Modelling Heat Transfer in Near-Wall Flows

Y. Nagano

Abstract

Recent developments in turbulence models for heat transfer are presented, focusing on near-wall behavior of thermal turbulence in flows with different Prandtl numbers. First, we outline a phenomenological two-equation heat-transfer model for gaseous flows along with an accurate prediction of wall turbulent thermal fields. The model reproduces the correct wall-limiting behavior of velocity and temperature under arbitrary wall thermal conditions. The model appraisal is given with four different typical thermal fields, which often occur in engineering applications, in wall turbulent shear flows. Secondly, we describe the methodology of how to construct a rigorous two-equation heat-transfer model with the aid of the most up-to-date direct numerical simulation (DNS) data for wall turbulence with heat transfer. The DNS data indicate that the near-wall profile of the dissipation rate, \( \varepsilon_\theta \), for the temperature variance, \( k_\theta = \langle \theta^2 \rangle / 2 \), is completely different from the previous model predictions. We demonstrate the results of a critical assessment of existing \( \varepsilon_\theta \) equations for both two-equation and second-order closure models. Based on these assessments, we construct a new dissipation rate equation for temperature variance, taking into account all the budget terms in the exact \( \varepsilon_\theta \) equation. Also, we present a similarly refined \( k_\theta \) equation, which is linked with this new \( \varepsilon_\theta \) equation, to constitute a new two-equation heat-transfer model. Comparisons of the refined model predictions with the DNS data for a channel flow with heat transfer are given, which shows excellent agreement for the profiles of \( k_\theta \) and \( \varepsilon_\theta \) themselves and the budget in the \( k_\theta \) and \( \varepsilon_\theta \) equations. The only limitation is that this refined model is applicable only to gaseous flows such as air streams. Thus, finally, we present the development of a heat-transfer model for a variety of Prandtl-number fluids. This model incorporates new velocity and time scales to represent various sizes of eddies in velocity and thermal fields with different Prandtl numbers. Fundamental properties of the reconstructed \( k_\theta - \varepsilon_\theta \) model are first verified in basic flows under arbitrary wall thermal boundary conditions and next in backward-facing-step flows at various Prandtl numbers through a comparison of the predictions with the DNS and measurements.

1 Introduction

The turbulence model for heat transfer is a set of differential equations which, when solved with the mean-flow and turbulence Reynolds-stress equations,
allows calculations of relevant correlations and parameters that simulate the behavior of thermal turbulent flows. Like the classification of turbulence models for the Reynolds stresses, phenomenological turbulent heat-transfer models are classified into zero-equation, two-equation, and heat-flux equation models. The zero-equation heat-transfer model is a typical and most conventional method for analyzing the turbulent heat transfer, in which the eddy diffusivity for heat $\alpha_t$ is prescribed via the known eddy viscosity $\nu_t$ together with the most probable turbulent Prandtl number $Pr_t$, so that $\alpha_t = \nu_t/Pr_t$. Thus, in this formulation the analogy is tacitly assumed between turbulent heat and momentum transfer (Launder 1988). Many previous experimental studies have, however, revealed that there are no universal values of $Pr_t$ even in simple flows (Kays 1994), e.g. at the same streamwise location, a value of $Pr_t$ close to the wall is different from that away from the wall (Cebeci 1973; Antonia 1980). The recent sophisticated renormalization group (RNG) theory for turbulence based on an iterative averaging method (Nagano and Itazu 1997) indicates that the turbulent Prandtl number changes according to the molecular Prandtl number (Itazu and Nagano 1998). This lack of universality restricts the applicability of a zero-equation model. On the other hand, a heat-flux equation model ought to be more universal, at least in principle. This model, however, is still rather primitive and extensive further research is in progress.

For the velocity field, the linear eddy viscosity $k-\varepsilon$ model of turbulence is still regarded as a powerful tool for predicting many engineering flow problems including jets, wakes, wall flows, reacting flows, and flows with centrifugal and Coriolis forces (Rodi 1984). For scalar turbulence, Nagano and Kim (1988) developed a corresponding two-equation model for heat transport (hereinafter referred to as the NK model). They modelled the eddy diffusivity for heat $\alpha_t$ using the temperature variance $k_\theta = (\bar{\theta}^2/2)$ and the dissipation rate of temperature fluctuations $\varepsilon_\theta$, together with $k$ and $\varepsilon$. The NK model is applicable to thermal fields where the real value of $Pr_t$ is unknown, and thus is more widely applicable than the conventional zero-equation model. A weakness is that the NK model has been developed mainly for heat transfer under uniform-wall-temperature conditions. Consequently, in order to analyze heat transfer problems under various wall thermal conditions, we need further improvements to the NK model or the development of a more sophisticated $k_\theta-\varepsilon_\theta$ heat-transfer model. Thus, a modified $k_\theta-\varepsilon_\theta$ model (Youssef, Nagano and Tagawa 1992), maintaining the original concept of the NK model has been developed. Using a Taylor-series expansion for the energy equation in the near-wall region, they have made it clear how the wall-limiting behavior of turbulence quantities in a thermal field varies with the thermal-wall condition, and then constructed the basic modelled equations to satisfy these requirements. This heat-transfer turbulence model was tested by application to turbulent boundary layers with four different wall thermal fields; namely, a uniform wall temperature, a uniform wall heat flux, a stepwise change in wall temperature, and a constant heat flux followed by an adiabatic wall.
Fortunately, recent direct numerical simulations (DNS) for wall shear flows provided the details of turbulent quantities near the wall (e.g., Kim et al. (1987), and Kasagi et al. (1992)). It was shown from these DNS data that the near-wall profiles of the dissipation rates of turbulent kinetic energy and temperature variance were completely different from the previous model predictions. Recently, Rodi and Mansour (1993), and Nagano and Shimada (1995a) have improved the $k$-$\varepsilon$ model using DNS databases in which all the budget terms in the exact $\varepsilon$ equation were incorporated in the modelled $\varepsilon$ equation. The performance of the existing $\varepsilon$-equation models was assessed by Nagano and Shimada (1994; 1995b), and a rational $\varepsilon$ equation was reconstructed by Nagano et al. (1994).

Similarly, two-equation heat-transfer models ($k_\theta$-$\varepsilon_\theta$) have been improved (Nagano et al. 1991; Youssef, Nagano and Tagawa 1992; Hattori, Nagano and Tagawa 1993; Sommer et al. 1992; Abe, Kondoh and Nagano 1995), since Nagano and Kim (1988) proposed the first model for wall turbulent shear flows. The two-equation heat-transfer model is a powerful tool for predicting the heat transfer in flows with almost complete dissimilarity between velocity and thermal fields. Also, the characteristic time scale for a thermal field needed in a second-order closure model may now be calculated with the two-equation heat-transfer model (Shikazono and Kasagi 1996).

In the present chapter, as the first example of modelling heat transfer, we develop a rigorous $k_\theta$-$\varepsilon_\theta$ model. In particular, modelling of the $\varepsilon_\theta$ equation is performed by taking into account all the budget terms in the exact $\varepsilon_\theta$ equation. First, we make a critical assessment of previous $\varepsilon_\theta$ equations for both two-equation and second-order closure models. Second, we reconstruct a more sophisticated $\varepsilon_\theta$ equation reflecting the assessment results. Then, we propose a set of $k_\theta$-$\varepsilon_\theta$ models to match with the present rigorous $\varepsilon_\theta$ equation. Finally, we verify a set of model equations using DNS data and experimental data. In order to show the performance of the proposed two-equation heat-transfer model, we analyze the case of a sudden change in thermal wall condition, which is hard to measure using conventional tools; and we then investigate the physical phenomena of the system using the results of analysis.

As mentioned above, after the first proposal by Nagano and Kim (1988), several $k_\theta$-$\varepsilon_\theta$ models have been proposed (Youssef et al. 1992; So and Sommer 1993; Hattori, Nagano and Tagawa 1993). Most of the previous models have adopted a dimensionless parameter $y^+ = u_\tau y/\nu$, which is normalized by the viscous length $\nu/u_\tau$ consisting of the friction velocity $u_\tau$ and the kinematic viscosity $\nu$, to represent the distance from the wall, $y$. However, as recently pointed out on a number of occasions, the viscous length $\nu/u_\tau$ becomes infinity (in other words $u_\tau$ becomes zero) at the separation and reattachment points, so that the introduction of a parameter without the viscous length is indispensable in order to extend the applicability of the turbulence model for complex engineering problems. Abe et al. (1994) introduced a new parameter...
Table 1: Constants and functions for the $k$-ε model.

| $\sigma^*_k$ | $1.4/f_t$ |
| $\sigma^*_\varepsilon$ | $1.3/f_t$ |
| $f_t$ | $1 + 6f_w$ |
| $f_w_1$ | $f_w(4)$ |
| $f_w_2$ | $f_w(26)$ |
| $C_{\varepsilon_1}$ | 1.45 |
| $C_{\varepsilon_2}$ | 1.9 |
| $C_{\varepsilon_3}$ | 0.005 |
| $C_{\varepsilon_4}$ | 0.5 |
| $f_2$ | $(1 + f'_2)(1 - f_w)\{1 - 0.6 \exp[-(R_t/45)^{1/2}]\}$ |
| $f'_2$ | $\exp(-2 \times 10^{-4} R_v^{13})[1 - \exp(-2.2 R_v^{0.5})]$ |
| $R_v$ | $(k/\varepsilon)[1/(1 + \nu_t/\nu)](1/R_t^{1/2})f_w$ |
| $f_\mu$ | $(1 - f_w)\{1 + (60/R_t^{3/4}) \exp[-(R_t/55)^{1/2}]\}$ |

using the Kolmogorov length $\eta = (\nu^3/\varepsilon)^{1/4}$ in place of the viscous length $\nu/\tau_\nu$ as the characteristic length, and succeeded in predicting the correct reattachment points in backward-facing-step flows at various Reynolds numbers. Thus, in this chapter, we use the same normalization as in Abe et al. (1994).

As the second example of modelling heat transfer, we construct a refined two-equation heat-transfer model based on information obtained from the DNS databases for turbulent heat and fluid flows with different Reynolds and Prandtl numbers. In addition, in modelling the turbulence transport processes, we consider length and time scales characterizing a range from small to larger eddies in both the velocity and thermal fields, and we try to combine the effects of those various scales in each field with the present two-equation heat-transfer model without employing the viscous length $\nu/\tau_\nu$.

2 Two-equation model of turbulence for velocity field

A velocity field is described with the following continuity and momentum equations

$$\frac{\partial U_i}{\partial x_i} = 0, \quad (2.1)$$

$$\frac{D U_i}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \nu \frac{\partial U_i}{\partial x_j} - u_i u_j \right), \quad (2.2)$$

where $D/Dt \equiv \partial/\partial t + U_j \partial/\partial x_j$.
In the \( k-\varepsilon \) model, the Reynolds stress \( \overline{u_i u_j} \) in (2.2) (see Nagano and Tagawa 1990a) can be obtained from the following set of equations

\[
-\overline{u_i u_j} = \nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij}k, \tag{2.3}
\]

\[
\nu_t = C_\mu f_\mu \frac{k^2}{\varepsilon}, \tag{2.4}
\]

\[
\frac{Dk}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \right) \frac{\partial k}{\partial x_j} \right] - \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} - \varepsilon, \tag{2.5}
\]

\[
\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \right) \frac{\partial \varepsilon}{\partial x_j} \right] - C_{\varepsilon 1} \frac{\varepsilon}{k} \frac{u_i u_j}{\partial x_j} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \tag{2.6}
\]

As indicated by Myong and Kasagi (1990), and by Nagano and Tagawa (1990a), imposing the rigid boundary condition (i.e. no-slip) at the wall does not necessarily lead to the correct asymptotic solutions of \( k \propto y^2, -\overline{uv} \propto y^3, \nu_t \propto y^3, \) and \( \varepsilon \propto y^0 \) for \( y \to 0 \), unless the wall-limiting behavior of turbulence is properly incorporated in the turbulence model adopted. In the present study, for the basic formulation in the \( k-\varepsilon \) model, we adopt the Nagano–Shimada model (Nagano and Shimada 1995a) (hereinafter referred to as the NS model), which reproduces strictly the limiting behavior of wall. The model also reproduces the turbulent energy and its dissipation rate (including their budgets) very closely for the cases of wall bounded flows:

\[
\frac{Dk}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \right) \frac{\partial k}{\partial x_j} \right] - \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} - \varepsilon + \max \left[ -0.5 \nu \frac{\partial}{\partial x_j} \left( k \frac{\partial \varepsilon}{\partial x_j} \right), 0 \right], \tag{2.7}
\]

\[
\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \right) \frac{\partial \varepsilon}{\partial x_j} \right] - C_{\varepsilon 1} \frac{\varepsilon}{k} \frac{u_i u_j}{\partial x_j} - C_{\varepsilon 2} f_2 \frac{\varepsilon^2}{k} + f_{w2} \nu u_t \left( \frac{\partial^2 U_i}{\partial x_j \partial x_k} \right)^2 + C_{\varepsilon 3} \nu \frac{k}{\partial x_j} \frac{\partial U_i}{\partial x_j} \frac{\partial^2 U_i}{\partial x_j \partial x_k} + C_{\varepsilon 4} \nu \frac{\partial}{\partial x_j} \left[ (1 - f_{w1}) \frac{\varepsilon}{k} \frac{\partial k}{\partial x_j} \right] f_{w1}. \tag{2.8}
\]

The NS model employed the wall friction velocity \( u_\tau \) in the wall reflection function \( f_w \). However, here we introduce the Kolmogorov velocity \( u_\varepsilon \) in this function described as followed:

\[
f_w(\xi) = \exp \left[ - \left( \frac{y^*}{\xi} \right)^2 \right], \tag{2.9}
\]

where \( y^* = u_\varepsilon y / \nu \) (\( = y / \eta \)) is the dimensionless distance from the wall based on the Kolmogorov velocity scale \( u_\varepsilon = (\nu \varepsilon)^{1/4} \) (or the Kolmogorov length
scale \( \eta = (\nu^3/\varepsilon)^{1/4} \). This model function is more useful for analysis of various complex flows, as confirmed by Abe et al. (1995) and Nagano et al. (1997). The model constants and functions were optimized for the proposed function. These are listed in Table 1. Note that, from (2.9), \( f_w(4) = \exp \left[ -\left( y^*/4 \right)^2 \right] \) and \( f_w(26) \) is similarly defined.

3 Two-equation model for heat transfer (DNS-based modelling)

3.1 Governing Equations

The energy equation may be written

\[
\frac{D\Theta}{Dt} = \frac{\partial}{\partial x_j} \left( \alpha \frac{\partial \Theta}{\partial x_j} - u_j \overline{\theta} \right). \tag{3.1}
\]

In (3.1), the term on the right-hand side, the turbulent heat flux \( \overline{u_j \theta} \), is described using the concept of eddy diffusivity for heat \( \alpha_t \) (see, Nagano and Kim 1988; Nagano et al. 1991),

\[
-\overline{u_i \theta} = \alpha_t \frac{\partial \Theta}{\partial x_i}, \tag{3.2}
\]

where

\[
\alpha_t = C_\lambda f_\lambda k \tau_m. \tag{3.3}
\]

As a time-scale equivalent to the relative ‘lifetime’ of the energy-containing eddies or temperature fluctuations, we adopt the mixed or hybrid time-scale \( \tau_m \) which is a function of the time-scale ratio \( R = \tau_\theta/\tau_u \), where \( \tau_u = k/\varepsilon \) and \( \tau_\theta = k_\theta/\varepsilon_\theta \) \( k_\theta = \overline{\theta^2}/2 \) are the dynamic and scalar time-scales, respectively. Obviously, \( \tau_m \) blends both thermal and mechanical contributions. The characteristic length scale (i.e. the spatial extent of a fluctuating temperature) can hence be written as \( L_m = k^{1/2} \tau_m \), and the eddy diffusivity for heat can be modelled as \( \alpha_t \propto k^{1/2} L_m = k \tau_m \). The present expression for \( \alpha_t \) can be regarded as a generalized form for the eddy diffusivity introduced by Nagano and Kim (1988). Accordingly, we incorporate near-wall effects on the thermal field in the model function \( f_\lambda \). The optimal value of eddy diffusivity for heat \( \alpha_t \) can be expressed as a function of the state of both velocity and thermal fields by solving the transport equations for \( k, \varepsilon, k_\theta, \) and \( \varepsilon_\theta \).

