A DISTORTION THEOREM FOR ANALYTIC MAPS OF ANNULI

C. E. CASTONGUAY AND H. G. HELFENSTEIN

1. Introduction. Every abstract open Riemann surface can be made "concrete" (in the terminology of **(1)**) by considering it as a covering surface (in general branched) of the complex plane $\mathfrak{W}$ by means of a suitable projection map p. Since this covering map is not unique, it seems natural to single out some such maps by an extremal property. The use of Riemannian metrics compatible with the conformal structure on the given surface $\mathfrak{R}$ for the study of $\mathfrak{R}$ is well known; from the point of view of differential geometry it suggests an investigation of the distortion caused by p between such a metric $ds_{\mathfrak{R}}$ and the Euclidean metric of $\mathfrak{W}$. A definition of integral or average distortion involving the square of the logarithm of the local distortion (already introduced by Nevanlinna in **(6)** for a proof of the Picard–Landau theorem) is used. It has the disadvantage of depending on the system of co-ordinates used on $\mathfrak{R}$; the corresponding invariant integral involving the area-element of $ds_{\mathfrak{R}}$, however, does not exist in general—not even for the natural metrics of constant curvature.

We deduce sharp inequalities for this distortion for doubly connected surfaces, and the minimizing maps are determined. (In this case there exist unbranched (smooth) covering maps into $\mathfrak{W}$.) The simply connected case has been treated in **(5)**. The new result is applied in particular to the natural locally hyperbolic metric. It is noteworthy that some of the minimizing projections are not one-to-one maps.

Applications to the Bergmann and Lindelöf metrics as well as to analytic maps of the pseudosphere and generalizations to surfaces of higher connectivity will be considered elsewhere.

2. Definitions. Let $\mathfrak{R}$ be a doubly connected Riemann surface—that is, the fundamental group of $\mathfrak{R}$ is infinite cyclic. Let $\mathfrak{W}$ denote the (finite) complex w-plane, and let

$$\dot{\mathfrak{E}} = \{z|0<|z|<\infty\} = \text{the punctured Euclidean plane,}$$
$$\dot{\mathfrak{H}} = \{z|0<|z|<1\} = \text{the punctured hyperbolic plane,}$$
$$\mathfrak{A}_q = \{z|\sqrt{q} < |z| < 1/\sqrt{q}\}, \qquad 0 < q < 1.$$

$\mathfrak{R}$ is conformally equivalent to one and only one of the above standard surfaces **(7)**. Denote this surface by $\mathfrak{A}$, and let j be a one-one conformal map

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of $\mathfrak{R}$ onto $\mathfrak{A}$. For any smooth covering map p: $\mathfrak{R} \to \mathfrak{W}$, the function $f(z) = p \circ j^{-1}(z)$ is holomorphic in $\mathfrak{A}$ and satisfies $f'(z) \neq 0$. Since all the surfaces $\mathfrak{A}$ are "schlichtartig," $\mathfrak{A}$ thus furnishes a global system of isothermic co-ordinates for $\mathfrak{R}$, whereby we may define an "integral distortion" for any such map p,

$$D_z[p] = \iint_{\mathfrak{A}} [\ln \delta(z)]^2 \, dx \, dy,$$

where

$$\delta(z) = \text{local distortion at } z = |dw|/ds_{\mathfrak{R}} = |f'(z)|\gamma(z),$$

$$\gamma(z) = |dz|/ds_{\mathfrak{R}} > 0 \text{ in } \mathfrak{A}, \qquad \text{and } z = x + iy,$$

with integration in the Lebesgue sense. Obviously $D_z[p] = 0$ entails $\delta(z) \equiv 1$ almost everywhere, i.e. absence of distortion.

In order to make a later statement (§3.5) independent of the co-ordinate system, we define, for an arbitrary real constant C,

$$D_z[p, C] = \iint_{\mathfrak{A}} [\ln \delta(z) - C]^2 \, dx \, dy.$$

In the following we restrict ourselves to the cases $\mathfrak{A} = \mathfrak{A}_q$ and $\mathfrak{A} = \dot{\mathfrak{H}}$. The preceding definition yields infinite values for $D_z[p]$ in the case of $\dot{\mathfrak{C}}$.

