The results in this addendum extend [1, Theorems 1.1 and 8.7].

Let \( h > 0 \) be an integer. We characterize algebraic number fields possessing class number \( h \) in terms of the sequence of rational primes.

Using the notation of [1], let \( k \) be an algebraic number field, let \([k : \mathbb{Q}] = f\), and let \( h(k) \) denote the class number of \( k \). Let \( \mathcal{E} \) be the ring of algebraic integers in \( k \). Then \( \mathcal{E} \) is a ring whose additive group \( \mathcal{E}, + \) is a free Abelian group of finite rank \( f \). For each rational prime \( p \) let \( E(p) = \mathbb{Z} + p\mathcal{E} \). Let \( G(p) \) be a reduced torsion-free rank-\( f \) Abelian group such that End\((G(p)) \cong E(p)\). These groups exist by Butler’s theorem [3, Theorem 1.2.6]. There is a torsion-free reduced group \( \overline{G}(p) \) of rank \( f \) such that \( \overline{G}(p)/G(p) \) is finite, and End\((G(p)) = \mathcal{E}\).

Let \( L(p) = \text{card}(u(\mathcal{E})/u(E(p))) \) where \( u(R) \) is the group of units in the ring \( R \). For an Abelian group \( H \) let \( h(H) \) be the number of isomorphism classes of groups \( L \) that are locally isomorphic to \( H \). (See [3].) Sequences \( s_n \) and \( t_n \) are asymptotically equal if \( \lim_{n \to \infty} s_n/t_n = 1 \).

The main theorem of this paper follows.

**Theorem 1.** Let \( k \) be an algebraic number field, let \([k : \mathbb{Q}] = f\), and let \( h(k) = h \). Then \( \{L(p)h(G(p)) \mid \text{rational primes } p\} \) is asymptotically equal to the sequence \( \{hp^{f-1} \mid \text{rational primes } p\} \).

**Proof.** In addition to the the stated notation we let:

1. \( \hat{m}_p = \text{card}(u(\mathcal{E}/p\mathcal{E})) \);  
2. \( \hat{n}_p = \text{card}(u(E(p)/p\mathcal{E})) \);  
3. \( L(p) = \text{card}(u(\mathcal{E})/u(E(p))) \).

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There are at most finitely many rational primes that ramify in \( k \), so let us avoid those primes. By [2, Theorem 8.4],

\[
L(p)h(G(p)) \frac{\hat{n}_p}{m_p} = h(\overline{G}(p)).
\]  

(1)

Because \( \text{End}(\overline{G}(p)) = \overline{E} \), [2, Corollary 3.2] implies that \( h(\overline{G}(p)) = h(\overline{E}) = h(k) = h \).

Hence

\[
L(p)h(G(p)) \frac{\hat{n}_p}{m_p} = h.
\]  

(2)

Since \( p \) does not ramify in \( k \), there are distinct prime ideals \( I_1, \ldots, I_g \) in \( \overline{E} \) and integers \( f_1, \ldots, f_g \) such that \( \sum_{i=1}^{g} f_i = f \),

\[
p\overline{E} = I_1 \cap \cdots \cap I_g,
\]

and \( [\overline{E}/I_i : \mathbb{Z}/p\mathbb{Z}] = f_i \) for each \( i = 1, \ldots, g \). Then

\[
\overline{E}/p\overline{E} = \overline{E} / I_1 \times \cdots \times \overline{E} / I_g
\]

so that

\[
u(\overline{E}/p\overline{E}) = \nu \left( \frac{\overline{E}}{I_1} \right) \times \cdots \times \nu \left( \frac{\overline{E}}{I_g} \right).
\]

Since \( \overline{E}/I_i \) is a finite field of characteristic \( p \),

\[
\hat{m}_p = (p^{f_1} - 1) \cdots (p^{f_g} - 1).
\]

(3)

Since \( E(p)/p\overline{E} \cong \mathbb{Z}/p\mathbb{Z}, \hat{n}_p = p - 1. \)

Form the polynomial of degree \( f - 1 \),

\[
x^{f-1} + Q_p(x) = \frac{(x^{f_1} - 1) \cdots (x^{f_g} - 1)}{x - 1}.
\]

(4)

The coefficients of \( (x^{f_2} - 1) \cdots (x^{f_g} - 1) \) are multinomial coefficients \( \binom{f-1}{r_1, \ldots, r_t} \) for some partitions \( r_1, \ldots, r_t \) of \( f - 1 \). These coefficients are bounded above by \( (f - 1)! \). The coefficients of \( Q_p(x) \) in (4) are then bounded above by \( f! \). Thus \( Q_p(x) \) has degree \( \leq f - 2 \), and the coefficients of \( Q_p(x) \) are bounded above by \( f! \). Hence

\[
\lim_p \frac{p^{f-1} + Q_p(p)}{p^{f-1}} = 1 + \lim_p \frac{Q_p(p)}{p^{f-1}} = 1.
\]

(5)

Now, \( p^{f-1} + Q_p(p) = \hat{m}_p/\hat{n}_p \) when \( p \) replaces \( x \) in (4), so by (2),

\[
\frac{L(p)h(G(p))}{p^{f-1} + Q_p(p)} = L(p)h(G(p)) \frac{\hat{n}_p}{\hat{m}_p} = h.
\]

(6)
Furthermore,

\[
\frac{L(p)h(G(p))}{p^{f-1}} = \frac{(L(p)h(G(p))/p^{f-1})}{(L(p)h(G(p))/p^{f-1} + Q_p(p))} \cdot \frac{L(p)h(G(p))}{p^{f-1} + Q_p(p)} = \frac{p^{f-1} + Q_p(p)}{p^{f-1}} \cdot h
\]

by (6). Using the limit in (5) we see that

\[
\lim_{p} \frac{L(p)h(G(p))}{hp^{f-1}} = 1.
\]

Therefore, \(\{L(p)h(G(p)) \mid \text{rational primes } p\}\) is asymptotically equal to \(\{hp^{f-1} \mid \text{rational primes } p\}\).

\[\square\]

**Corollary 2.** Let \(k\) be a quadratic number field, and let \(h(k) = h\). Then \(\{L(p)h(G(p)) \mid \text{rational primes } p\}\) is asymptotically equal to the sequence \(\{hp \mid \text{rational primes } p\}\).

**Theorem 3.** Let \(k\) be an algebraic number field and let \(h > 0\) be an integer. The following are equivalent.

1. \(h(k) = h\).
2. The sequence \(\{L(p)h(G(p)) \mid \text{rational primes } p\}\) is asymptotically equal to the sequence \(\{hp^{f-1} \mid \text{rational primes } p\}\).

**Proof.** \(1 \Rightarrow 2\). This is Theorem 1.

\(2 \Rightarrow 1\). The sequence \(\{L(p)h(G(p)) \mid \text{rational primes } p\}\) is asymptotically equal to the sequence \(\{hp^{f-1} \mid \text{rational primes } p\}\) for some integer \(h > 0\). Then by Theorem 1 and part 2,

\[
\lim_{p} \frac{L(p)h(G(p))}{h(k)p^{f-1}} = 1 = \lim_{p} \frac{L(p)h(G(p))}{hp^{f-1}}.
\]

Hence \(h(k) = h\) which completes the proof. \[\square\]

**Corollary 4.** Let \(k\) be a quadratic number field and let \(h > 0\) be an integer. The following are equivalent.

1. \(h(k) = h\).
2. The sequence \(\{L(p)h(G(p)) \mid \text{rational primes } p\}\) is asymptotically equal to the sequence \(\{hp \mid \text{rational primes } p\}\).

**References**


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