, UT, 2000)*, Contemporary Mathematics, vol. 312 (American Mathematical Society, Providence, RI, 2002), 235–264.

Usu91 S. Usui, *Period maps and their extensions*, Sci. Rep. College Gen. Ed. Osaka Univ. **40** (1991), 21–37.

Ver01 A. Verra, *The degree of the Gauss map for a general Prym theta-divisor*, J. Algebraic Geom. **10** (2001), 219–246.

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