We offer the following suggestion for an invariant definition of distortion which is patterned after a somewhat similar situation in **(2)**. (By distinguishing between $\mathfrak{R}$ and $\mathfrak{A}$ we take into account that in differential geometry a surface with a metric is generally not given by its simplest conformally equivalent representation; cf. **(3)**.) If k is any other one-one conformal map of $\mathfrak{R}$ onto $\mathfrak{A}$, then there exists a unique conformal automorphism h of $\mathfrak{A}$ such that the diagram

$$\begin{tikzcd}
\mathfrak{A} \arrow[drr, "g = f \circ h^{-1}"] & & \\
& \mathfrak{R} \arrow[ul, "k"'] \arrow[dl, "j"] \arrow[r, "p"] & \mathfrak{W} \\
\mathfrak{A} \arrow[uu, "h"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

is commutative. Accordingly, relative to the new system of co-ordinates $h(z) = \zeta = \xi + i\eta$, we have

$$D_\zeta[p] = \iint_{h(\mathfrak{A})} [\ln \delta(\zeta)]^2 \, d\xi \, d\eta.$$

The local distortion being invariant, we find that

$$D_\zeta[p] = \int\int_{\mathfrak{A}} [\ln \delta(z)]^2 \left|\frac{d\zeta}{dz}\right|^2 dx\, dy.$$

To make the measure of distortion independent of the co-ordinate system we may define

$$D[p] = \inf_h D_{h(z)}[p],$$

where h ranges over all conformal automorphisms of $\mathfrak{A}$.

The group of conformal automorphisms of $\mathfrak{A}_q$ consists of the transformations $\zeta(z) = e^{i\tau}z$ and $\zeta(z) = e^{i\tau}z^{-1}$, τ real; hence

$$D[p] = \min\left\{ \int\int_{\mathfrak{A}_q} \ln^2\delta(z) dx\, dy,\ \int\int_{\mathfrak{A}_q} \ln^2\delta(z)|z|^{-2}\, dx\, dy \right\}.$$

Even with this simplification one can hardly compute $D[p]$ for a general metric $ds_{\mathfrak{A}}$; hence we restrict ourselves to the consideration of distortion relative to one system of co-ordinates in the annuli $\mathfrak{A}_q$. The situation is simpler for $\dot{\mathfrak{H}}$, since all the conformal automorphisms of this surface are of the form $\zeta(z) = e^{i\tau}z$. Hence

$$D_\zeta[p] = D_z[p] = D[p].$$

3. Distortion theorems.

1. Let $\mathfrak{R} = \mathfrak{A}_q$, and let $\phi(z) = \ln \gamma(z)$ have in $\mathfrak{A}_q$ the Fourier expansion

$$\phi(z) = \sum_{-\infty}^{+\infty} a_k(r)e^{ik\theta},$$

with $z = re^{i\theta}$ and

$$(1)\qquad a_k(r) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(z)e^{-ik\theta}\, d\theta \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots.$$

We shall assume later that ϕ is at least piecewise smooth in θ, so that it may be represented by its Fourier series. Define for all integers k the following quantities depending on the given metric of $\mathfrak{A}_q$:

$$(2)\qquad \begin{aligned} L_k &= \int_{\surd q}^{1/\surd q} |a_k(r)|^2\, rdr, & M_k &= \int_{\surd q}^{1/\surd q} a_k(r)r^{k+1}dr,\\ N_k &= \int_{\surd q}^{1/\surd q} a_k(r)r^{-k+1}dr, & \alpha_k &= \int_{\surd q}^{1/\surd q} r^{2k+1}dr,\\ \beta &= \int_{\surd q}^{1/\surd q} \ln r\cdot rdr, & \gamma &= \int_{\surd q}^{1/\surd q} \ln^2 r\cdot rdr,\\ \delta &= \int_{\surd q}^{1/\surd q} a_0(r)\ln r\cdot rdr, & \Delta_k &= \begin{cases} \alpha_k\,\alpha_{-k} - {\alpha_0}^2 & \text{if } k \neq 0,\\ \alpha_0\,\gamma - \beta^2 & \phantom{\text{if }} k = 0.\end{cases} \end{aligned}$$