The exact transport equations for \( k_\theta \) and \( \varepsilon_\theta \) are symbolically expressed as follows (Nagano and Kim 1988):

\[
\frac{Dk_\theta}{Dt} = D_{k_\theta} + T_{k_\theta} + P_{k_\theta} - \varepsilon_\theta, \tag{3.4}
\]

\[
\frac{D\varepsilon_\theta}{Dt} = D_{\varepsilon_\theta} + T_{\varepsilon_\theta} + P^1_{\varepsilon_\theta} + P^2_{\varepsilon_\theta} + P^3_{\varepsilon_\theta} + P^4_{\varepsilon_\theta} - \Upsilon_{\varepsilon_\theta}, \tag{3.5}
\]
The terms on the right-hand sides in (3.4) and (3.5) are identified as

Molecular diffusion
\[ D_k = \alpha \frac{\partial^2 k}{\partial x_j \partial x_j}, \quad D_\varepsilon = \alpha \frac{\partial^2 \varepsilon}{\partial x_j \partial x_j}, \]

Turbulent diffusion
\[ T_k = - \frac{\partial u_j k'}{\partial x_j}, \quad T_\varepsilon = - \frac{\partial u_j \varepsilon'}{\partial x_j}, \]

Mean gradient production
\[ P_k = -u_j \frac{\partial \Theta}{\partial x_j}, \quad P_\varepsilon = -2\alpha \frac{\partial \Theta}{\partial x_k} \frac{\partial \Theta}{\partial x_j} \frac{\partial U_j}{\partial x_k}, \]

Gradient production
\[ P_3 = -2\alpha u_j \frac{\partial \theta}{\partial x_j} \frac{\partial^2 \Theta}{\partial x_k \partial x_j}, \]

Turbulent production
\[ P_4 = -2\alpha \frac{\partial u_j}{\partial x_k} \frac{\partial \theta}{\partial x_j} \frac{\partial \theta}{\partial x_j}, \]

Destruction
\[ \Upsilon_\varepsilon = 2\alpha^2 \left( \frac{\partial^2 \theta}{\partial x_k \partial x_j} \right)^2, \]

where \( k' = \theta^2/2 \) and \( \varepsilon' = \alpha(\partial \theta/\partial x_j)^2 \), respectively.

### 3.2 Wall-Limiting Behavior of Velocity and Temperature

The behavior of the turbulent quantities of velocity and thermal fields near the wall can be inferred from a Taylor series expansion in terms of \( y \), together with the continuity, momentum and energy equations, namely,

\[
\begin{align*}
U^+ &= y^+ + a_1 y^{+2} + a_2 y^{+3} + \cdots \\
u &= b_1 y + b_2 y^2 + b_3 y^3 + \cdots \\
v &= c_1 y^2 + c_2 y^3 + \cdots \\
w &= d_1 y + d_2 y^2 + d_3 y^3 + \cdots \\
k &= \frac{u_i u_i}{2} = (b_1^2 + d_1^2)/2 |y|^2 + (b_1 b_2 + d_1 d_2) y^3 + \cdots \\
\bar{uv} &= b_1 c_1 y^3 + (b_1 c_2 + c_1 b_2) y^4 + \cdots \\
v_\omega &= \nu(\partial^2 k/\partial y^2)_w = \nu(b_1^2 + d_1^2) \\
\Theta^+ &= P \rho y^+ + g_1 y^{+2} + g_2 y^{+3} + \cdots \\
\theta &= \theta_w + h_1 y + h_2 y^2 + \cdots \\
k_\theta &= \bar{\theta}^2/2 = \bar{\theta}_w^2/2 + [(h_1^2 + 2h_2 \theta_w)/2] y^2 + \cdots \\
v_\theta &= c_1 \theta_w y^2 + (c_2 \theta_w + c_1 h_1) y^3 + \cdots \\
\varepsilon_{\theta w} &= \alpha(\partial^2 k_\theta/\partial y^2)_w = \alpha(h_1^2 + 2h_2 \theta_w) \end{align*}
\]
In (3.7), in view of the correspondence between \( k \) and \( k_{\theta} \) profiles near the wall, a smooth change in temperature variance \( k_{\theta} \) in the immediate vicinity of the wall is assumed, i.e. \( \partial k_{\theta}/\partial y \big|_w = 0 \), which is exact in the case of both uniform wall temperature and uniform wall heat flux. From (3.7), in the vicinity of the wall, we obtain the following relations: \( U^+ = y^+ \), \( u \propto y \), \( v \propto y^2 \), \( w \propto y \), \( \Theta^+ = Py^+ \), and \( \theta \propto y^{p/2} \) (where, \( p = 2 \) : without fluctuations in wall temperature, \( p = 0 \) : with \( \theta_w \) fluctuations). From (3.7), in the vicinity of the wall, we obtain the following relations:

\[
\begin{align*}
U^+ & = y^+, \\
& = u \propto y, \\
& = v \propto y^2, \\
& = w \propto y, \\
\Theta^+ & = Py^+, \\
& = \theta \propto y^{p/2},
\end{align*}
\]

Note that, as may be seen from (3.2), \( v_{\theta} \) and \( \alpha_t \) vary as the same power of \( y \) near the wall. Consequently, from the wall-limiting behavior of turbulence, we have the following two regimes according to the wall thermal conditions

\[
\begin{align*}
\alpha_t & \propto y^3 \quad \text{for } p = 2 \quad (\text{without } \theta_w \text{ fluctuations}) \\
& \propto y^2 \quad \text{for } p = 0 \quad (\text{with } \theta_w \text{ fluctuations})
\end{align*}
\]  

(3.8)

As discussed later, the eddy diffusivity \( \alpha_t \) should be modelled to satisfy the above requirements consistently.

### 3.3 Assessment of Modelled \( \varepsilon_{\theta} \) Equations

#### 3.3.1 Modelled \( \varepsilon_{\theta} \) equations

The modelled dissipation rate equations for temperature variance used in the current \( k_{\theta}-\varepsilon_{\theta} \) models for wall shear flows are written in one of two ways:

\[
\begin{align*}
\frac{D\varepsilon_{\theta}}{Dt} & = \alpha \frac{\partial^2 \varepsilon_{\theta}}{\partial x_j \partial x_j} + T_{\varepsilon_{\theta}} + C_{P_1} f_{P_1} \frac{\varepsilon_{\theta}}{k_{\theta}} P_{k_{\theta}} + C_{P_2} f_{P_2} \frac{\varepsilon_{\theta}}{k} P_k \\
& - C_{D_1} f_{D_1} \frac{\varepsilon_{\theta}^2}{k_{\theta}} - C_{D_2} f_{D_2} \frac{\varepsilon_{\theta} \varepsilon}{k} + \text{additional term},
\end{align*}
\]  

(3.9)

\[
\begin{align*}
\frac{D\varepsilon_{\theta}}{Dt} & = \alpha \frac{\partial^2 \varepsilon_{\theta}}{\partial x_j \partial x_j} + T_{\varepsilon_{\theta}} + C_{P_1} f_{P_1} \frac{\varepsilon_{\theta}}{k_{\theta}} P_{k_{\theta}} + C_{P_2} f_{P_2} \frac{\varepsilon_{\theta}}{k} P_k \\
& - C_{D_1} f_{D_1} \frac{\varepsilon_{\theta}^2}{k_{\theta}} - C_{D_2} f_{D_2} \frac{\varepsilon_{\theta} \varepsilon}{k} + \text{additional term},
\end{align*}
\]  

(3.10)

where \( P_k = -\bar{u}_i \bar{u}_j (\partial U_i/\partial x_j) \) is the production rate of \( k \). The turbulent diffusion term \( T_{\varepsilon_{\theta}} \) in (3.5) is generally modelled as follows:

\[
T_{\varepsilon_{\theta}} = \left\{ \begin{array}{ll}
\frac{\partial}{\partial x_j} \left( \frac{\alpha_i}{\sigma_{\phi}} \frac{\partial \varepsilon_{\theta}}{\partial x_j} \right) & : \text{at a two-equation level}, \\
\frac{\partial}{\partial x_j} \left( C_s f_R \frac{k}{\varepsilon} u_i u_j \frac{\partial \varepsilon_{\theta}}{\partial x_i} \right) & : \text{at a second-order closure level}.
\end{array} \right.
\]  

(3.11)

The \( T_{\varepsilon_{\theta}} \) term is also modelled in the same manner as in (3.11).
Table 2: Existing $\varepsilon_\theta$ and $\tilde{\varepsilon}_\theta$ equation models.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_\lambda$</td>
<td>0.11</td>
<td>0.1</td>
</tr>
<tr>
<td>$C_\ast$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$C_{P_1}$</td>
<td>0.9</td>
<td>0.85</td>
</tr>
<tr>
<td>$C_{P_2}$</td>
<td>0.72</td>
<td>0.64</td>
</tr>
<tr>
<td>$C_{D_1}$</td>
<td>1.1</td>
<td>1.0</td>
</tr>
<tr>
<td>$C_{D_2}$</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>$\sigma_\phi$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\tau_{m}$</td>
<td>$(k/\bar{\varepsilon})(\bar{R})^{1/2}$</td>
<td>$(k/\bar{\varepsilon}) \left[ (2R)^2 + 3.4(2R)^{1/2}/R_t^{3/4} \right]$</td>
</tr>
<tr>
<td>$f_\lambda$</td>
<td>$[1 - \exp(-y^+/A_\lambda)]^2$</td>
<td>$[1 - \exp(-y^+/A_\lambda)]^2$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$A_\lambda$</td>
<td>$(30.5/\sqrt{Pr})(C_f/2St)$</td>
<td>$26/\sqrt{Pr}$</td>
</tr>
<tr>
<td>$f_R$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$f_{P_1}$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_{P_2}$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_{D_1}$</td>
<td>1.0</td>
<td>$[1 - \exp(-y^+/5.8)]^2$</td>
</tr>
<tr>
<td>$f_{D_2}$</td>
<td>1.0</td>
<td>$(1/C_{D_2})(C_{\varepsilon 2}f_\varepsilon - 1) [1 - \exp(-y^+/6)]^2$</td>
</tr>
<tr>
<td>Additional term</td>
<td>$\alpha \alpha_t (1 - f_\lambda)(\partial^2 \Theta/\partial x_j \partial x_k)^2$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$\bar{R} = (k_\theta/\tilde{\varepsilon}_\theta)/(k/\bar{\varepsilon})$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C_{\varepsilon 2} = 1.9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f_\varepsilon = 1 - \exp[-(R_t/6.5)^2]$</td>
</tr>
</tbody>
</table>

In (3.10), quantities $\bar{\varepsilon}$ and $\tilde{\varepsilon}_\theta$, called the isotropic dissipation rates of $k$ and $k_\theta$, are defined by the following equations, respectively:

$$\bar{\varepsilon} = \varepsilon - \hat{\varepsilon},$$  \hspace{1cm} (3.12)

$$\tilde{\varepsilon}_\theta = \varepsilon_\theta - \hat{\varepsilon}_\theta,$$  \hspace{1cm} (3.13)

where $\hat{\varepsilon} = 2\nu(\partial^2 \bar{k}/\partial y)^2$, $\tilde{\varepsilon}_\theta = 2\alpha(\partial \sqrt{\Delta k_\theta}/\partial y)^2$, and $\Delta k_\theta = k_\theta - k_{\theta w}$.

3.3.2 Assessment procedure

In (3.9) and (3.10), $\varepsilon_\theta$ and $\tilde{\varepsilon}_\theta$ are the only unknown variables, and all turbulence quantities except $\varepsilon_\theta$ and $\tilde{\varepsilon}_\theta$ are supplied directly from the DNS data; i.e., $U_i$, $u_i u_j$, $k$, $\varepsilon$, $\bar{\varepsilon}$, $\Theta$, and $k_\theta$ are not calculated from any modelled equation, but are given as the ‘true’ values from the DNS data.

We perform the model assessment in a fully developed channel flow with heat transfer (Hattori and Nagano 1998) for which a trustworthy DNS database (Kasagi et al. 1992) is available. The Reynolds number based on the friction velocity and a channel half-width, $Re_\tau$, is 150 and the Prandtl number is 0.71.
Table 2: (continued)

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$C_\lambda$</td>
<td>—</td>
<td>0.1</td>
</tr>
<tr>
<td>$C_s$</td>
<td>0.3</td>
<td>—</td>
</tr>
<tr>
<td>$C_{P1}$</td>
<td>0.8</td>
<td>1.9</td>
</tr>
<tr>
<td>$C_{P2}$</td>
<td>0.3</td>
<td>0.6</td>
</tr>
<tr>
<td>$C_{D1}$</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>$C_{D2}$</td>
<td>0.3</td>
<td>0.9</td>
</tr>
<tr>
<td>$\sigma_\phi$</td>
<td>—</td>
<td>1.6</td>
</tr>
<tr>
<td>$\tau_m$</td>
<td>—</td>
<td>$(k/\varepsilon) \left{ f_R + [3(2R)^{1/2} / (R_t^{3/4} Pr)] f_d \right}$</td>
</tr>
<tr>
<td>$f_\lambda$</td>
<td>—</td>
<td>$[1 - \exp (-y^<em>/A_\mu)] [1 - \exp (-y^</em>/A_\lambda)]$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>—</td>
<td>14</td>
</tr>
<tr>
<td>$A_\lambda$</td>
<td>—</td>
<td>$A_\mu / \sqrt{Pr}$</td>
</tr>
<tr>
<td>$f_R$</td>
<td>$2R / (0.7 + R)$</td>
<td>$2R / (0.5 + R)$</td>
</tr>
<tr>
<td>$f_{P1}$</td>
<td>1.0</td>
<td>$[1 - \exp (-y^*)]^2$</td>
</tr>
<tr>
<td>$f_{P2}$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_{D1}$</td>
<td>1.0</td>
<td>$[1 - \exp (-y^*)]^2$</td>
</tr>
<tr>
<td>$f_{D2}$</td>
<td>1.0</td>
<td>$(1 / C_{D2}) (C_{c2} f_e - 1) [1 - \exp (-y^*/5.7)]^2$</td>
</tr>
<tr>
<td>Additional term</td>
<td>$2\alpha C_{w2} (k_\theta / \varepsilon_\theta) \bar{u}<em>2 (\partial^2 \Theta / \partial y^2)^2 - \bar{\varepsilon}</em>\theta \bar{\varepsilon}<em>\theta / k</em>\theta$</td>
<td>—</td>
</tr>
<tr>
<td>$C_{w2}$</td>
<td>$\max[0.1, 0.35 - 0.21 Pr]$</td>
<td>$f_d = \exp[-(R_t / 200)^2]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C_{c2} = 1.9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f_e = 1 - \exp[-(R_t / 6.5)^2]$</td>
</tr>
</tbody>
</table>

3.3.3 Models for assessment

We assess the six temperature dissipation rate equations proposed by Nagano and Kim (NK) (1988), Hattori, Nagano and Tagawa (HNT) (1993) and Shikazono and Kasagi (SK) (1996), which are the $\tilde{\varepsilon}_\theta$ equations, and those by Nagano, Tagawa and Tsuji (NTT) (1991), Abe, Kondoh and Nagano (AKN) (1995) and Sommer, So and Lai (SSL) (1992), which are the $\varepsilon_\theta$ equations. The abbreviations in parentheses are introduced for ease of reference. The details of the above six modelled equations are listed in Table 2. It should be mentioned that the SSL model has partly introduced $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_\theta$ to prevent divergence in the calculation caused by finite values of $\varepsilon$ and $\varepsilon_\theta$ at the wall, while the NTT and AKN models avoid it by introducing the proper $f_{D1}$ and $f_{D2}$ functions. In the SSL model, the turbulent heat-flux $\bar{u}_i \bar{\Theta}$ in the $P_{k_\theta}$ term is modelled using $\alpha_t$, but the Reynolds shear stress $\bar{u}_i \bar{u}_j$ and turbulent diffusion term $T_{\varepsilon_\theta}$ are modelled at a second-order closure level [see (3.11)]. The AKN model puts $f_{P1} = f_{D1}$ to avoid divergence in the calculation of flows with complete dissimilarity between velocity and thermal fields. In the SK model, where the $k_\theta$-$\varepsilon_\theta$ model is employed to calculate the time scale of the thermal field, the tur-
Table 2: (continued)