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Clearly, $\alpha_k > 0$ and $\Delta_k > 0$, for all k, by the Schwarz inequality. Finally, we need the following combinations:

$$(3) \qquad A_k = \frac{\alpha_0 N_k - \alpha_{-k} M_k}{\Delta_k}, \qquad k \neq 0,$$

$$\chi = \frac{\beta M_0 - \alpha_0 \delta}{\Delta_0} = \text{“characteristic exponent'' of the givin metric,}$$

$A_0 = a$ nearest integer to χ,

$$B_0 = -\frac{N_0 + \beta A_0}{\alpha_0},$$

$$C_k = \begin{cases} L_k + \dfrac{1}{\Delta_k}\{\alpha_0(M_k \overline{N_k} + \overline{M_k} N_k) - \alpha_{-k}|M_k|^2 - \alpha_k|N_k|^2\}, & k \neq 0, \\ L_0 + \dfrac{1}{\Delta_0}\{2\beta\delta M_0 - \alpha_0\delta^2 - \gamma M_0^2\}, & k = 0. \end{cases}$$

With these notations we prove

THEOREM 1. (a) *If ϕ is piecewise smooth in θ and square-integrable over $\mathfrak{A}_q$, then for any smooth covering map p: $\mathfrak{A}_q \to \mathfrak{W}$ the following inequality holds:*

$$(4) \qquad D_z[p] \geqslant 2\pi\left\{\frac{\Delta_0}{\alpha_0}|A_0 - \chi|^2 + \sum_{-\infty}^{+\infty} C_k\right\}.$$

(b) *If in addition $\partial^2\phi/\partial\theta^2$ exists, is piecewise continuous and bounded in $\mathfrak{A}_q$, then*

$$(5) \qquad g(z) = \exp(B_0 + i\psi_0)\cdot z^{A_0}\cdot\exp\left(2\sum_{-\infty}^{+\infty}{}' A_k z^k\right)$$

is an analytic (and one-valued) function in $\mathfrak{A}_q$, with ψ_0 an arbitrary real constant and $\sum'$ denoting summation over $k \neq 0$.

(c) *Under the conditions (b), the inequality (4) is sharp if and only if*

$$\oint_{|z|=1} g(z)dz = 0,$$

with equality holding then if and only if $p = \int g(z)dz \circ j$, ψ_0 and a complex constant of integration being arbitrary.

Proof of (a). Given $p = f \circ j$: $\mathfrak{A}_q \to \mathfrak{W}$, set

$$\Phi(z) = \ln|f'(z)| = \operatorname{Re}\log f'(z).$$

Under the projection $j = e \circ j$: $\mathfrak{A}_q \to \mathfrak{W}$, where $e(z) \equiv z$, we have $\Phi(z) \equiv 0$ and so $D_z[j] < \infty$, since $\ln\delta = \phi$ is square-integrable over $\mathfrak{A}_q$. Consequently we shall establish inequality (4) among smooth projection maps with finite distortion, i.e. we shall assume that $\ln\delta = \Phi + \phi$ is square-integrable over $\mathfrak{A}_q$. $\Phi(z)$ being harmonic in $\mathfrak{A}_q$, $\Phi + \phi$ is piecewise smooth in θ; let

$$\sum_{-\infty}^{+\infty} b_k(r)e^{ik\theta}$$

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be its Fourier representation. By the Lebesgue-Fubini theorem and the completeness relation we obtain

(6)
$$\begin{aligned} D_z[p] &= \int_{\sqrt{q}}^{1/\sqrt{q}} rdr \cdot \int_0^{2\pi} (\phi + \phi)^2\, d\theta \\ &= 2\pi \int_{\sqrt{q}}^{1/\sqrt{q}} \sum_{-\infty}^{+\infty} |b_k(r)|^2\, rdr. \end{aligned}$$

Let

$$h_n(r) = \sum_{-n}^{+n} |b_k(r)|^2 r \quad \text{and} \quad \lim_{n\to\infty} h_n(r) = h(r).$$

From $h_n(r) \leqslant h(r)$ and (6), we find that

$$\int_{\sqrt{q}}^{1/\sqrt{q}} h_n(r)dr \leqslant \int_{\sqrt{q}}^{1/\sqrt{q}} h(r)dr < \infty.$$

$(h_n(r))_{n=0,1,2,\ldots}$ is therefore an increasing sequence of integrable real-valued functions; applying the Monotone Convergence theorem **(9)**, we may write (6) as

$$\begin{aligned} D_z[p] &= 2\pi \int_{\sqrt{q}}^{1/\sqrt{q}} h(r)dr = 2\pi \lim_{n\to\infty} \int_{\sqrt{q}}^{1/\sqrt{q}} h_n(r)dr \\ &= 2\pi \lim_{n\to\infty} \sum_{-n}^{+n} \int_{\sqrt{q}}^{1/\sqrt{q}} |b_k(r)|^2 rdr \end{aligned}$$

(7)
$$= 2\pi \sum_{-\infty}^{+\infty} \int_{\sqrt{q}}^{1/\sqrt{q}} |b_k(r)|^2 rdr.$$

The function

$$\Phi = \sum_{-\infty}^{+\infty} [b_k(r) - a_k(r)] e^{ik\theta}$$

is harmonic. Integrating $\nabla^2\Phi = 0$, we find that

(8)
$$b_k(r) = \begin{cases} E_k r^k + F_k r^{-k} + a_k(r) & \text{if } k \neq 0, \\ E_0 \ln r + F_0 + a_0(r) & \phantom{\text{if }} k = 0, \end{cases}$$

where E_k, F_k are complex constants depending on $f(z)$. From (7) and (8) we obtain now

(9)
$$\begin{aligned} D_z[p] = 2\pi \Bigg\{ & \frac{\Delta_0}{\alpha_0} |E_0 - \chi|^2 + \frac{1}{\alpha_0} |\beta E_0 + \alpha_0 F_0 + N_0|^2 \\ & + C_0 + \sum_{-\infty}^{+\infty}{}' \left[\frac{\Delta_k}{\alpha_{-k}} \left| E_k - \frac{\alpha_0 N_k - \alpha_{-k} M_k}{\Delta_k} \right|^2 \right. \\ & \left. + \frac{1}{\alpha_{-k}} |\alpha_0 E_k + \alpha_{-k} F_k + N_k|^2 + C_k \right] \Bigg\}. \end{aligned}$$

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Since Φ and $\Phi + \phi$ are real-valued and $\bar{b}_k = b_{-k}$, we have $\bar{a}_k = a_{-k}$. Hence from (8)

$$\bar{E}_k = F_{-k}, \quad k \neq 0; \qquad \bar{E}_0 = E_0, \qquad \bar{F}_0 = F_0. \tag{10}$$

Calculating the harmonic conjugate of

$$\Phi(z) = E_0 \ln r + F_0 + \sum_{-\infty}^{+\infty}{}' (E_k r^k + F_k r^{-k}) e^{ik\theta}$$

and using (10), we find that $f'(z)$ is of the form

$$f'(z) = \exp(F_0 + i\psi_0) z^{E_0} \cdot \exp\left(2 \sum_{-\infty}^{+\infty}{}' E_k z^k\right), \tag{11}$$

with ψ_0 a real constant. Since $f(z)$ was one-valued analytic, we conclude that E_0 must be an integer. (a) now follows from (9).