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_\lambda$</td>
<td>0.11</td>
</tr>
<tr>
<td>$C_s$</td>
<td>0.11</td>
</tr>
<tr>
<td>$C_{P1}$</td>
<td>0.9</td>
</tr>
<tr>
<td>$C_{P2}$</td>
<td>0.72</td>
</tr>
<tr>
<td>$C_{D1}$</td>
<td>1.1</td>
</tr>
<tr>
<td>$C_{D2}$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\sigma_\phi$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\tau_m$</td>
<td>$(k/\varepsilon)\left{ (2R)^{1/2} + \left[ 0.1(2R)^{1/2}/R_i^{1/4} \right] \right} \times (f_{\varepsilon t}/f_\lambda)$</td>
</tr>
<tr>
<td>$f_{\lambda}$</td>
<td>$[1 - \exp \left(-y^+/A_{\lambda}\right)]^2$</td>
</tr>
<tr>
<td>$A_{\mu}$</td>
<td>$[1 - \exp \left(-y^+/A_{\mu}\right)]\left[ 1 - \exp \left(-y^+/A_{\lambda}\right) \right]/30(1 + 11.8P^+)\left[A_{\mu}/Pr^{1/3}\right]$</td>
</tr>
<tr>
<td>$A_{\lambda}$</td>
<td>30</td>
</tr>
<tr>
<td>$f_R$</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_{P1}$</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_{P2}$</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_{D1}$</td>
<td>$\varepsilon_0/\varepsilon_\theta$</td>
</tr>
<tr>
<td>$f_{D2}$</td>
<td>(1/$C_{D2}$)($C_{D2}f_\varepsilon - 1$)</td>
</tr>
<tr>
<td>Additional term</td>
<td>$f_{\varepsilon t} \left[ (C_{D1} - 2)(\varepsilon_0/k_\theta)\varepsilon_\theta + C_{D2}(\varepsilon_\theta/k)\varepsilon_\theta \right] - (\varepsilon_0^2/(2k_\theta) + (1 - C_{P1})(\varepsilon_0/k_\theta)P_\theta^2)$</td>
</tr>
<tr>
<td></td>
<td>$f_{\varepsilon t} = \exp\left(-(R_t/80)^2\right)$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_0 = \varepsilon_\theta - 2\alpha(k_\theta/y^2)$</td>
</tr>
<tr>
<td></td>
<td>$P_\theta^2 = \varepsilon_\theta(\partial^2\Theta/\partial x_j \partial x_k)$</td>
</tr>
<tr>
<td></td>
<td>$\alpha_{\lambda}(1 - f_\omega)(\partial^2\Theta/\partial x_j \partial x_k)^2$</td>
</tr>
<tr>
<td></td>
<td>$f_d = \exp\left[-(R_t/120)^2\right]$</td>
</tr>
<tr>
<td></td>
<td>$f_w = [1 - \exp\left{-y^+/\left(30/Pr^{1/3}\right)\right}]^2$</td>
</tr>
<tr>
<td></td>
<td>$P^+ = \nu(\partial P/\partial x)/\rho u_\varepsilon^3$</td>
</tr>
<tr>
<td></td>
<td>$C_{D2} = 1.9$</td>
</tr>
<tr>
<td></td>
<td>$f_\varepsilon = 1 - 0.3\exp\left[-R_t^2\right]$</td>
</tr>
</tbody>
</table>

Bululent diffusion and production terms are modelled at a second-order closure level.

A characteristic time scale $\tau_m$, whose importance was demonstrated by Nagano and Kim (1988), has been used in all the two-equation heat-transfer models for wall shear flows. It can be shown that the eddy diffusivity for heat $\alpha_t$ is governed near the wall by the Kolmogorov microscale in the NTT, HNT and AKN models, and by the Taylor microscale in the SSL model.

3.3.4 Assessment results

The results of assessment for $\tilde{\varepsilon}_\theta$- and $\varepsilon_\theta$-equation models at a two-equation level and those at a second-order closure level are shown in Figure 1, where in $\varepsilon_\theta$-equation modelling $\varepsilon_\theta$ is obtained from $\varepsilon_\theta = \tilde{\varepsilon}_\theta + \hat{\varepsilon}_\theta$. The resultant characteristic time scale $\tau_\varepsilon$ is assessed in Figure 2. To assess the NK model, the time scale $\tau_\nu$ in $\alpha_t$ is given by $k/\varepsilon$ from the DNS, because the NK model is usually combined with the Nagano and Hishida model (1987) ($\tilde{\varepsilon}$-equation model).
As can be seen from Figures 1 and 2, the results of assessment for $\tilde{\varepsilon}_\theta$- and $\varepsilon_\theta$-equation models indicate that none of the four models can reproduce accurately the DNS behavior. Especially, predicted $\varepsilon_\theta$ tends to increase in the buffer layer ($5 < y^+ < 40$). In Figures 1(c) and 2(c), only the SK model qualitatively and quantitatively reproduces a trend similar to DNS. However, the constants $C_{P1}$ and $C_{P2}$ in the SK model do not satisfy the relation for a
Figure 2: Profiles of time scale $\tau_\theta$: (a) in $\tilde{\varepsilon}_\theta$ equations at a two-equation level; (b) in $\varepsilon_\theta$ equations at a two-equation level; (c) in $\tilde{\varepsilon}_\theta$ and $\varepsilon_\theta$ equations at a second-order closure level.

‘constant stress and constant heat flux layer’, namely

$$\frac{\kappa^2}{Pr_t} + \frac{C_{P1} - C_{D1}}{R} + C_{P2} - C_{D2} = 0,$$

where $\kappa = 0.39 - 0.41$, $Pr_t = 0.9$, $C_\mu = 0.09$ and $R = 0.5$ are typical values in wall-bounded flows.
Next, we discuss the gradient of $\varepsilon_\theta$ at the wall. The near-wall behavior of $\varepsilon_\theta$ without $\theta_w$ fluctuations can be inferred from a Taylor series expansion in terms of $y$ as follows (Youssef et al. 1992):

$$\varepsilon_\theta = h_1 + 4h_2y + O(y^2),$$  \hspace{1cm} (3.15)

where the coefficients $h_1$ and $h_2$ are independent of the $y$ coordinate. On the other hand, from (3.4), the molecular diffusion term balances the dissipation term at $y = 0$:

$$\varepsilon_\theta = \alpha \frac{\partial^2 k_\theta}{\partial y^2} \quad \text{with} \quad \alpha \frac{\partial^2 k_\theta}{\partial y^2} = h_1 + 6h_2y + O(y^2).$$  \hspace{1cm} (3.16)

From (3.15) and (3.16), the coefficient $h_2 \left[= (1/4)(\partial \varepsilon_\theta / \partial y)|_w \right]$ should be zero. This can be seen, of course, in the DNS data.

In theory, the wall-limiting behavior of $\tilde{\varepsilon}_\theta$ must be $\tilde{\varepsilon}_\theta \propto y^2$. In the $\tilde{\varepsilon}_\theta$ equation of the NK and HNT models, however, the molecular diffusion term balances with $C_{D1}\tilde{\varepsilon}_\theta^2/k_\theta$ term at $y = 0$. As a result, the wall-limiting behavior of $\tilde{\varepsilon}_\theta$ becomes $\tilde{\varepsilon}_\theta \propto y^1$. Therefore, in the $\tilde{\varepsilon}_\theta$ equation, adding the extra term to reproduce the correct wall-limiting behavior of $\tilde{\varepsilon}_\theta$ is of the first importance to obtain the correct profile of $\varepsilon_\theta$ near the wall. The SK model has an additional term to balance the molecular diffusion term in the $\tilde{\varepsilon}_\theta$ equation at the wall, as suggested by Kawamura and Kawashima (1994).

The budget data for the $\tilde{\varepsilon}_\theta$ and $\varepsilon_\theta$ equations are shown in Figure 3. The budget in the $\tilde{\varepsilon}_\theta$ equation is represented by $\alpha(\partial^2 \varepsilon_\theta / \partial y^2) = \alpha(\partial^2 \tilde{\varepsilon}_\theta / \partial y^2) + 2\alpha^2(\partial^2[(\partial \sqrt{k_\theta}/\partial y)]/\partial y^2)$. Obviously, the two-equation model predictions are not in agreement with the DNS data. As seen from Figure 3(c), the sum total of the budget in the SK model is the closest to the DNS. This is a consequence of the smaller model constants $C_{P2}$ and $C_{D2}$ used, which render the production and destruction terms smaller in magnitude. In the $\varepsilon_\theta$-equation models, the NTT model is rather close to the DNS. This is because the NTT model has no additional production term.

From these assessments it becomes clear that solutions for $\varepsilon_\theta$ are significantly influenced by any additional production term, the values ascribed to the model constants, and the formulation of the characteristic time scale.

### 3.4 Construction of a Rigorous $k_\theta$-$\varepsilon_\theta$ Model

#### 3.4.1 Modelling the eddy diffusivity for heat $\alpha_t$

Thermal eddy diffusivity $\alpha_t$ given by (3.3) must be adequately modelled with the dominant characteristic velocity and time scales responsible for scalar transfer. Thus, it is important to reflect the influence of the time scales for both velocity and thermal fields. Previous $\alpha_t$ models have been based on the concept of a single time scale, e.g., the assumption of the turbulent Prandtl
Figure 3: Budgets of modelled $\tilde{\varepsilon}_\theta$ equations ($P_1^{\tilde{\varepsilon}_\theta} + P_2^{\tilde{\varepsilon}_\theta} + P_3^{\tilde{\varepsilon}_\theta} + P_4^{\tilde{\varepsilon}_\theta} + T^{\tilde{\varepsilon}_\theta} - \Upsilon^{\tilde{\varepsilon}_\theta}$) and $\varepsilon_\theta$ equations ($P_1^{\varepsilon_\theta} + P_2^{\varepsilon_\theta} + P_3^{\varepsilon_\theta} + P_4^{\varepsilon_\theta} + T^{\varepsilon_\theta} - \Upsilon^{\varepsilon_\theta}$): (a) two-equation level $\tilde{\varepsilon}_\theta$ equations; (b) two-equation level $\varepsilon_\theta$ equations; (c) second-order closure level $\tilde{\varepsilon}_\theta$- and $\varepsilon_\theta$ equations.

number $Pr_t$ or of a mixed time scale, e.g., $\tau_m = \sqrt{\tau_u \tau_\theta}$ or $\tau_m = \tau_\theta^2 / \tau_u$. However, as is frequently pointed out, the former fails to predict heat transfer in flows with a dissimilarity between velocity and thermal fields, while the latter compromises the accuracy of the predicted near-wall turbulence quantities because it relates $\tau_u$ and $\tau_\theta$, which characterize large-scale motions, to the region adjacent to the wall where the dissipative motion is dominant. Therefore, a
further development of \( \alpha_t \), reflecting the effect of various time scales in velocity and thermal fields, is needed.

Recently, Abe et al. (1995) have proposed the following multiple-time-scale \( \tau_m \) using the hybrid time scale \( \tau_h = \tau_u R / (C_m + R) \) (i.e., \( 1/\tau_h = 1/\tau_u + C_m/\tau_\theta \) with \( C_m \) as a model constant):

\[
\tau_m = \frac{k}{\varepsilon} \left\{ \frac{2R}{0.5 + R} + \frac{\sqrt{2R}}{Pr} \frac{3}{R_t^{3/4}} \exp \left[ -\left( \frac{R_t}{200} \right)^2 \right] \right\},
\]

where \( R_t = k^2 / (\nu \varepsilon) \) is the turbulent Reynolds number and \( Pr \) is the molecular Prandtl number.

Using the multiple-time-scale similar to (3.17), we adopt the following representation for \( \alpha_t \) to satisfy the wall-limiting behavior of thermal turbulence indicated by (3.8):

\[
\alpha_t = C_\lambda f_\lambda k \tau_m = C_\lambda f_\lambda k \left\{ \frac{k}{\varepsilon} \left[ \frac{2R}{0.5 + R} + \frac{\sqrt{2R}}{Pr^{4/3}} \frac{26}{R_t^{3/4}} \exp \left( -\frac{R_t}{220} \right) \right] \right\},
\]

\[
f_\lambda = \left[ 1 - f_w(A_{\mu}) \right]^{1/2} \left[ 1 - f_w(A_\lambda) \right]^{1/2}
\]

where \( R_h = k\tau_h / \nu = R_t[2R/(0.5+R)] \) is the turbulent Reynolds number based on the harmonic-averaged time scale \( \tau_h = (k/\varepsilon)[2R/(0.5 + R)] \). Note that \( \tau_h \) becomes identical to \( \tau_u = k/\varepsilon \) in local equilibrium flows with \( R = 0.5 \).

### 3.4.2 Modelling the \( \varepsilon_\theta \) equation

As shown in the previous section, none of the existing \( \tilde{\varepsilon}_\theta \) and \( \varepsilon_\theta \) models at a two-equation level give qualitative and quantitative agreement with the DNS. Hence, we will construct an \( \varepsilon_\theta \)-equation model based on the NTT model by taking into account all the budget terms in the exact \( \varepsilon_\theta \) equation.

(a) Modelling of \( P_{\varepsilon_\theta}^1, P_{\varepsilon_\theta}^2, P_{\varepsilon_\theta}^4 \) and \( \Upsilon_{\varepsilon_\theta} \)

The \( P_{\varepsilon_\theta}^1, P_{\varepsilon_\theta}^2, P_{\varepsilon_\theta}^4 \) and \( \Upsilon_{\varepsilon_\theta} \) terms can be modelled in a way similar to the NK (Nagano and Kim 1988) and NTT models (Nagano et al. 1991):

\[
P_{\varepsilon_\theta}^1 + P_{\varepsilon_\theta}^2 + P_{\varepsilon_\theta}^4 - \Upsilon_{\varepsilon_\theta} = -C_{P1} f_{P1} \frac{\varepsilon_\theta}{k \theta} \frac{\partial \Theta}{\partial x_j} - C_{D1} f_{D1} \frac{\varepsilon_\theta^2}{k \theta} - C_{P2} f_{P2} \frac{\varepsilon_\theta}{k} \frac{\partial U_i}{\partial x_j} - C_{D2} f_{D2} \frac{\varepsilon_\theta^2 \varepsilon_\theta}{k}.
\]

The DNS data indicates that the \( P_{\varepsilon_\theta}^1 \) and \( P_{\varepsilon_\theta}^2 \) terms exert a great influence on the production of \( \varepsilon_\theta \) near the wall. The modelling given by (3.20), however, is
based on $P^4_{\varepsilon_\theta}$ and $\Upsilon_{\varepsilon_\theta}$, so that the influence of other terms is not sufficiently reflected. Therefore, we model the contributions from the $P^4_{\varepsilon_\theta}$ and $P^2_{\varepsilon_\theta}$ terms using an order-of-magnitude analysis, as done in modelling $\varepsilon$ by Rodi and Mansour (1993) and Nagano and Shimada (1995a). With $k_\theta$ and $\ell_\theta = \sqrt{k_\theta} \tau_\theta$ (thermal turbulence length scale), we can estimate an order of magnitude of the $P^4_{\varepsilon_\theta}$, $P^2_{\varepsilon_\theta}$ and $P^4_{\varepsilon_\theta}$ terms as

$$
P^1_{\varepsilon_\theta} = O \left( \frac{k\sqrt{k_\theta}}{\ell_\theta} \left( \frac{\lambda_\theta}{\lambda} \right) G \right), \tag{3.21}
$$

$$
P^2_{\varepsilon_\theta} = O \left( \frac{\sqrt{k_\theta}}{\ell_\theta} S \right), \tag{3.22}
$$

$$
P^4_{\varepsilon_\theta} = O \left( \frac{kk_\theta}{\ell_\theta^2} \left( \frac{\ell_\theta}{\lambda} \right) \right),
$$

where $G = [(\partial \Theta/\partial x_j)(\partial \Theta/\partial x_j)]^{1/2}$ represents the mean temperature gradient, $S = [(\partial U_i/\partial x_j)(\partial U_i/\partial x_j)]^{1/2}$ is the mean strain rate, and $\lambda = \sqrt{k_U/\varepsilon}$ and $\lambda_\theta = \sqrt{k_\theta \alpha/\varepsilon_\theta}$ are the Taylor microscales for the velocity and temperature fields, respectively. The above relations give $P^1_{\varepsilon_\theta}/P^4_{\varepsilon_\theta} \approx (\lambda_\theta/\sqrt{k_\theta})G = G/(\varepsilon_\theta/\alpha)^{1/2}$, $P^2_{\varepsilon_\theta}/P^4_{\varepsilon_\theta} \approx (\lambda/\sqrt{k})S = S/(\varepsilon/\nu)^{1/2}$. Consequently, we define the parameters $R_T$ and $R_U$ as

$$
R_T = \left[ \frac{(\partial \Theta/\partial x_j)^2}{(\partial \theta/\partial x_j)^2} \right]^{1/2} = \frac{G}{(\varepsilon_\theta/\alpha)^{1/2}}, 
$$

$$
R_U = \left[ \frac{(\partial U_i/\partial x_j)^2}{(\partial u_i/\partial x_j)^2} \right]^{1/2} = \frac{S}{(\varepsilon/\nu)^{1/2}}. \tag{3.23}
$$

These parameters represent the ratio of the gradient of mean flow to that of fluctuating components. Apparently, the relations $P^1_{\varepsilon_\theta}/P^4_{\varepsilon_\theta} \approx R_T$ and $P^2_{\varepsilon_\theta}/P^4_{\varepsilon_\theta} \approx R_U$ hold. Since the structure of turbulent shear flows near the wall is governed mainly by the gradient of the mean flow (see, e.g., Hinze (1975)), contributions of $P^4_{\varepsilon_\theta}$ and $P^2_{\varepsilon_\theta}$ terms must appear when $R_T > 1$ and $R_U > 1$. We replace the mean temperature gradient $G$ and the strain rate parameter $S$ with the well-known relations for the constant heat flux layer $[G = (q_w/\rho c_p)/(\alpha + \alpha_t) = u_\tau \theta_\tau/(\alpha + \alpha_t)]$ and the constant stress layer $[S = (\tau_w/\rho)/(\nu + \nu_t) = u_\tau^2/(\nu + \nu_t)]$. Then, (3.22) and (3.23) lead to

$$
R_T = \frac{u_\tau \theta_\tau}{(\nu/Pr + \alpha_t)(Pr \varepsilon_\theta/\nu)^{1/2}} f_w(6), \tag{3.24}
$$

$$
R_U = \frac{u_\tau^2}{(\nu + \nu_t)(\varepsilon/\nu)^{1/2}} f_w(6). \tag{3.25}
$$
The contributions of \( P_{\varepsilon \theta}^1 \) and \( P_{\varepsilon \theta}^2 \) can now be included in the model functions \( f_{P1} \) and \( f_{P2} \) as follows:

\[
\begin{align*}
\begin{cases}
    f_{P1} &= (1 - f'_{P1}) f_p, \\
    f'_{P1} &= \exp(-7 \times 10^{-5} R_T^{10})[1 - \exp(-2.2 R_T^{1/2})], \\
    f_{P2} &= (1 - f'_{P2}) f_p, \\
    f'_{P2} &= \exp(-7 \times 10^{-5} R_U^{10})[1 - \exp(-2.2 R_U^{1/2})],
\end{cases}
\end{align*}
\]  

(3.26)

where \( f_p \) is introduced for correcting overproduction near the wall, and the wall reflection function \( f_w(6) \) in (3.24) and (3.25) is given by (2.9) with \( \xi = 6 \). In the following equations (3.28) and (3.29), \( f_w(12) \) and \( f_w(3) \) are similarly defined.