Proof of (b). Writing

$$\sum_{-\infty}^{+\infty}{}' A_k z^k = \sum_1^{\infty} A_k z^k + \sum_1^{\infty} A_{-k} w^k$$

with $w = 1/z$, we first establish the uniform convergence of

$$\sum_1^{\infty} A_k z^k$$

in $|z| < 1/\sqrt{q}$; the second term may be treated in a similar way. Let

$$|\partial^2 \phi / \partial \theta^2| < D < \infty \qquad \text{in } \mathfrak{A}_q,$$

with D independent of r and θ. Twice repeated integration by parts of (1) yields

$$|a_k(r)| < Dk^{-2}, \qquad k \geqslant 1.$$

It is then easy to see that

$$|M_k|, |N_k| < Dk^{-3} q^{-(\frac{1}{2}k+1)}$$

for large k. Also, for k large enough, we have

$$\alpha_0 < \alpha_{-k}, \qquad \alpha_k > \tfrac{1}{5} k^{-1} q^{-(k+1)}, \qquad \Delta_k > \tfrac{1}{2} \alpha_k \alpha_{-k}.$$

Hence we conclude, for k sufficiently large, say $k \geqslant k_0(q)$,

$$\begin{aligned} |A_k| &= \frac{1}{\Delta_k} |\alpha_0 N_k - \alpha_{-k} M_k| \\ &< \frac{2}{\alpha_k \alpha_{-k}} |\alpha_0 N_k - \alpha_{-k} M_k| \\ &< \frac{2}{\alpha_k} [|N_k| + |M_k|] \\ &< 20 D k^{-2} q^{\frac{1}{2}k}. \end{aligned}$$

Therefore for $|z| < 1/\sqrt{q}$ and $k \geqslant k_0$,

$$|A_k z^k| < 20Dk^{-2},$$

and the uniform convergence follows.

Proof of (c). The vanishing of the period $\oint_{|z|=1} g(z)dz$ is necessary and sufficient for

$$\int_1^z g(t)dt$$

to be one-valued and analytic in $\mathfrak{A}_q$. The rest is clear from (9), where the variables E_k, F_k appear within squares multiplied by positive coefficients.

2. COROLLARY. *Suppose that* $\gamma(z) = \gamma(r)$. *Then inequality* (4) *holds with* $C_k = 0$, $k \neq 0$; *it is sharp if and only if* $A_0 \neq -1$, *with equality holding then if and only if*

$$p = \left[\frac{\exp(B_0 + i\psi_0)}{A_0 + 1} z^{A_0+1} + \psi_1\right] \circ j,$$

with ψ_0 *and* ψ_1 *arbitrary constants, real and complex respectively.*

Proof. From $a_k(r) \equiv 0$, $k \neq 0$, we conclude that

$$M_k = N_k = L_k = C_k = 0, \qquad k \neq 0.$$

Hence $g(z) = \exp(B_0 + i\psi_0).z^{A_0}$, and $\int g(t)dt$ is one-valued analytic in $\mathfrak{A}_q$ if and only if $A_0 \neq -1$.

3. If $\gamma(z) = \gamma(r)$ and $A_0 = -1$, then the inequality (4) cannot be sharp—that is, in this case there exists no map with minimal distortion. The right-hand side of (4) may yet be, however, the greatest lower bound for the distortions of the projection maps. Knowing that for any such $p = f \circ j$, f is an analytic map such that (11) holds with E_0 an integer, we see that this question is equivalent to the following: With $E_0 = A_0 = -1$, is it possible to choose the complex constants E_k, $k \neq 0$, in such a way that

$$\sum_{-\infty}^{+\infty}{}' \frac{\Delta_k}{\alpha_{-k}} |E_k|^2$$

is arbitrarily small and $\oint_{|z|=1} f'(z)dz = 0$? This problem remains open.

4. Suppose now that $\mathfrak{R} = \dot{\mathfrak{H}}$ and $\gamma(z) = \gamma(r)$. Defining for $\dot{\mathfrak{H}}$ the quantities α_0, β, M_0, etc. as before (except for the appropriate limits in the integrals) we can state

THEOREM 2. *If* $\phi(r)$ *is square-integrable over* $\dot{\mathfrak{H}}$, *then for any smooth* $p: \dot{\mathfrak{H}} \to \mathfrak{W}$, *we have*

$$D[p] \geqslant 2\pi\left\{\frac{\Delta_0}{\alpha_0} |A_0 - \chi|^2 + C_0\right\}.$$

This inequality is sharp if and only if $A_0 \neq -1$.

Proof. We obtain

$$D[p] = 2\pi \int_0^1 \left[|b_0(r)|^2 + \sum_{-\infty}^{+\infty}{}' \; |E_k r^k + F_k r^{-k}|^2 \right] rdr$$
$$\geqslant 2\pi \int_0^1 |b_0(r)|^2 \, rdr,$$

and the proof carries on as before.

5. Under the same assumptions as before, one can deduce similarly as for $D_z[p]$ an inequality $D_z[p, C] \geqslant M$ for arbitrary real constant C, with M denoting a non-negative finite lower bound independent of C and p. Letting $N = \exp(M/|\mathfrak{A}|)^{\frac{1}{2}}$ (where $|\mathfrak{A}|$ denotes the Euclidean area of $\mathfrak{A}$) we obtain from the mean-value theorem the following invariant inequality for the local distortion: For every real constant C and every covering map p there exists a point $P_0 \in \mathfrak{R}$ such that either

$$\delta_p(P_0) \geqslant Ne^C \quad \text{or} \quad \delta_p(P_0) \leqslant \frac{1}{N} e^C.$$

Eliminating C we deduce the following: *Let a be any positive real number. Then no open interval (a, aN^2) contains all local distortions δ_p of any covering map p.*

4. Applications.

1. Every Riemann surface $\mathfrak{R}$ has a universal covering surface $\mathfrak{R}$ which is either the Riemann sphere $\mathfrak{S}$, the Euclidean plane $\mathfrak{E}$, or the hyperbolic plane $\mathfrak{H}$ (Uniformization Theorem, **7**). Correspondingly, $\mathfrak{R}$ is said to be of elliptic, parabolic, or hyperbolic type. Any projection map p: $\mathfrak{R} \to \mathfrak{R}$ induces on $\mathfrak{R}$ a natural Riemannian metric of constant curvature (locally spherical, locally Euclidean, or locally hyperbolic metric.

$\dot{\mathfrak{H}}$ is of hyperbolic type. If $\mathfrak{H}$ is represented by the upper half ζ-plane with line-element $ds_{\mathfrak{H}} = |d\zeta|/\mathrm{Im}(\zeta)$, then a covering map of $\mathfrak{H}$ onto $\dot{\mathfrak{H}}$ is given by $z = e^{i\zeta}$, and the induced locally hyperbolic metric on $\dot{\mathfrak{H}}$ becomes

$$ds_{\dot{\mathfrak{H}}} = \frac{|dz|}{|z| \ln(1/|z|)} . \tag{12}$$

Each $\mathfrak{A}_q$ is also of hyperbolic type. With the same representation for $\mathfrak{H}$, the following is a projection $\mathfrak{H} \to \mathfrak{A}_q$:

$$z = \sqrt{q} \exp\!\left(\frac{\ln(1/q)}{i\zeta} \ln \zeta\right),$$

with induced metric on $\mathfrak{A}_q$:

$$ds_{\mathfrak{A}_q} = \frac{\pi |dz|}{\ln(1/q)|z| \cos(\pi \ln|z|/\ln q)} . \tag{13}$$

Since these expressions are invariant with respect to the group of conformal automorphisms of $\dot{\mathfrak{H}}$ and $\mathfrak{A}_q$, they represent essentially the only complete Riemannian metrics of constant curvature possible on these surfaces. In both cases $\gamma(z) = \gamma(r)$; from the calculations that follow it will become evident that $\phi(r)$ is square-integrable so that we are justified in applying the preceding distortion theorems. We shall demonstrate the existence of minimizing conformal representations for the surfaces $\dot{\mathfrak{H}}$ and $\mathfrak{A}_q$ with metrics (12) and (13) respectively by estimating A_0 in each case.

2. THEOREM 3. *For $\dot{\mathfrak{H}}$ with metric* (12), *any smooth covering map p: $\dot{\mathfrak{H}} \to \mathfrak{W}$ satisfies $D[p] \geqslant \pi(\frac{1}{6}\pi^2 - 1)$, with equality holding only for*

$$p(z) = \exp(\xi + 1 + i\psi_0)\cdot z^2 + \psi_1,$$

where ξ denotes Euler's constant, and ψ_0 and ψ_1 are arbitrary constants, real and complex respectively.