(b) Modelling of \( P_{\varepsilon \theta}^3 \)

The \( P_{\varepsilon \theta}^3 \) term is negligibly small in comparison with \( P_{\varepsilon \theta}^1, P_{\varepsilon \theta}^2, P_{\varepsilon \theta}^4 \) and \( \Upsilon_{\varepsilon \theta} \). However, when compared with the sum of these terms, i.e., \( P_{\varepsilon \theta}^1 + P_{\varepsilon \theta}^2 + P_{\varepsilon \theta}^4 - \Upsilon_{\varepsilon \theta} \), the \( P_{\varepsilon \theta}^3 \) term becomes of the same order, so modelling \( P_{\varepsilon \theta}^3 \) is also important.

In the present model, we adopt the following form similar to (2.8) in the \( k-\varepsilon \) model:

\[
P_{\varepsilon \theta}^3 = \alpha \alpha_t f_w(12) \left( \frac{\partial^2 \Theta}{\partial x_j \partial x_k} \right)^2 + C_{\varepsilon \theta} \kappa \varepsilon f_R \frac{\partial k}{\partial x_j} \frac{\partial \Theta}{\partial x_j} \frac{\partial^2 \Theta}{\partial x_j \partial x_k},
\]  

(3.28)

where \( f_R = 2R/(0.5 + R) \). It should be noted that the hybrid turbulent Reynolds number \( R_h \) and the corresponding time scale \( \tau_h \) in (3.18) can be written as \( R_h = f_R R_t \) and \( \tau_h = f_R \tau_u \).

(c) Modelling of \( T_{\varepsilon \theta} \)

A gradient-type diffusion plus convection by large-scale motions may effectively represent turbulent diffusion for a scalar (see, e.g., Hinze (1975)). Thus, considering the relation \( \varepsilon \theta \simeq 2(\varepsilon/k)k_0 \) at \( R \simeq 0.5 \) and the near-wall-limiting behavior of \( T_{\varepsilon \theta} \), we write \( T_{\varepsilon \theta} \) as

\[
T_{\varepsilon \theta} = \frac{\partial}{\partial x_j} \left( \frac{\alpha_t}{\sigma_\phi} \frac{\partial \varepsilon \theta}{\partial x_j} \right) + C_{\varepsilon \theta} \alpha \frac{\partial}{\partial x_j} \left\{ [1 - f_w(3)]^{3/2} \frac{\varepsilon}{k} \frac{\partial k_0}{\partial x_j} f_w(3) \right\}. \tag{3.29}
\]

(d) Modelled \( \varepsilon \theta \)-equation

To sum up, the proposed \( \varepsilon \theta \)-equation can be written as

\[
\frac{D\varepsilon \theta}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \alpha + \frac{\alpha_t}{\sigma_\phi} \right) \frac{\partial \varepsilon \theta}{\partial x_j} \right] - C_{P1} f_{P1} \frac{\varepsilon \theta}{k_0} u_j \theta \frac{\partial \Theta}{\partial x_j} - C_{P2} f_{P2} \frac{\varepsilon \theta}{k} u_i u_j \frac{\partial U_i}{\partial x_j} - C_{D1} f_{D1} \frac{\varepsilon \theta^2}{k_0} - C_{D2} f_{D2} \frac{\varepsilon \theta \varepsilon}{k} + \alpha \alpha_t f_w(12) \left( \frac{\partial^2 \Theta}{\partial x_j \partial x_k} \right)^2.
\]
\[ + C_{P3} \frac{\alpha}{\varepsilon} f_R \frac{\partial k}{\partial x_k} \frac{\partial \Theta}{\partial x_j} \frac{\partial^2 \Theta}{\partial x_j \partial x_k} + C_{\varepsilon_0} \alpha \frac{\partial}{\partial x_j} \left\{ [1 - f_w(3)]^{3/2} \varepsilon \frac{\partial k}{k} \frac{\partial x_j}{k} f_w(3) \right\}. \] (3.30)

The wall reflection function \( f_w(\xi) \) is given by (2.9).

(e) Model functions and constants

From (3.30), the molecular diffusion term balances the dissipation terms at \( y = 0 \):

\[ \alpha \frac{\partial^2 \varepsilon_\theta}{\partial y^2} = C_{D1} f_{D1} \varepsilon_{\theta 0}^2 + C_{D2} f_{D2} \varepsilon_{\theta 0} \varepsilon. \] (3.31)

Considering the limiting behavior of wall turbulence, \( f_{D2} \propto y^2 \) and \( f_{D1} \propto y^2 \) (without \( \theta_w \) fluctuations) or \( f_{D1} \propto y^n \) where \( n > 0 \) (with \( \theta_w \) fluctuations) are required to satisfy (3.31). In free turbulence, as described next (see (3.41)), the limiting behavior requires

\[ C_{D2} f_{D2} = C_{\varepsilon 2} f_\varepsilon - 1. \] (3.32)

In the present model, the following equations are thus proposed to meet the requirements for both wall and free turbulence

\[ f_{D1} = 1 - \exp \left[ - (y^* / 12)^2 \right] = 1 - f_w(12) \] (3.33)

\[ f_{D2} = (1/C_{D2})(C_{\varepsilon 2} f_\varepsilon - 1) \left[ 1 - f_w(12) \right] \] (3.34)

with \( f_\varepsilon = 1 - 0.3 \exp[-(R_t / 6.5)^2] \).

The constants appearing in the present two-equation heat-transfer model are determined as follows. Firstly, we specify a value of \( C_\lambda \) in (3.18) defining the eddy diffusivity for heat, \( \alpha_t \). In the log-law region where the molecular diffusion is negligible, i.e. \( f_\mu = f_\lambda = 1 \), \( C_\lambda \) may be given from (2.4) and (3.18), together with the turbulent Prandtl number \( Pr_t = \nu_t / \alpha_t \), by

\[ C_\lambda = C_\mu / (Pr_t f_R) \quad \text{with} \quad f_R = 2R / (0.5 + R) \] (3.35)

thus, substituting the typical values of \( C_\mu = 0.09 \), \( R = 0.5 \), and \( Pr_t = 0.9 \) (Nagano and Kim 1988; Launder 1988), we obtain \( C_\lambda = 0.10 \).

We determine the constants \( C_{D1} \) and \( C_{D2} \) in the equation for \( \varepsilon_\theta \), (3.30) from the decay law of homogeneous turbulence. In a homogeneous decaying turbulent flow, (2.5), (2.6), (3.4) and (3.30) become simply

\[ U \frac{dk}{dx} = -\varepsilon, \] (3.36)

\[ U \frac{d\varepsilon}{dx} = -C_{\varepsilon 2} f_\varepsilon \frac{\varepsilon^2}{k}, \] (3.37)

\[ U \frac{dk_\theta}{dx} = -\varepsilon_\theta, \] (3.38)

\[ U \frac{d\varepsilon_\theta}{dx} = -C_{D1} f_{D1} \frac{\varepsilon^2_\theta}{k_\theta} - C_{D2} f_{D2} \frac{\varepsilon_\theta \varepsilon}{k}, \] (3.39)
where the $x$ axis is taken in the flow direction. On the other hand, it is known that the time-scale ratio $R = (k_\theta/\varepsilon_\theta)/(k/\varepsilon)$ does not change in the flow direction in homogeneous grid-generated turbulence (Newman et al. 1981; Warhaft and Lumley 1978), thus, rewriting (3.39) in terms of $R$ and substituting (3.37)–(3.39) into this equation, we obtain

$$U \frac{d\varepsilon_\theta}{dx} = \frac{1}{R} \left( \frac{\varepsilon^2 k_\theta}{k^2} - C_{\varepsilon_2 f_\varepsilon} \frac{\varepsilon^2 k_\theta}{k^2} - \frac{\varepsilon_\theta \varepsilon}{k} \right) = - \frac{\varepsilon^2}{k_\theta} - (C_{\varepsilon_2 f_\varepsilon} - 1) \frac{\varepsilon_\theta \varepsilon}{k}. \quad (3.40)$$

Equations (3.39) and (3.40) give the following relations

$$C_{D1} f_{D1} = 1,$$
$$C_{D2} f_{D2} = C_{\varepsilon_2 f_\varepsilon} - 1. \quad (3.41)$$

Equation (3.41) is also valid for the initial period ($f_\varepsilon = f_{D1} = f_{D2} = 1$) in decaying turbulent flows, and hence we have $C_{D1} = 1$ and $C_{D2} = C_{\varepsilon_2 f_\varepsilon} - 1 = 0.9$.

The model constants $C_{P1}$ and $C_{P2}$ for the production terms in the $\varepsilon_\theta$-equation (3.30) are determined by considering the characteristics of the log-law region (constant stress-heat-flux layer) in wall turbulence. In this region, the convection terms in the transport equations $k$, $\varepsilon$, $k_\theta$, and $\varepsilon_\theta$ can all be ignored, and the production terms for $k$ and $k_\theta$ balance with the respective dissipation terms, thus, with (3.18), rewriting (3.30) gives

$$\frac{C_\lambda}{\sigma_\phi} \frac{\partial}{\partial y} \left( \frac{k^2}{\varepsilon} f_R \frac{\partial \varepsilon_\theta}{\partial y} - C_{\varepsilon_1} \frac{\varepsilon_\theta}{k_\theta} \frac{\partial \Theta}{\partial y} - C_{P2} \frac{\varepsilon_\theta}{k} \frac{\partial U}{\partial y} - C_{D1} \frac{\varepsilon_2 k_\theta}{k} - C_{D2} \frac{\varepsilon_\theta \varepsilon}{k} \right) = 0. \quad (3.42)$$

With the above-mentioned characteristics of constant stress-heat-flux layer, the following relation is obtained from (3.42)

$$C_{P2} = (C_{D1} - C_{P1})/R + C_{D2} - (\kappa^2/Pr_t)/(\sigma_\phi C_{\mu}^{1/2}), \quad (3.43)$$

where $\kappa$ is the von Kármán constant. Equation (3.43) is similar to the well-known relation in the $k$-$\varepsilon$ model given by

$$C_{\varepsilon_1} = C_{\varepsilon_2} - \kappa^2/(\sigma_\varepsilon C_{\mu}^{1/2}). \quad (3.44)$$

The value $C_{P2} = 0.77$ is then obtained if we substitute the foregoing values of $C_{D1}$, $C_{D2}$, $R$, $Pr_t$, and $C_{\mu}$ for (3.43), together with $\kappa = 0.39 - 0.41$ and $C_{P1} = 0.9$ which is determined on the basis of computer optimization. Note that the present value of $C_{P1} = 0.9$ is exactly the same as the NK model constant. (It is noted that Jones and Musonge (1988) developed a transport equation for $\varepsilon_\theta$ similar to the NK model and assigned the value of $C_{P1} = 0.85$ and $C_{P2} = 0.7$.)

The model constants and functions in the present $\varepsilon_\theta$-equation at a two-equation level are listed in Table 3.
(f) Second-order closure modelling

In second-order closure modelling, the turbulent diffusion term \( T_{\varepsilon\theta} \) and the production term \( P_{\varepsilon\theta} \) should be slightly modified, since the second-order closure model needs neither \( \nu_t \) nor \( \alpha_t \). Hence, the gradient parameters \( R_T \) and \( R_U \) are changed as follows:

\[
R_T = \frac{u_{\tau}\theta_T + \overline{v\theta}}{(\varepsilon_\theta\nu/P_T)^{1/2} f_w(6)}, \tag{3.45}
\]

Table 3: Model constants and functions in the present \( \varepsilon_\theta \) models.

| \( C_\lambda \) | 0.1 |
| \( C_s \) | — |
| \( C_{P1} \) | 0.9 |
| \( C_{P2} \) | 0.77 |
| \( C_{P3} \) | 0.05 |
| \( C_{D1} \) | 1.0 |
| \( C_{D2} \) | 0.9 |
| \( C_{\varepsilon\theta} \) | 1.6 |
| \( \sigma_\phi \) | 1.8 |
| \( \tau_m \) | \((k/\varepsilon)\left\{f_R + [26(2R)^{1/2}/(P_t^{A/3}R_t^{3/4})]f_d\right\}\) |
| \( f_\lambda \) | \([1 - f_w(A_\mu)]^{1/2}/[1 - f_w(A_\lambda)]^{1/2}\) |
| \( f_w(\xi) \) | \(\exp\left[-(y^*/\xi)^2\right]\) |
| \( A_\mu \) | \(A_\mu/Pr^{1/3}\) |
| \( f_R \) | \(2R/(0.5 + R)\) |
| \( f_{P1} \) | \((1 - f'_{P1})f_p[1 - f_w(12)]\) |
| \( f_{P2} \) | \((1 - f'_{P2})f_p\) |
| \( f_{D1} \) | \(1 - f_w(12)\) |
| \( f_{D2} \) | \((1/C_{D2})(C_{\varepsilon2}f_\varepsilon - 1)[1 - f_w(12)]\) |
| Additional term \( (= P_{\varepsilon\theta}^3) \) | \(f_d\exp(-R_h/220)\) |
| \( C_{\varepsilon2} \) | 1.9 |
| \( f_\varepsilon \) | \(1 - 0.3\exp[-(R_t/6.5)^2]\) |
| \( f'_{P1} \) | \(\exp(-7 \times 10^{-5}R_t^{10})[1 - \exp(-2.2R_t^{1/2})]\) |
| \( f'_{P2} \) | \(\exp(-7 \times 10^{-5}R_u^{10})[1 - \exp(-2.2R_u^{1/2})]\) |
| \( f_p \) | \(1 + 0.75\exp[-(R_h/40)^{1/2}]\) |
| \( R_T \) | \(f_w(6)u_{\tau}\theta_T/[(\alpha + \alpha_t)(\varepsilon_\theta/\alpha)^{1/2}]\) |
| \( R_U \) | \(f_w(6)u_{\tau}^2/[(\nu + \nu_t)(\varepsilon/\nu)^{1/2}]\) |
Figure 4: Assessment of the proposed $\varepsilon_\theta$-equation models: (a) profiles of $\varepsilon_\theta$ near the wall; (b) profiles of time scale $\tau_\theta$ near the wall; (c) budget of the proposed $\varepsilon_\theta$-equation models ($P^1_{\varepsilon_\theta} + P^2_{\varepsilon_\theta} + P^3_{\varepsilon_\theta} + P^4_{\varepsilon_\theta} + T_{\varepsilon_\theta} = \Upsilon_{\varepsilon_\theta}$).

$$R_U = \frac{u^2_T}{(\varepsilon u)^{1/2}} f(w) (6).$$

(3.46)

The model functions $f'_{P1}$ and $f'_{P2}$ in (3.26) and (3.27) are defined by

$$f'_{P1} = \exp(-7 \times 10^{-5} R_T^{10}) [1 - \exp(-1.1 R_T^{1/2})],$$
$$f'_{P2} = \exp(-7 \times 10^{-5} R_U^{10}) [1 - \exp(-1.1 R_U^{1/2})].$$

(3.47)
The turbulent diffusion term, \( T_{\varepsilon_\theta} \), can be written as

\[
T_{\varepsilon_\theta} = \frac{\partial}{\partial x_k} \left( C_s \frac{k}{\varepsilon} f_R \frac{\partial \varepsilon_\theta}{\partial x_j} \right) + C_{\varepsilon_\theta} \alpha \frac{\partial}{\partial x_j} \left\{ [1 - f_w(3)]^{3/2} \frac{\varepsilon k_\theta}{k} f_w(3) \right\},
\]

where \( C_s = 0.11 \), and \( f_R \) and \( C_{\varepsilon_\theta} \) are exactly the same as in the two-equation heat-transfer model.