Proof. Substituting $\phi(r) = \ln r + \ln \ln (1/r)$, we obtain for the "characteristic exponent" (3), $\chi = 1$, and thus $A_0 = 1$. (For some of the non-elementary integrals appearing in (2) see Appendix.)

3. THEOREM 4. *For an annulus with metric* (13), *any p: $\mathfrak{A}_q \to \mathfrak{W}$ satisfies the following inequality which is best possible:*

$$D_z[p] \geqslant 2\pi\left\{\frac{\Delta_0(s)}{\alpha_0(s)}\,|A_0(s) - \chi(s)|^2 + C_0(s)\right\}, \tag{14}$$

where the symbols occurring have the following meaning:

$$s = (1/\pi)\ln(1/q), \qquad \alpha_0 = \sinh(\pi s),$$

$$\Delta_0 = \tfrac{1}{4}[\sinh^2(\pi s) - (\pi s)^2], \qquad \psi(z) = \Gamma'(z)/\Gamma(z),$$

$$\chi(s) = -\frac{2s\sinh^2(\pi s)}{\sinh^2(\pi s) - (\pi s)^2}\operatorname{Im}[\psi'(is)] - 1, \tag{15}$$

$$C_0(s) = -\sinh(\pi s)\left\{\frac{\pi^2}{6} + \frac{1}{2}\operatorname{Re}[\psi'(is)] + \frac{1}{2}s^{-2} + \frac{\sinh^2(\pi s)\cdot s^2}{\sinh^2(\pi s) - (\pi s)^2}\operatorname{Im}^2[\psi'(is)]\right\}, \tag{16}$$

$$A_0(s) = \begin{cases} 1 & s > \sigma, \\ 0 & \text{for } \sigma > s > 0, \\ 0 \text{ or } 1 & s = \sigma; \end{cases} \tag{17}$$

$$\operatorname{Re} = \text{real part of}, \qquad \operatorname{Im} = \text{imaginary part of}.$$

The number σ is the (*single*) *solution of the transcendental equation $\chi(s) = \frac{1}{2}$ in $s > 0$ and satisfies $0.33 < \sigma < 0.34$.*

Hence for large conformal modulus (i.e. q near 1: narrow annuli) the minimizing maps are similarities, but wide annuli (q small) are mapped on double-sheeted rings. Thus, surprisingly, the minimal coverings are either "schlicht" or double-sheeted, and no higher numbers of sheets occur. There is one single annulus corresponding to $s = \sigma$ where both simple and double covering lead to the same minimal distortion.

Proof of Theorem 4. The following proof makes use of some lemmas whose treatment is postponed to §4.4–6. Taking into account some non-elementary integrals occurring in (2) and (3) which are evaluated in §5, we obtain (15) and (16). In order to prove (17) we show first that $-1/2 < \chi(s) < 3/2$ for all $s > 0$, whence A_0 is either 0 or 1. Using the expansion in partial fractions

$$\psi(z) = -\xi + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{z+k}\right),$$

we rewrite (15) in the form

$$c(s) = \chi(s) + 1 = 4b(s)\Sigma(s),$$

with

$$b(s) = \frac{\sinh^2(\pi s)}{\sinh^2(\pi s) - (\pi s)^2}, \qquad \Sigma(s) = \sum_{n=1}^{\infty} \frac{n}{(n^2+s^2)^2}.$$

We now estimate $\Sigma(k)$ for a positive integer k by its Euler-MacLaurin expansion, truncating after the third term, within the following error (see §4.5):

$$\Sigma(k) = \frac{1}{2}\frac{1}{1+k^2} + \frac{5}{12}\frac{1}{(1+k^2)^2} + \frac{1}{3}\frac{1}{(1+k^2)^3} \pm \frac{5}{9}\frac{1}{k^6}. \tag{18}$$