The \( P^3_{\varepsilon_\theta} \) term may be written [see (3.28)] as

\[
P^3_{\varepsilon_\theta} = C_{P3} \alpha \frac{k}{\varepsilon} f_R \frac{\partial^2 \Theta}{\partial x_k \partial x_\ell} \frac{\partial^2 \Theta}{\partial x_\ell \partial x_j} + C_{P4} \alpha \frac{k}{\varepsilon} f_R \frac{\partial u_j \partial u_\ell}{\partial x_k} \frac{\partial \Theta}{\partial x_\ell} \frac{\partial \Theta}{\partial x_j},
\]

with \( C_{P3} = 0.1 \) and \( C_{P4} = 0.25 \).

(g) **Assessment of proposed \( \varepsilon_\theta \) equation models**

Figure 4 shows the solutions obtained from new \( \varepsilon_\theta \) equations. As shown previously, the existing two-equation-level models have never reproduced the correct near-wall behavior of \( \varepsilon_\theta \), whereas the present predictions give excellent agreement with the DNS data. Owing to the inclusion of the model for \( \alpha_t \), the proposed model at a two-equation level gives predictions slightly different from those at a second-order closure level. The overall predictions, however, are much better than with the existing models. It should also be noted that, for the budget balance in the \( \varepsilon_\theta \) equation (Figure 4(c)), excellent agreement is now achieved.

### 3.4.3 Modelling the \( k_\theta \) equation

Figure 5 shows the budget of temperature variance predicted by the NTT model (Nagano et al. 1991), which is the basis for the proposed model and the
Table 4: Model constants and functions in the present $k_\theta$ model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_\theta$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\sigma_h^\star$</td>
<td>$1.8/[1 + 0.5f_w(28)]$</td>
</tr>
</tbody>
</table>

AKN model (Abe et al. 1995). Obviously, the model predictions are different from DNS data near the wall because of the solution given by the $\varepsilon_\theta$ equation and the modelling of the turbulent diffusion term in the $k_\theta$ equation. Therefore, in the $k_\theta$ equation given by (3.4), it is the turbulent diffusion term $T_{k_\theta}$ that should be modelled. We adopt the foregoing turbulent diffusion modelling, and write $T_{k_\theta}$ as

$$T_{k_\theta} = \frac{\partial}{\partial x_j} \left( \frac{\alpha_t \partial k_\theta}{\sigma_h^\star \partial x_j} \right) + C_\theta \frac{\partial}{\partial x_j} \left\{ \sigma_{\overline{u_k u_\ell}} d_k n_\ell e_j [1 - f_w(28)]^{1/2} \sqrt{k} k_\theta [f_w(28)]^{1/2} \right\},$$

(3.50)

where $d_k$, $n_\ell$ and $e_j$ are unit vectors in the streamwise, wall-normal and $x_j$ directions, respectively, and $\sigma_{\overline{u_k u_\ell}}$ is a sign function, first introduced by Nagano and Tagawa (1990b). The sign function $\sigma_{\overline{u_k u_\ell}}$ is necessary to make a model independent of the coordinate system, and is defined as

$$\sigma_x = \begin{cases} 
1 & (x \geq 0), \\
-1 & (x < 0).
\end{cases}$$

(3.51)

The final formulation of the $k_\theta$-equation model is written as follows:

$$\frac{D k_\theta}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \alpha + \frac{\alpha_t}{\sigma_h^\star} \right) \frac{\partial k_\theta}{\partial x_j} \right] + C_\theta \frac{\partial}{\partial x_j} \left\{ \sigma_{\overline{u_k u_\ell}} d_k n_\ell e_j [1 - f_w(28)]^{1/2} \sqrt{k} k_\theta [f_w(28)]^{1/2} \right\} + P_{k_\theta} - \varepsilon_\theta.$$

(3.52)

The model functions and constants in the present $k_\theta$ model are listed in Table 4.

### 3.5 Discussion of Predictions with Proposed Models

In general, a turbulence model must give predictions of good accuracy in both fundamental internal and external flows. If the model does not indicate good agreement for both cases, one can hardly rely on it to predict complex flows of technological interest. In this study, the modelling takes into account all the key turbulence quantities and their budgets obtained by DNS results, so that we can confirm the precision of the model prediction in both fields. Then we assess the proposed model performance in flow fields for different thermal boundary conditions at the wall.
3.5.1 Numerical scheme

The numerics sometimes affect the results of the turbulence models, both in the algorithm chosen and in the number and distribution of grid points (Kline 1980). Therefore, special attention was paid to the numerics to enable a more meaningful model appraisal. The numerical technique used is a finite-volume method, as used by Hattori and Nagano (1995). The coordinate for regions of very large gradients should be expanded near the wall. Thus, for internal flows, a transformation is introduced so that \( \eta = (y/h)^{1/2} \). For external flows, the following nonuniform grid (Nagano and Tagawa 1990a) across the layer is employed:

\[
 y_j = \Delta y_1 (K^j - 1)/(K - 1),
\]

(3.53)

where \( \Delta y_1 \), the length of the first step, and \( K \), the ratio of two successive steps, are chosen as \( 10^{-5} \) and 1.03, respectively. For both internal and external flows, 201 cross-stream grid points were used to obtain grid-independent solutions. To confirm numerical accuracy, the cross-stream grid interval was cut in half for internal flow cases. No significant differences were seen in the results.

The boundary conditions are \( U_w = k_w = k_\theta_w = 0, \varepsilon_w = 2\nu(\partial\sqrt{k}/\partial y)^2, \varepsilon_\theta_w = 2\alpha(\partial\sqrt{\Delta k}/\partial y)^2 \) and \( \Theta_w \) or \( q_w \) are determined by experimental or DNS data at a wall, \( \partial U/\partial y = \partial k/\partial y = \partial \varepsilon/\partial y = \partial \Theta/\partial y = \partial k_\theta/\partial y = \partial \varepsilon_\theta/\partial y = 0 \) at the axis for internal flows (symmetry); \( U = U_e, k = \varepsilon = k_\theta = \varepsilon_\theta = 0 \) and \( \Theta = \Theta_e \) at the outer edge of the boundary layer, where \( U_e \) and \( \Theta_e \) are prescribed from experiments.

The criterion for convergence is

\[
 \max |X^{(i+1)} - X^{(i)}|/X^{(i)} < 10^{-5},
\]

(3.54)

where \( X = U, k, \varepsilon, \Theta, k_\theta, \) and \( \varepsilon_\theta \), and \( i \) denotes the number of iterations. The computations were performed on a personal computer and a DEC Alpha workstation.

3.5.2 Channel flow with heat transfer (constant-heat-flux wall and constant-temperature wall)

It is important to predict the velocity field precisely for relevant temperature field prediction. The framework of the proposed \( k-\varepsilon \) model is based on the NS model, which has been confirmed to show highly accurate prediction of wall-bounded turbulent flows (Nagano and Shimada 1995a). In this study, however, the wall reflection function is, as noted above, now based on (2.9), so that the model was tested in the channel flow calculated for the DNS conditions of both Moser et al. (1999) \( (Re_\tau = 395) \) and of Kasagi et al. (1992) \( (Re_\tau = 150) \) shown in Figure 6. From Figure 6, it can be seen that the mean velocity and turbulent energy are predicted quite successfully for both cases.

Next, we assess the constructed two-equation heat-transfer model with the \( k-\varepsilon \) model in a fully developed channel flow under both constant-temperature
Figure 6: Channel flow predictions: (a) mean velocity; (b) turbulent energy.

(Kim and Moin 1989) \( (Re_T = 180 \text{ and } Pr = 0.71) \) and constant-heat-flux wall conditions (Kasagi et al. 1992) \( (Re_T = 150 \text{ and } Pr = 0.71) \). Comparisons of the predicted mean temperature, turbulent heat flux, temperature variance, near-wall behavior of temperature variance and turbulent heat flux with DNS are shown in Figures 7(a), 7(b), 7(c), 7(d) and 7(e) respectively. The model predictions are in almost perfect agreement with the DNS data and reproduce exactly the wall-limiting behavior near the wall for both thermal wall conditions, i.e., \( \theta_w = 0 \) and \( \theta_w \neq 0 \). Figures 8 and 9 show the predicted budget of temperature variance and its dissipation rate, compared with the DNS data. Obviously, agreement of each term in both budgets with DNS is also very good. An important point of the present study is the modelling for the turbulent diffusion term, \( \nabla T \), in the \( k_\theta \) equation. From a comparison of Figure 5 with Figure 8, the calculated budget of the proposed model is seen to improve on previous models near the wall \( (y^+ < 15) \). These facts indicate that the modelling of a gradient-type diffusion plus convection by large-scale motions is effective for the turbulent diffusion term, and that the proposed modelling is appropriate for construction of a set of heat-transfer models.
3.5.3 Boundary-layer flows with uniform-temperature or uniform-heat-flux wall

In the following, we assess the present two-equation heat-transfer model in boundary-layer flows under different thermal conditions. The most basic situations encountered are the heat transfer from a uniform-temperature or uniform-heat-flux wall. The results of thermal field calculations under a constant-wall-temperature or constant-wall-heat-flux condition along a flat plate, compared with experimental data of Gibson et al. (uniform-temperature wall) (1982) and of Antonia et al. (uniform-heat-flux wall) (1977), are shown in Figure 10. It is known that the NTT model for reference gives good prediction of turbulent thermal fields under these wall thermal conditions (Youssef et
Figure 8: Budget of temperature variance in channel flow.

Figure 9: Calculated budget of $\varepsilon_\theta$ in channel flow.

al. 1992), and the present predictions also indicate good agreement with the experimental data.

3.5.4 Constant wall temperature followed by adiabatic wall

The next test case for which calculations have been performed is concerned with a more complex thermal field in a boundary layer along a uniformly heated wall followed by an adiabatic wall. Figure 11 shows a comparison of
Figure 10: (a) mean temperature; (b) turbulent heat flux; (c) rms temperature.

the predicted results with the experimental data (Reynolds et al. 1958) of temperature differences between the wall and the free-stream $\Delta \Theta (= \Theta_e - \Theta_w)$. It can be seen that the proposed model gives generally good predictions for the rapidly changing thermal field. Also by comparison, the present model gives no prediction inferior to the AKN model (Abe et al. 1995).
Figure 11: Comparison of the predicted variations of wall temperature with the measurement.

Figure 12: Comparison of the predicted rms temperature profiles and measurements (sudden decrease in wall heat flux).

3.5.5 Constant heat flux followed by adiabatic wall

To further verify the effectiveness of the present model for calculating various kinds of turbulent thermal fields, we have carried out the calculation of a boundary layer flow along a uniform heat-flux wall followed by an adiabatic wall, which has been reported in detail by Subramanian and Antonia (1981). The calculated distributions of rms temperature fluctuations normalized by temperature difference between the free stream and the wall, $\Delta \Theta_c$, at a step change in surface thermal condition, are shown in Figure 12, compared with the experimental data (Subramanian and Antonia 1981) and the prediction of the AKN model. Both models indicate a slight underprediction of the peak value of rms temperature. The proposed model, however, shows the variation of physical phenomena of rms temperature in the thermal layer along a uniform heat-flux followed by an adiabatic wall. In particular, the rapid decrease
in temperature fluctuations from the inner region has been captured by the proposed model.

3.5.6 Double-pulse heat input

As a final test case, we have calculated the more complex thermal case, where the heat input is spatially intermittent in a double-pulse manner. Then we have investigated the mechanism of turbulent heat transfer in such a rapidly changing thermal layer.

The temperature difference between the free stream and the wall, \( \Delta \Theta = \Theta_w - \Theta_e \), and the Stanton number reported by Reynolds et al. (1958), are shown in Figure 13 compared with the prediction of the present model. It is indicated that both the velocity and the thermal fields are well predicted, and the turbulent heat transfer characteristics in the thermal entrance region are reproduced very well. Figure 14(a) shows how the turbulent near-wall thermal layer changes when the heat input is intermittent, where \( \Delta \Theta = \Theta - \Theta_e \) is normalized by the temperature difference between the wall and the free stream, \( \Delta \Theta_e = \Theta_{wc} - \Theta_e \), just before the first heat input/cutoff point. It can be seen that a very abrupt decrease and increase in mean fluid temperature occurs in the wall region, which is a consequence of the no-heat-input condition followed by heat input, i.e., \( \partial \Theta / \partial y \bigg|_w = 0 \rightarrow \partial \Theta / \partial y \bigg|_w = \text{constant} \). Within a short distance from the abrupt change-over, the mean temperature profile becomes uniform over most of the thermal layer. The following discussion deals with how these phenomena affect other turbulent quantities. Figure 14(b) shows the distribution of turbulent heat flux normalized by \( u^\tau \) and \( \Delta \Theta_c \). Just after the first heat input/cutoff point, with vanishing mean temperature gradient near the wall, the turbulent heat flux, \( \langle \tilde{v}\tilde{\theta} \rangle \), decreases rapidly. Just before the second heat input point, \( \langle \tilde{v}\tilde{\theta} \rangle \) has greatly decayed with its maximum occurring in the outer layer. Over the reheated wall, \( \langle \tilde{v}\tilde{\theta} \rangle \) again shows a rapid increase near the

![Figure 13](https://www.cambridge.org/core/core/terms. https://doi.org/10.1017/CBO9780511755385.008)
Figure 14: Variations of turbulent quantities for double-pulse heat input: (a) mean temperature; (b) turbulent heat flux; (c) rms temperature.

This is qualitatively consistent with the experimental result (Antonia et al. 1977) for the thermal entrance region of the boundary layer on a flat plate.

Next, variations of the rms temperature are shown in Figure 14(c). Just after the heat cutoff point, distributions of the rms temperature tend to be similar to the experimental evidence obtained by Subramanian and Antonia.
Figure 15: Budget of temperature variance for double-pulse heat input: (a) \(x = 0.517\) [m]; (b) \(x = 0.579\) [m]; (c) \(x = 0.876\) [m]; (d) \(x = 0.936\) [m].

(1981) and discussed in the previous section. It can be seen that just before the second heat input point, the rms temperature remains in the outer region only, and that it increases very rapidly near the wall beyond that point. Figures 15(a)–(d) show budgets of temperature variance at locations just before the first heat cutoff \((x = 0.517\) m), just after the heat cutoff \((x = 0.579\) m), just before the second heat input \((x = 0.876\) m), and just after the second heat input \((x = 0.936\) m), respectively. In these figures, each term is normalized by the peak value of the production \(P_{k_\theta}\) at the respective locations. Since the mean temperature gradient vanishes near the wall as shown in Figure 14(a) at \(x = 0.579\) m, the peak value of the production term tends to increase in the outer region and the rapidly decreasing temperature fluctuation is restrained by an increase of the convective term there. Consequently, the fluctuating temperature is transported actively by the turbulent diffusion from the outer region to the wall, though the dissipation also increases away from the wall. Since the molecular diffusion and the dissipation preserve the near-wall structure and no temperature fluctuations are created by the mean temperature gradient, \(\sqrt{\kappa_\theta}\) is virtually nonexistent just before the second heat input point, as shown in Figure 14(c). From the above, after the first cutoff point, it is understandable that the near-wall structure of thermal turbulence is preserved mainly by diffusion from the outer to the inner region, and the temperature
fluctuation decreases remarkably. Then, just after the second heat input point, the near-wall profile of temperature fluctuation returns rapidly to the unperturbed initial profile. The remaining fluctuation in the outer region does not participate in the reproduction. Since the proposed model is rigorously constructed by considering the budget profiles of turbulence quantities obtained by DNS, we may expect that the model could be used to investigate the detailed mechanism of heat transfer in complex applications, as illustrated in this section.

4 Two-equation Model for heat transfer (effects of Prandtl number)

4.1 Construction of $k_\theta-\tilde{\varepsilon}_\theta$ Model

4.1.1 Eddy diffusivity $\alpha_t$ for $k_\theta-\tilde{\varepsilon}_\theta$ model

As mentioned in the foregoing, the eddy diffusivity for heat, $\alpha_t$, is generally given by (3.3). Features of $\alpha_t$ in (3.3) with (3.17) are summarized as follows: in the near-wall region, $\alpha_t$ yields the relation $\alpha_t \propto \sqrt{k\eta}\sqrt{R/Pr} \propto \nu_t\sqrt{R/Pr}$, so it is possible to adequately capture the behavior of dissipative motions and, in the region far from the wall, the $\alpha_t$ model consists only of time scales of the energy-containing eddies through the hybrid time scale $\tau_h$ (see section 3.4.1). In the present $k_\theta-\tilde{\varepsilon}_\theta$ model (Nagano and Shimada 1996), we adopt a multiple-time-scale similar to (3.17):

$$\tau_m = \frac{k}{\varepsilon} \left( \frac{2R}{C_m + R} + \sqrt{\frac{2R B_\lambda}{Pr R_\lambda f_{\eta_\theta}}} \right),$$

(4.1)

where $C_m$ is the weighting constant of the composite time scale, $B_\lambda$ is the model constant that represents the effectiveness of dissipation eddies, and $f_{\eta_\theta}$ is the model function limiting the $B_\lambda$-affected region.

As for the model function $f_\lambda$ in (3.3), we introduce the following formulation:

$$f_\lambda = 1 - \exp \left(-A_\lambda \tilde{y}_\theta^* n \tilde{y}_\theta^{2-n}\right) = 1 - \exp \left(-A_\lambda^* \tilde{y}_\theta^*\right),$$

where

$$A_\lambda^* = A_\lambda (1 + C_\eta \sqrt{Pr})^n.$$  \hspace{1cm} (4.2)

Here, $A_\lambda$ and $C_\eta$ denote model constants, and the dimensionless parameter $\tilde{y}_\theta^*$ is defined using the mixed length scale $\tilde{\eta}_m$ as

$$\tilde{y}_\theta^* = \frac{y}{\tilde{\eta}_m} = (1 + C_\eta \sqrt{Pr})\tilde{y}_\theta^*,$$

$$\tilde{\eta}_m = \left( \frac{1}{\eta} + \frac{C_\eta}{\eta_\theta} \right)^{-1} = \frac{\tilde{\eta}}{1 + C_\eta \sqrt{Pr}}.$$  \hspace{1cm} (4.3)
The length scale $\tilde{\eta} = (\nu^3/\varepsilon)^{1/4}$ represents the Kolmogorov microscale defined previously, and $\tilde{\eta}_0 = (\alpha^2 \nu/\varepsilon)^{1/4}$ is the Batchelor microscale (Batchelor 1959). $\tilde{y}$* is defined as $\tilde{y}^* = y/\tilde{\eta}$. From (4.3), the characteristic length scale $\tilde{\eta}_m$ gives weight to the Kolmogorov microscale $\tilde{\eta}$ for lower Prandtl number fluids ($Pr < 1$), while for higher Prandtl number fluids the Batchelor microscale $\tilde{\eta}_0$ becomes dominant. [For low Prandtl number fluids, the so-called Obukhov microscale (Obukhov 1949) $\tilde{\eta}_0 = (\alpha^3/\varepsilon)^{1/4} = \tilde{\eta} Pr^{-3/4}$ could be used as the characteristic length scale of dissipation eddies. However, it is easily understood that $\tilde{\eta}$ is always smaller than $\tilde{\eta}_0$ because of the relation $\tilde{\eta}_0/\tilde{\eta} = Pr^{-3/4}$.] It should also be noted that $f_\lambda$ given by the above equation differs slightly from (3.19).

This is because we here incorporate Prandtl-number effects on the thermal field in the model function $f_\lambda$. Also note that the relation between $\varepsilon$ and $\bar{\varepsilon}$ is represented by (3.12).

We note that the fact that the sum of the power of $\tilde{y}_\theta^*$ and $\tilde{y}^*$ in (4.2) is 2 results from a restriction in the wall-limiting behavior of $\alpha_t$ (Youssef et al. 1992; Abe et al. 1995). We determine the power $n$, the constant $A_\lambda$, and the weighting constant $C_\eta$ from the following procedure: we calculate algebraically $A_\lambda^* = A_\lambda(1 + C_\eta \sqrt{Pr})^n$ so that the maximum values of $k_\theta$ at $Pr = 0.1, 0.71$ and 2.0 agree with those of the DNS data, while the remaining constants and functions are fixed. As a result, we obtain $n = 1/4, A_\lambda = 7 \times 10^{-4}$, and $C_\eta = 2$.

The weighting constant $C_m$ plays an important role in giving weight to a shorter time scale between $\tau_u$ and $\tau_\theta$. Thus, we consider the relationship between $\tau_u$ and $\tau_\theta$ before determining $C_m$. It is clear that, in decaying homogeneous flows with high Prandtl number fluids, the time-scale ratio $R(= \tau_\theta/\tau_u)$ is greater than unity in the high $R_t$ region (Iida and Kasagi 1993). Also, it is easily verified that for turbulent wall flows, $R$ strictly equals $Pr$ at the wall. (Note that $R = Pr$ is ensured for the case of $\theta_w = 0$). Hence, the characteristic time scale suitable for high Prandtl number fluids is $\tau_u$. In the case of low Prandtl number fluids, on the other hand, since the DNS data (Iida and Kasagi 1993; Kasagi et al. 1992; Kasagi and Ohtsubo 1992) indicate that $R$ is always smaller than unity irrespective of the types of flow fields, $\tau_\theta$ is appropriate for the characteristic time scale. When $Pr \simeq 1$ (e.g., air flow), the wavenumbers related to peak intensities of both velocity and temperature fluctuations are close to each other, so that the influence of both $\tau_u$ and $\tau_\theta$ becomes significant. For the reasons mentioned above, the weighting constant $C_m$ would be expected to change with $Pr$, and hence we decide that $C_m = 0.2/Pr^{1/4}$.

As for the model function $f_{\eta_0}$, Sato et al. (1994) pointed out that $f_{\eta_0}$ exerts a significant influence on the prediction of high Prandtl number fluids in which the dissipation scale becomes much smaller. Thus, we write the model function $f_{\eta_0}$ using the foregoing parameter $\tilde{\eta}_\theta^*$ as $f_{\eta_0} = \exp[-(\tilde{\eta}_\theta^*/25)^{3/4}]$.

Finally, we consider the model constant $B_\lambda$ representing the effectiveness of dissipation eddies. In order to obtain the correct wall-limiting behavior of $\alpha_t$ independent of wall thermal conditions, and to reproduce the fact that the
ratio of the respective time scales for dissipation eddies in the velocity and thermal fields becomes equal to $\sqrt{R/Pr}$ (Youssef et al. 1992; Abe et al. 1995), $B_{\lambda}$ is set to the value $120/(1 + 2\sqrt{Pr})^{1/4}$. Here, the value of 120 is chosen so that the wall-limiting behavior of the calculated $\bar{\nu}^{*}\theta$ agrees with that of the DNS. As a result, the near-wall behavior of $\alpha_{t}$ is proportional to $\nu_{t}\sqrt{R/Pr}$, similar to that given by Abe et al. (1995).

To sum up, the proposed $\alpha_{t}$ model is written as follows:

$$\alpha_{t} = C_{\lambda}f_{\lambda} \frac{k^{2}}{\varepsilon} \left\{ \frac{2R}{0.2/Pr^{1/4} + R} + \sqrt{\frac{2R}{Pr}} \frac{120}{(1 + 2\sqrt{Pr})^{1/4}} \frac{1}{\tau_{t}} \exp \left[ -\left( \frac{\tilde{y}_{\theta}^{*}}{25} \right)^{3/4} \right] \right\},$$

$$f_{\lambda} = 1 - \exp(-7 \times 10^{-4} y_{\theta}^{*1/4} \tilde{y}^{*7/4}),$$

$$\tilde{y}_{\theta} = (1 + 2\sqrt{Pr})\tilde{y}^{*}. \tag{4.4}$$

The model constant $C_{\lambda}$ is assigned the standard value 0.10 (see (3.35)).

### 4.1.2 Modelling the $k_{\theta}$-equation

The $k_{\theta}$-equation necessary to determine the time scale $\tau_{\theta}$ (or the time scale ratio $R$) is given by (3.4). As mentioned before, the only term to be modelled in (3.4) is the turbulent diffusion $T_{k_{\theta}}$, which plays a significant role in the accurate prediction of $\varepsilon_{\theta}$.

The turbulent diffusion term $T_{k_{\theta}}$ is generally modelled using the generalized gradient diffusion hypothesis (GGDH). However, as pointed out in section 3, GGDH modelling for $T_{k_{\theta}}$ causes an imbalance in the budget for the $k_{\theta}$-equation and produces an incorrect behavior of $\varepsilon_{\theta}$. This discrepancy is mainly due to the fact that GGDH modelling represents the turbulent diffusion caused by relatively small-scale (higher wavenumber) eddies. (It can be readily understood that the formulation of the modelled turbulent diffusion through GGDH is similar to that of the molecular diffusion $D_{k_{\theta}}$ which is a small-scale phenomenon in a turbulent flow.) As a result, an underestimation of turbulent diffusion occurs in the buffer layer where relatively lower wavenumber eddies are dominant, and the behavior of $\varepsilon_{\theta}$ is incorrectly reproduced. Thus, in order to obtain the correct behavior of $\varepsilon_{\theta}$, the effect of large-scale structures must be reflected in the turbulent diffusion modelling.

In the present study, the turbulent diffusion term, including the contribution from lower wavenumber eddies, is modelled by using the following proposal similar to that of Hattori and Nagano (1998) (see (3.50)):

$$T_{k_{\theta}} = \frac{\partial}{\partial x_{\ell}} \left[ \left( \frac{\alpha_{t}}{\sigma_{h}} \right) \frac{\partial k_{\theta}}{\partial x_{\ell}} \right] + C_{\theta} \frac{\partial}{\partial x_{\ell}} \left( \sqrt{k} \ k_{\theta} f_{w\theta} \sigma_{uiuj}c_{Si}c_{Nj}c_{\ell} \right), \tag{4.5}$$

$$\sigma_{h} = \sigma_{h0}/f_{h},$$

where $\sigma_{h0}$, $C_{\theta}$, $f_{h}$, and $f_{w\theta}$ are the model constants and functions, respectively. (Note that, as already mentioned, the sign function $\sigma_{uiuj}$ and the unit vectors
Table 5: Model constants and functions of the proposed $k_\theta-\tilde{\varepsilon}_\theta$ model.

<table>
<thead>
<tr>
<th>$C_\lambda$</th>
<th>$C_m$</th>
<th>$\sigma_h$</th>
<th>$C_\theta$</th>
<th>$\sigma_\phi$</th>
<th>$C_{P1}$</th>
<th>$C_{P2}$</th>
<th>$C_{D1}$</th>
<th>$C_{D2}f_{D2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.2</td>
<td>1.6</td>
<td>0.1</td>
<td>1.8</td>
<td>0.825</td>
<td>0.9</td>
<td>1.0</td>
<td>(4.15)</td>
</tr>
<tr>
<td>$C_{D3}^*$</td>
<td>$C_\tau$</td>
<td>$f_\lambda$</td>
<td>$f_h$</td>
<td>$f_{w\theta}$</td>
<td>$f_{P1}$</td>
<td>$f_{P2}$</td>
<td>$f_{D1}$</td>
<td>$f_{D3}^*$</td>
</tr>
<tr>
<td>0.025</td>
<td>3.0</td>
<td>(4.2)</td>
<td>(4.6)</td>
<td>(4.7)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$f_w$</td>
</tr>
</tbody>
</table>

e_{Si}, e_{Nj}, e_\ell are needed to make the model independent of the coordinate system.) The model constants $\sigma_{h0}$ and $C_\theta$ are assigned the values 1.6 and 0.1, respectively, and the model functions $f_h$ and $f_{w\theta}$ are given as follows:

$$f_h = \left(\frac{C_m + R}{2R}\right) \left\{1 + \frac{5}{Pr} \exp\left(-\frac{\tilde{R}_t}{100}\right)\right\},$$

(4.6)

$$f_{w\theta} = (1 - f_w)^2 f_w^{1/4},$$

where $f_w$ is the wall reflection function similar to (2.9), $\tilde{R}_t = k^2/(\nu \tilde{\varepsilon})$ is the turbulent Reynolds number based on $\tilde{\varepsilon}$, and $(C_m + R)/2R = \tau_u/(2\tau_h)$ in (4.6) is necessary for $f_h$ to ensure the balance in the order of magnitude between the modelled turbulent diffusion term through GGDH $[\partial(\alpha_t/\sigma_h \cdot \partial k_\theta/\partial x_j)/\partial x_j \sim O(\alpha_t/\sigma_h \cdot \theta^2/\ell^2)]$, and the strict turbulent diffusion term $[T_{k_\theta} \sim O(\theta^2/\ell)]$ in the $k_\theta$-equation in the local equilibrium state. The final formulation of the $k_\theta$-equation model is written as follows:

$$\frac{Dk_\theta}{Dt} = \frac{\partial}{\partial x_j} \left[\left(\alpha + \frac{\alpha_t}{\sigma_h}\right) \frac{\partial k_\theta}{\partial x_j} + C_\theta \left(\sqrt{k} \ f_{w\theta} \sigma_{w\theta} e_{Si} e_{Nj} e_\ell\right)\right] + P_{k_\theta} - \varepsilon_\theta.$$ 

(4.8)

One may note that (4.8) is almost identical to (3.52).

4.1.3 Modelling the $\tilde{\varepsilon}_\theta$-equation

In the two-equation modelling of a thermal field, the dissipation rate $\varepsilon_\theta$ of temperature variance must be determined from its transport equation. The use of the $\varepsilon_\theta$-equation involves the same problems as those already mentioned in the use of the $\varepsilon$-equation, so instead of the $\varepsilon_\theta$-equation, we adopt the following $\tilde{\varepsilon}_\theta$-equation similar to that proposed by Nagano and Kim (1988):

$$\frac{D\tilde{\varepsilon}_\theta}{Dt} = \frac{\partial}{\partial x_j} \left[\left(\alpha + \frac{\alpha_t}{\sigma_\phi}\right) \frac{\partial \tilde{\varepsilon}_\theta}{\partial x_j}\right] + \frac{\tilde{\varepsilon}_\theta}{k} (C_{P1} f_{P1} P_{k_\theta} - C_{D1} f_{D1} \varepsilon_\theta) + \frac{\tilde{\varepsilon}_\theta}{k} (C_{P2} f_{P2} P_k - C_{D2} f_{D2} \varepsilon) + \alpha \alpha_t (1 - f_\lambda) \left(\frac{\partial^2 \Theta}{\partial x_j \partial x_k}\right)^2.$$ 

(4.9)
where $\sigma_\phi$, $C_{P1}$, $C_{P2}$, $C_{D1}$, $C_{D2}$ are model constants, and $f_{P1}$, $f_{P2}$, $f_{D1}$, $f_{D2}$ are model functions. In the present study, these model constants and functions are basically identical to those adopted by previous $k\theta$-$\varepsilon_\theta$ models (Nagano and Kim 1988; Youssef et al. 1992; So and Sommer 1993; Hattori et al. 1993) except for some model functions, and are listed in Table 5. The relation between $\varepsilon_\theta$ and $\bar{\varepsilon}_\theta$ is defined by (3.13).

In the proposed model, we use the above $\bar{\varepsilon}_\theta$-equation (4.9) as is, but its relative time scales are reconsidered below. The physical role of a transport equation is to correctly express how transported turbulence quantities change through the eddy motions with various scales. Indeed, the previous modelling of the source and sink terms included in (4.9) was made by the formulation $\bar{\varepsilon}_\theta \times (\text{time scale})^{-1}$. Thus, we classify terms in the transport equation into the following three groups:

(i) Comparatively slow motion due to the existence of mean velocity and temperature gradients (contributing to the generation of turbulence quantities);

(ii) Relatively rapid motion due to the effect of large (energy-containing) eddies;

(iii) Remarkably rapid motion caused by dissipation eddies.

First, we consider the time scale dominating the motion (i). After investigating the coherent structure of wall turbulence using the DNS database (Robinson 1991), it was clarified that the dissipation reaches its maximum in the ‘internal shear layer’ (or near-wall shear layer (Robinson 1991)) occurring near the wall. This means that a close relationship exists between the production process of $\varepsilon$ (or $\varepsilon_\theta$) and the formation of the internal shear layer. Also, it is well known that an internal shear layer occurs when surrounding high-speed fluid impacts on the upstream edge of a kinked low-speed streak and/or on the low-speed fluid ejected away from a wall by streamwise vortices in the viscous sublayer, and that the lifetime of the internal shear layer is strongly influenced by the characteristic time scale of the mean field. This also means that the characteristic time scale in producing $\varepsilon$ or $\varepsilon_\theta$ can be closely related to the lifetime of an internal shear layer (or the time scale of the mean field). Thus, the characteristic time scales for the production of $\varepsilon$ and $\varepsilon_\theta$ are expected to be represented through those related to mean velocity and thermal fields, e.g., mean velocity gradient $\tau_U = 1/(\partial U/\partial y)$.

Put another way, the shear rate parameter $S \equiv \sqrt{2S_{ij}S_{ij}}$, where $S_{ij} \equiv (\partial U_i/\partial x_j + \partial U_j/\partial x_i)/2$ denotes the mean velocity gradient (strain) tensor, seems to be suitable as the characteristic time scale to be considered. However, since turbulence must be generated by the production terms $P_k$ and $P_{k\theta}$ in the $k$- and $k\theta$-equations through interactions between mean and fluctuating
fields, the time scales contributing to the generation of turbulence should be directly expressed by using $P_k$ and $P_{k\theta}$ as

$$
\tau_U = \frac{1}{P_k/k} = \frac{1}{k} \left( \frac{-u_i u_j}{k} \frac{\partial U_i}{\partial x_j} \right)^{-1},
$$

$$
\tau_\Theta = \frac{1}{P_{k\theta}/k_{\theta}} = \frac{1}{k_{\theta}} \left( \frac{-u_j \theta}{k_{\theta}} \frac{\partial \Theta}{\partial x_j} \right)^{-1}.
$$

(4.10)

It is interesting to note that (4.10) is clearly identified with the time scales of the production processes previously introduced by Nagano and Kim (1988) [see (4.9)].

Next, let us examine fluid motion (ii). It is already well known that the motion of energy-containing eddies makes little contribution to generating turbulence but rather acts to supply energy to smaller eddies through the energy-cascade process. The characteristic time scales of the energy-containing eddies for both velocity and thermal fields are closely connected with those of eddies with wavenumbers related to the maximum values of the spectral distributions for $k$ and $k_{\theta}$, and these are generally represented as follows:

$$
\tau_u = k/\varepsilon, \quad \tau_\theta = k_{\theta}/\varepsilon_{\theta}.
$$

(4.11)

Note that the effects of (4.11) are already explicitly included in (4.9).

Finally, we discuss the motion (iii) with rapid change. It is known that almost all the energy fed by the mean field is accumulated in the energy-containing eddies and, subsequently, that it is successively supplied to smaller eddies (or dissipation eddies) through the deformation work of larger eddies. The dissipation eddies have vorticity much higher than that of the energy-containing eddies, so that the characteristic time scale of the former eddies becomes shorter compared with that of the latter. Almost all the modelled $\varepsilon_\theta$- or $\tilde{\varepsilon}_\theta$-equations proposed so far have applied only $\tau_u$ and $\tau_\theta$ in (4.11) to all eddy motions. In the vicinity of the wall, however, it is clear that there are always dissipation eddies. Also, we can easily imagine that, in flows with $Pr \gg 1$ or $Pr \ll 1$, the size of the dissipative (or destruction) eddies in one field develops at a similar rate to that of energy-containing eddies in another field. Therefore, the dissipation-eddy time scales become very significant factors in model construction. The representative time scale of the dissipation eddies in a velocity field is generally expressed by the following well-known Kolmogorov time scale:

$$
\tau_{nu} = \sqrt{\nu/\varepsilon}.
$$

(4.12)

In a similar way, the representative time scale of the dissipation eddies in a thermal field can be defined as follows:

$$
\tau_{n\theta} = \sqrt{\alpha/\varepsilon_{\theta}} \sqrt{k_{\theta}/k} = \sqrt{R/Pr} \tau_{nu}.
$$

(4.13)
It should be noted that the Taylor microscales \( \lambda = \sqrt{\nu k/\varepsilon} \) and \( \lambda_\theta = \sqrt{\kappa \alpha k/\varepsilon_\theta} \) for velocity and thermal fields, which are often used as the scales of the smallest eddies, are strongly linked to the characteristic time scales of dissipation eddies as \( \lambda = \sqrt{\kappa \tau_{\eta u}} \) and \( \lambda_\theta = \sqrt{\alpha \tau_{\eta \theta}} \).

In the present study, as mentioned previously, the characteristic time scales of dissipation eddies [(4.12) and (4.13)] are written through the following composite time scale \( \tau_{\eta m} \):

\[
\frac{1}{\tau_{\eta m}} = \frac{1}{\tau_{\eta u}} + C_\tau \frac{1}{\tau_{\eta \theta}} \left( 1 + C_\tau \sqrt{\frac{\Pr}{R}} \right). \tag{4.14}
\]

where \( C_\tau \) is a weighting constant. Now, (4.14) is not directly incorporated into the present \( \tilde{\varepsilon}_\theta \)-equation [(4.9)], but is indirectly reflected in the model constant \( C_{D2} \) and the model function \( f_{D2} \) as

\[
C_{D2}f_{D2} = C_{D2}^*f_{D2}^* \left[ 1 + C_{D3}^*f_{D3}^* \sqrt{\frac{\Pr}{R}} \left( 1 + C_\tau \sqrt{\frac{\Pr}{R}} \right) \right], \tag{4.15}
\]

with

\[
C_{D2}^*f_{D2}^* = (C_{\varepsilon_2}f_2 - 1) \left[ 1 - \exp \left( -y^*2 \right) \right]. \tag{4.16}
\]

Here, \( C_{D3}^* \), \( C_\tau \), and \( f_{D3}^* \) are \( C_{D3}^* = 0.025 \), \( C_\tau = 3.0 \), and \( f_{D3}^* = f_w \), respectively. [We would like to emphasize that the proposed \( \tilde{\varepsilon}_\theta \)-equation model with (4.15) can improve the accuracy of prediction without modifying the existing numerical scheme.] Equation (4.16) is established to reproduce the limiting behavior of free turbulence (see Youssef et al. (1992)). The term in square brackets on the right-hand side in (4.16) is needed to ensure that the molecular diffusion term in (4.9) will strictly balance with the destruction term directly related to the thermal field, i.e., \( C_{D1}f_{D1}\tilde{\varepsilon}_\theta \tilde{\varepsilon}_\theta/\kappa_\theta \), with the correct wall-limiting behavior of \( \tilde{\varepsilon}_\theta \propto y^2 \) (Kawamura and Kawashima 1994; Shikazono and Kasagi 1996).

### 4.2 Model Performance in Thermal Fields

In this section, we examine the validity of the proposed \( k_\theta-\tilde{\varepsilon}_\theta \) model for various flow conditions as follows.

- **Case A.** Channel flows with internal heat source.
- **Case B.** Channel flows with uniform wall heat flux.
- **Case C.** Channel flows with injection and suction.
- **Case D.** Reynolds and Prandtl number dependence for internal flows.
- **Case E.** Boundary layer flows under arbitrary wall thermal boundary conditions.
Table 6: Boundary conditions of the proposed \( k_\theta \tilde{\varepsilon}_\theta \) model for various flow fields.

<table>
<thead>
<tr>
<th>Case</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta_w )</td>
<td>const</td>
<td>( \alpha \left( \frac{\partial \Theta}{\partial y} \right)_w )</td>
<td>( -\frac{q_w}{\rho c_p} )</td>
<td>( k_{\theta_{wave}} )</td>
<td>( \tilde{\varepsilon}<em>{\theta</em>{wave}} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \Theta_{w_{wave}} )</td>
<td>const</td>
<td>( \alpha \left( \frac{\partial \Theta}{\partial y} \right)_w )</td>
<td>( -\frac{q_w}{\rho c_p} )</td>
<td>( k_{\theta_{wave}} )</td>
<td>( \tilde{\varepsilon}<em>{\theta</em>{wave}} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \Theta_{w_{wave}} )</td>
<td>on suction side</td>
<td>( \alpha \left( \frac{\partial \Theta}{\partial y} \right)_w )</td>
<td>( -\frac{q_w}{\rho c_p} )</td>
<td>( k_{\theta_{wave}} )</td>
<td>( \tilde{\varepsilon}<em>{\theta</em>{wave}} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \Theta_{w_{wave}} )</td>
<td>on injection side</td>
<td>( \alpha \left( \frac{\partial \Theta}{\partial y} \right)_w )</td>
<td>( -\frac{q_w}{\rho c_p} )</td>
<td>( k_{\theta_{wave}} )</td>
<td>( \tilde{\varepsilon}<em>{\theta</em>{wave}} )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
Case F. Backward-facing step flow for various Prandtl number fluids.

The numerical scheme and the grid system are the same as for the \( k-\overline{\varepsilon} \) model, as already mentioned. In the analysis of case D, however, 201 grid points were used for the calculation of relatively low Reynolds and/or Prandtl number fluids, but 1001 finer grid points were used to calculate thermal fields in high Reynolds and/or Prandtl number fluids with reference to the recent work by Sato et al. (1994). The wall-boundary conditions are collectively listed in Table 6. (Symmetric boundary conditions are imposed on various thermal turbulence quantities at the centerline of internal flows, while free-stream conditions are imposed for external flows, Hattori et al. (1993).)

4.2.1 Channel flow with internal heat source

First of all, we confirm the validity of the proposed \( k_\theta-\overline{\varepsilon}_\theta \) model in channel flows with internal heat source. Figures 16(a)–16(c) show the thermal-field turbulence quantities, compared with the DNS data of Kim and Moin (1989) at \( Re_\tau = 180 \). These figures reveal the excellent agreement between the predictions and the DNS data, and the Prandtl number dependence is adequately captured. Predicted budget profiles in various Prandtl number fluids are given in Figure 17. From these figures, we immediately notice the different contributions of each term in the \( k_\theta \)-equation under different \( Pr \) conditions. For example, the maximum value of the production term at a relatively high Prandtl number \( (Pr = 2.0) \) is located around \( y^+ = 10 \), which is identified with the interface between the viscous sublayer and the buffer layer in a velocity field, whereas for relatively low Prandtl number fluid \( (Pr = 0.1) \), the production term becomes a maximum at \( y^+ \simeq 30 \), which is almost equal to the interface between the buffer layer and the log-law region; this means that these differences are closely related to the development of the thermal boundary layer.

4.2.2 Channel flow with a uniform wall heat flux

Next, the proposed model is applied to flow with thermal boundary conditions at the wall different from the above case. The calculated profiles of thermal turbulence in a channel flow with a uniform wall heat flux are shown in Figure 18, in comparison with the DNS data of Kasagi et al. (1992) for air \( (Pr = 0.71) \) and those of Kasagi and Ohtsubo (1992) for mercury \( (Pr = 0.025) \). The mean temperature profiles in Figure 18(a) indicate that the present model efficiently predicts low Prandtl number fluids. We notice from Figure 18(b) that, despite the slight discrepancy in the predicted heat flux \( \overline{v\theta}^+ \) for mercury, the overall agreement between the predictions and the DNS is quite good. (Note that the slight discrepancy in the predicted \( \overline{v\theta}^+ \) at \( Pr = 0.025 \) exerts little influence on the behavior of the mean temperature profile because of the dominance
Figure 16: Thermal turbulence quantities in various Prandtl number fluids (channel flow with internal heat source): (a) mean temperature; (b) temperature fluctuation intensities; (c) turbulent heat flux.

Also, it can be seen from Figure 19, which shows the budget of the $k_\theta$-equation for air flow ($Pr = 0.71$), that the obtained profiles agree fairly well with the DNS data, and, in particular, that the predicted $\varepsilon_\theta$, which is given by $\varepsilon_\theta = \bar{\varepsilon}_\theta + 2\alpha (\partial/\partial y) (\Delta k_\theta/\partial y)^2$, can...
Figures 20(a) and 20(b) show the mean temperature normalized by $\Theta_w|_{suc}$, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/CBO9780511755385.008

4.2.3 Heat transfer in channel flow with injection and suction

We consider a channel flow with injection and suction, in which the thermal boundary condition on one wall is quite different from that on the other. Figures 20(a) and 20(b) show the mean temperature normalized by $\Theta_w|_{suc}$ –
Figure 18: Thermal turbulence quantities in air and mercury channel flows with uniform wall heat flux: (a) mean temperature; (b) temperature fluctuation intensities; (c) turbulent heat flux.

$\Theta_{w|\text{inj}}$ and turbulent heat flux $\overline{v\theta^+}$ normalized by $u_\tau$ and $\theta_\tau$ of each wall, together with the DNS data (Sumitani and Kasagi 1995). It is readily seen from the figures that, though the present $k_\theta-\varepsilon_\theta$ model shows small overpredictions of $\Theta$ on the injection side and of $\overline{v\theta^+}$ on the suction side, compared with the DNS,
Figure 19: Budget profile of the $k\theta$-equation in channel flow with uniform wall heat flux at $Pr = 0.71$.

Figure 20: Mean temperature and turbulent heat flux profiles in channel flow under a different wall temperature condition with wall transpiration: (a) mean temperature; (b) turbulent heat flux.

the model captures the essential characteristics of this complex thermal field. Thus, heat transfer control by injection and suction can be analyzed by the present model. One should note that the constant heat-flux layer ($-\bar{v}\theta^+ \approx 1$) does not exist in this type of flow.

4.2.4 Reynolds and Prandtl number dependence for internal flows

(a) Reynolds number dependence

Here we examine the performance of the proposed model over a wide range of Reynolds and Prandtl numbers. First, we compute mercury pipe flows ($Pr = 0.025$) under the constant-wall-temperature condition to investigate the Reynolds number dependence of the proposed model and compare the predictions with the available experimental data (Borishansky et al. 1964; Hochreiter and Sesonske 1974). As seen from Figure 21(a), the predicted $\Theta^+$ profile at
$Re_m = 10^5$ agrees with the experimental data (Borishansky et al. 1964) quite well. Also, the figure indicates that there are systematic variations of $\Theta^+$ with varying $Re_m$. A thermal field at low Reynolds number ($Re_m = 5 \times 10^3$) is mainly dominated by heat conduction, while as $Re_m$ increases, heat conduction is limited within the near-wall sublayer ($y^+ < 30$), and the effect of turbulent convection governs the remainder. We note that these variations are closely related to the activity of turbulence motion inducing strong turbulent heat flux ($\vec{v}\Theta$) at high $Re_m$ seen from Figure 21(b).

Figure 22 shows temperature fluctuations in a high-Reynolds-number ($Re_m = 50000$) mercury pipe flow (where $r_0$ in the figure denotes the radius of the pipe). It is clear from the figure that, though a little overprediction is seen in the central region of the pipe, the peak value of the predicted temperature fluctuation is in reasonable agreement with the measurements (Hochreiter and Sesonske 1974).

The Reynolds number dependence of turbulent heat transfer coefficient or the Nusselt number $Nu$ in various Prandtl number fluids ($Pr = 0.004 - 100$) is thoroughly investigated in Figure 23. Comparisons are made with several semi-empirical formulas for a pipe flow under a constant-wall-temperature condition (Bhatti and Shah 1987) [Note that Gnielinski’s formula in the figure implies the modified Petukhov’s correlation (see Bhatti and Shah (1987)).] Obviously, the predicted $Nu$ profiles at higher Prandtl numbers ($Pr \geq 7$) are in fairly good agreement with the Sandall et al. formula over a wide range of Reynolds numbers, and complete correspondence in air flow ($Pr = 0.71$) is obtained between the prediction and the Kays-Crawford formula. For the lower Prandtl number cases ($Pr = 0.004$ and $Pr = 0.025$), the Chen-Chiou formula gives the best fit with the present model. Note that as $Pr$ approaches zero, $Nu \simeq 5.5$ is obtained from the present model in the lower limit of Reynolds number, whereas some recent empirical formulas (Bhatti and Shah 1987) give $Nu \simeq 5.0$ for flows under a constant-wall-temperature condition.
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Figure 22: Temperature fluctuation in mercury pipe flow at high Reynolds number ($Re_m \approx 50000$).

Figure 23: Reynolds number dependence of turbulent heat transfer coefficient in pipe flows under a constant-wall-temperature condition at different Prandtl numbers ($0.004 \leq Pr \leq 100$).

(b) Prandtl number dependence

Next, we investigate the validity of the present model in various Prandtl number fluids. First, the model performance for low Prandtl number fluids ($Pr \leq 1$) at a constant Reynolds number ($Re_m = 10000$) and constant wall temperature is discussed. Thermal turbulence quantities of mean temperature $\Theta^+$, turbulent heat flux $\tilde{w}^{\theta^+}$, temperature fluctuation $\sqrt{k_\theta/\left(\Theta_w - \Theta_0\right)}$, and turbulent Prandtl number $Pr_t$ in pipe flow are presented in Figures 24(a)–24(d).

Owing to the lack of experimental data for various turbulence quantities at the corresponding Reynolds number, simply the results obtained by the proposed model under varying $Pr$ are discussed. It is readily found from those figures that there are systematic variations with varying $Pr$; for example, $\Theta^+$ shown in Figure 24(a) reveals that the conduction sublayer becomes increasingly thick...
Figure 24: Prandtl number dependence of various thermal turbulence quantities in pipe flow at $Re_m = 10000$: (a) mean temperature; (b) turbulent heat flux; (c) temperature fluctuation; (d) turbulent Prandtl number.

as $Pr$ decreases (e.g., the outer edge of the conduction sublayer is $y^+ \simeq 8$ at $Pr = 0.71$ while $y^+ \simeq 150$ at $Pr = 0.025$), and for $Pr > 0.1$, the logarithmic temperature distributions can clearly be recognized. It is also clear from Figure 24(b) of $\overline{v\theta}^+$ that the peak location of $\overline{v\theta}^+$ shifts toward the center of the pipe with decreasing $Pr$ and that the peak at $Pr < 0.1$ is around $y^+ \simeq 150$ in close accordance with the peak of temperature fluctuation mentioned below. A look at the temperature fluctuation profiles [Figure 24(c)] immediately reveals some distinct features. For example, its flattened distributions over the whole flow region can be seen for $Pr \leq 0.1$. When $Pr$ increases from 0.1, on the other hand, the temperature fluctuation, reaches its maximum in the vicinity of the wall. As for the turbulent Prandtl number $Pr_t$, some interesting trends can be seen from Figure 24(d). It is apparent that the predicted $Pr_t$ at $Pr = 0.71$ can correctly capture the tendency of the DNS obtained for air channel flow at a similar Reynolds number. It is also clear that the predicted $Pr_t$ tends to vary systematically as $Pr$ decreases: $Pr_t$ gradually decreases in the near-wall region ($1 < y^+ < 10$) and increases in the logarithmic region ($y^+ > 40$) from the standard value at $Pr = 0.71$. This can be recognized by investigating the molecular and turbulent transport terms of the energy equation. For fully developed turbulent flows, the energy equation reduces to
Figure 25: The mean temperature profile at a high Prandtl number fluid ($Pr = 95$, channel flow at $Re_m = 10000$).

Figure 26: Turbulent heat and mass transfer at various Prandtl and Schmidt numbers (pipe flow at $Re_m = 10000$).

\[
\frac{q}{q_w} = \left( \frac{1}{Pr} + \frac{1}{Pr_t \nu} \right) \frac{d\Theta^+}{dy^+}. \tag{4.17}
\]

The first term in brackets on the right-hand side denotes the molecular conduction and the second denotes the turbulent diffusion. The mean temperature distribution near the wall can be expressed as

\[
\Theta^+ = Pr y^+. \tag{4.18}
\]

This equation signifies that the heat transport near the wall is dominated by the molecular conduction, as confirmed by many experiments. In low Prandtl number fluids, it is confirmed from Figures 21 and 24 that the region where the molecular conduction predominates is extended to the logarithmic region of the velocity field. In this region, $\nu_t/\nu$ in (4.17) is large, so the turbulent Prandtl number should become larger to render the effect of the turbulent diffusion negligible. In the viscous sublayer, on the other hand, $\nu_t/\nu$ is much smaller than unity so the value of $Pr_t$ may be on the order of unity. It should be noted that the fact that the obtained $Pr_t$ at any $Pr$ in the immediate vicinity of the
wall approaches a constant value (about 1.21) is due to the near-wall behavior of the proposed $\nu_t$ and $\alpha_t$ models irrespective of the variation of $Pr$, i.e.,

$$Pr_t \simeq \frac{C_\mu A_\mu B_\mu}{C_\lambda A_\lambda B_\lambda \sqrt{2R/Pr}} \simeq 1.21,$$

as $y^+ \to 0$ (with $R \to Pr$).

(4.19)

Next, before evaluating the performance of the proposed model in higher Prandtl number fluids ($Pr \geq 1.0$), we discuss the predictive accuracy at extremely high Prandtl numbers. Owing to the difficulty of carrying out turbulent heat transfer experiments in higher Prandtl number fluids, the measurements reported up to now have been very few, and generally limited to mean temperature profiles; hence, the predictive accuracy of the present model is verified through a comparison of the prediction with the following measurements of mean temperature profiles. The predicted mean temperature profile in a fully developed channel flow ($Re_m = 10000$) of engine oil ($Pr = 95$) is shown in Figure 25 with the corresponding experimental data. It can be seen from the figure that the mean temperature profiles for both the prediction and the experiment show abrupt changes in the region adjacent to the wall ($y^+ < 10$). It is also clear that, though the beginning of the predicted almost constant mean temperature region is displaced slightly away from the wall in comparison with the measurement, the value obtained at the center of the channel agrees well with the experimental one. This means that either the wall friction temperature $\theta_\tau$ or the mean temperature gradient at the wall can be accurately obtained from the proposed model, and this is especially significant in accurately calculating the heat transfer rate mentioned below.

After an evaluation of the proposed model at high $Pr$ through a comparison of the prediction with the measurement, we now thoroughly examine the performance of the present model in various high $Pr$ fluids. In the high $Pr$ case, just as for a low $Pr$, systematic variations for various thermal turbulence quantities are confirmed from Figures 24(a)-24(d). We note that it is sufficient to show results up to $Pr = 10$, because fundamental characteristics of thermal turbulence at much higher Prandtl numbers can faithfully be captured at $Pr = 10$. It is found from Figure 24(a) that $\Theta^+$ becomes almost constant within the fully turbulent region as $Pr$ increases; this implies the thinning of the thermal boundary layer. The heat flux $\overline{w'\theta'}$ and temperature fluctuation $\sqrt{\overline{\theta'^2}}/(\Theta_w - \Theta_0)$ profiles shown in Figures 24(b) and 24(c) also exhibit systematic variations with increasing $Pr$: the constant heat flux layer ($\overline{w'\theta'} \simeq 1$) emerges in the wall region; and the maximum $k_\theta$ location shifts toward the wall. It should be emphasized that the present model, as already mentioned, can mimic the trend of both experiment and various empirical formulas for various Reynolds number flows with high accuracy, so that the predicted trend in various Prandtl number fluids also seems to capture the actual flow fields well. As for $Pr_t$ at high $Pr$, firstly, we immediately notice the opposite trend
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Figure 27: Thermal turbulence quantities at various streamwise locations in backward-facing step flow ($Re_H = 28000$, $\delta_0/H = 1.1$) with uniform wall heat flux for $Pr = 0.71$: (a) mean temperature; (b) turbulent heat flux.

with increasing $Pr$ [see Figure 24(d)] as found for low $Pr$, in particular, an abrupt increase of $Pr_t$ in the near-wall region ($1 < y^+ < 10$). Kays (1994) has pointed out that it is necessary for $Pr_t$ in high Prandtl number fluids to abruptly increase in the near-wall region as the wall is approached. Therefore, we conclude that such a behavior of the predicted $Pr_t$ is in the right direction.

Now, we discuss the dependence of the Prandtl number (or the Schmidt number $Sc$) on turbulent heat and mass transfer coefficients. Predictions of the Nusselt number $Nu$ and the Sherwood number $Sh$ in a pipe at $Re_m = 10000$ using the present model are presented in Figure 26. Since the predicted result corresponds fairly well with the various existing experimental data (see Side-man and Pinczewski (1975)) and empirical formulas (Harriott and Hamilton 1965; Azer and Chao 1961), the applicability of the present model to flows for a wide range of Prandtl numbers is duly verified.

4.2.5 Backward-facing step flow for various Prandtl number fluids

(a) Comparison of turbulence quantities in air flow

Finally, we assess the proposed model in complicated flow fields with heat transfer. The backward-facing step flow with a uniformly heated bottom wall downstream of the step, measured by Vogel and Eaton (1984), is arguably the best data for assessing the performance of the proposed model. The mean temperature $(\Theta - \Theta_\infty)/(\Theta_w - \Theta_\infty)$, turbulent heat flux $\overline{v\theta^+}$, and Stanton number $St = q_w/[\rho c_p U_0(\Theta_w - \Theta_0)]$ (where $c_p$ denotes specific heat at constant pressure) are illustrated in Figures 27 and 28, in comparison with the experimental data (Vogel and Eaton 1984) of $\delta_0/H = 1.1$, where $H$ is the step...
height and $\delta_0$ is the upstream boundary-layer thickness at the step location. Note that the predicted results for the corresponding velocity field are given in Figure 29. Again, in each figure, $X^*$ denotes the streamwise distance normalized by the reattachment length $X_R$, i.e., $X^* = (X - X_R)/X_R$. As seen from the mean temperature profiles in Figure 27(a), there is good agreement between the present prediction and the measurement at various streamwise locations, though the $Pr_t$-constant model gives consistently high values of $St$. This indicates that the conventional approach using a $Pr_t$-constant model has serious problems which cannot be overlooked in calculating a thermal field in complex flows of industrial interest. From the profiles of $\overline{v\theta}$ in Figure 27(b), we immediately notice quite different characteristics between the recirculation ($X^* < 0$) and redeveloping ($X^* > 0$) regions. Within the recirculation region, the prominent peak of $\overline{v\theta}$ appears in the shear layer near separation as well as in the vicinity of the heated wall, as found experimentally by Vogel and Eaton (1984).

The first peak near the heated wall is mainly caused by the steepness of mean temperature shown in Figure 27(a), while the second peak is induced by the strong turbulent motion in the separated shear layer caused by the steep mean-velocity gradient, as seen from Figure 29; this means that the two peaks of $\overline{v\theta}$ originate from quite different physical phenomena. Farther downstream, in the recirculation region, the second peak decreases, but the level within the recirculation region, in particular at $y/H \simeq 0.5$, increases. Vogel and Eaton (1984) have reported that in the middle of the recirculation region, the majority of the thermal transport from a heated wall is effected by large organized motions of the fluid. In the present study, the thermal transport by large organized motions is adequately reflected in the modelled $k_{\theta}$-equation through the turbulent diffusion model. Thus, the present model can properly reproduce the same tendency of $\overline{v\theta}$ as in the experiment, as already known from the agreement between the predicted and measured $\Theta$ profiles. Within the redeveloping region ($X^* > 0$) downstream of the reattachment, profiles of $\overline{v\theta}$ appear very similar to those of a flat plate boundary layer with the first peak remaining constant. The Stanton number $St$ distributions (Figure 28) demonstrate that the prediction is in qualitative and quantitative agreement with the experiment (Vogel and Eaton 1984) within the degree of experimental uncertainty. On the other hand, the behavior of the $Pr_t$-constant model is considerably different from the experimental one. In particular, the $Pr_t$-based model predicts a peak value of $St$ about twice the experiment. The Lauder and Sharma model reportedly gives a similar trend (Chieng and Launder 1980).

(b) Turbulent heat transfer for various Prandtl number fluids
We investigate the Prandtl number effect of turbulent heat transfer in backward-facing step flows. Figures 30 and 31 show the respective local maximum value
of $St$ and the mean Nusselt number $\overline{Nu}$ in the recirculation region (Kondoh et al. 1993) defined by

$$\overline{Nu} = \frac{\int_0^{X_R}Nu(x)dx}{X_R},$$

over a wide range of Prandtl numbers ($Pr = 1 \times 10^{-3} - 10^{2}$). (Here, local maximum value means the value that emerges around the reattaching point.) It is readily confirmed from Figure 30 that there are three sub-regions in the figure: (i) $Pr < 0.1$; (ii) $0.1 \leq Pr \leq 1$ and (iii) $1 < Pr$. Note that in the numerical analysis of laminar heat transfer in backward-facing step flows (Kondoh et al. 1993), similar characteristics with three different modes of behavior can be seen. Categories (i) and (iii) indicate a similar Prandtl number dependence varying approximately with $Pr^{0.5}$. On the other hand, within the middle range of Prandtl number [category (ii)], a quite different behavior is seen; $St_{\text{max}}$ becomes constant independent of the Prandtl number. In contrast to the $St_{\text{max}}$ profile, the configuration of $\overline{Nu}$ indicates that $\overline{Nu}$ varies in proportion to $Pr$ for $Pr < 1.0$, while the approximate relation $\overline{Nu} \propto Pr^{0.5}$ is obtained for $Pr > 1.0$. Furthermore, we find that, as $Pr$ decreases, $\overline{Nu}$ asymptotes to a constant value of 220; this means that heat conduction becomes overwhelming in the limit of low Prandtl number, as shown earlier in Figure 23 for pipe flow. Now, to further examine this uniqueness of $St_{\text{max}}$, it is interesting to note that...
we present contour maps of the mean temperature ($\Theta - \Theta_\infty$) in Figure 32 in which the contour value is normalized by the respective maximum temperature ($\Theta_{\text{max}} - \Theta_\infty$). The reattachment point is indicated by an arrow. It should be mentioned that the difference between the maximum and ambient temperature for $Pr > 10$ is too locally concentrated near the step corner to be drawn. In category (i) ($Pr < 0.1$), as seen from Figure 32(a), the thermal boundary layer remains thick even near the reattachment point, e.g., the thickness at $Pr = 0.01$ is about $0.5H$. This indicates that the thermal field downstream of the step is substantially governed by heat conduction. As $Pr$ increases from 0.1, i.e., in regime (ii), within the recirculation region [see Figure 32(b)], the obtained temperature distribution is markedly affected by the flow pattern in the central recirculation bubble: thermal convection becomes more and more dominant. For example, the temperature distribution in the recirculation region just behind the step wall is formed by an upward flow along the step wall, and hence the deterioration of turbulent heat transfer occurs because of the gradual variation of mean temperature, as seen from Figure 28. It is also found from the figures that $\Theta_{\text{max}}$ appears at the intersection between the central and the secondary recirculation bubbles, and that the thermal-boundary-layer thickness downstream of the reattachment point becomes rapidly thinner.
with increasing $Pr$. The thinning of the thermal boundary layer immediately induces a decrease in wall temperature or an increase of heat-transfer rate. This improvement of turbulent heat transfer strongly depends on the Prandtl number. The thickness appears to change in proportion to $Pr$, and hence a constant $St_{\text{max}}$ (or $Nu_{\text{max}} \propto Pr$) results. When the Prandtl number further increases [regime (iii)], as shown in Figure 32(c), though $\Theta_{\text{max}}$ appears at the intersection mentioned above, the temperature distribution associated with the upward flow along the step wall in the recirculation region is diminished. Furthermore, it is clear that the minimum-wall-temperature region spreads around the reattachment point. Therefore, the profiles of $St$ at high $Pr$ represent a plateau around this region (not shown).

## 5 Conclusions

Using the DNS data for turbulent wall shear flows with heat transfer, we have shown the methodology of how to construct a rigorous near-wall model for the temperature variance and its dissipation rate equations. In the $k \theta$- and $\varepsilon \theta$-equations, the turbulent diffusion terms are represented by gradient-type diffusion plus convection by large-scale motions. In the $\varepsilon \theta$-equation, all of the production and destruction terms are modelled to reproduce the correct behavior of $\varepsilon \theta$ near the wall. Note that, in order to obtain the correct wall-limiting behavior of $\varepsilon \theta$, it is of prime importance to have the correct $k \theta$ profile near the wall. It is also shown that the present model works very well for calculating the heat transfer under different thermal conditions. Furthermore, the present model reproduces the budget profiles of turbulence quantities as accurately as DNS. Thus, we anticipate a practical application of the present model in revealing the underlying physics of turbulent heat transfer in complex flows of technological interest.

In $k \theta$-$\varepsilon \theta$ modelling, the contributions from various eddies are also taken into consideration. Results obtained from the proposed $k \theta$-$\varepsilon \theta$ model in channel...
flows under arbitrary wall thermal boundary conditions show that the Prandtl number dependence identified by DNS is satisfactorily captured. The Reynolds and Prandtl (or Schmidt) number dependence of the proposed $k_\theta-\varepsilon_\theta$ model is thoroughly investigated in a pipe flow, and we have demonstrated that the predicted heat (and mass) transfer rate coincides with reliable empirical formulas and experimental data with high accuracy. Of course, this agreement cannot be achieved without a high-performance $k-\varepsilon$ model.

References


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