If $I = [s', s'']$ is any finite closed subinterval of $[0, \infty)$, since $b(s)$ is strictly increasing (see §4.4) and $\Sigma(s)$ is decreasing, we have

$$\min_{s\in I} c(s) > 4\Big[\min_{s\in I} b(s)\Big]\cdot\Big[\min_{s\in I} \Sigma(s)\Big] = 4b(s')\Sigma(s''),$$

$$\max_{s\in I} c(s) < 4\Big[\max_{s\in I} b(s)\Big]\cdot\Big[\max_{s\in I} \Sigma(s)\Big] = 4b(s'')\Sigma(s').$$

We shall show that $c(s) > \frac{1}{2}$ by partitioning $[0, \infty)$ into appropriate subintervals $[s', s'']$ and checking that $4b(s')\Sigma(s'') > \frac{1}{2}$ for each such interval. This is easily done for each of the intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, 3/2]$, and $[3/2, 2]$. Thus $c(s) > \frac{1}{2}$ for $0 < s < 2$. $b(0)$ is needed for the first of these calculations; we define

$$b(0) = \lim_{s\to 0} b(s) = \frac{3}{\pi^2}.$$

Suppose now that $s \in [n, n+1]$, n an integer $\geqslant 2$. Then

$$
\begin{aligned}
c(s) &> 4b(n)\Sigma(n+1) > 4n^2\Sigma(n+1) \\
&> \frac{2n^2}{n^2+2n+2} - \frac{20}{9}\cdot\frac{1}{n^4} \qquad \text{from (18)} \\
&> \tfrac{1}{2} \qquad \text{for } n \geqslant 2.
\end{aligned}
$$

Thus $c(s) > \frac{1}{2}$ for $s > 0$.

The proof of $c(s) < 5/2$ is similar. A partition of $[0, 10]$ into intervals of length $1/10$ proves sufficiently fine to check that $\max c(s) < 5/2$ for $0 < s < 10$. If $s \in [n, n+1]$, n an integer $\geqslant 10$, then using $b(n) \approx n^2$, since $\sinh(\pi s) > 10^7$, we obtain

$$
\begin{aligned}
c(s) &< 4b(n+1)\Sigma(n) \approx 4(n+1)^2\Sigma(n) \\
&< 2 + \frac{12n+5}{3(1+n^2)} + \frac{10n+4}{3(1+n^2)^2} + \frac{8n}{3(1+n^2)^3} \\
&\qquad + \frac{(n+1)^2}{n^6}\cdot\frac{20}{9} \qquad \text{(from (18))} \\
&< 2.43 \qquad \text{for } n \geqslant 10.
\end{aligned}
$$

Therefore $c(s) < 5/2$ for $s > 0$. From the above calculations and (16) we remark that $C_0(s)$ varies approximately as $\sinh(\pi s)$; from

$$
\frac{\Delta_0}{\alpha_0} = \frac{\sinh^2(\pi s) - (\pi s)^2}{4\sinh(\pi s)}
$$

we see that the right-hand side of (14) also varies roughly as $\sinh(\pi s)$.

A partition process, similar to those used above, with mesh diameter $1/1000$, was programmed for an IBM 1650 electronic computer and yielded the following results:

$$
\begin{aligned}
c(s) &< 3/2 \qquad \text{for } s \in [0, 0.33], \\
c(s) &> 3/2 \qquad \text{for } s \in [0.35, 3].
\end{aligned}
$$

In §4.6 $c'(s)$ is shown to be positive in the intervals $[0.33, 0.35]$ and $[3, \infty)$. Also, $c(0.34) > 3/2$ by direct calculation. Combining these results with the previous remarks, we conclude that there exists a value σ, $0.33 < \sigma < 0.34$, such that $c(\sigma) = 3/2$ and

$$
1/2 < c(s) < 3/2 \quad \text{for } 0 < s < \sigma, \qquad 3/2 < c(s) < 5/2 \quad \text{for } \sigma < s.
$$

Knowing $c'(s) > 0$ in $[0.33, 0.34]$, the critical value of the conformal modulus $q_c = e^{-\pi\sigma}$ can be computed as accurately as desired.

4. Lemma. *$b'(s) > 0$ for $s > 0$.*

Proof. Because

$$
b'(s) = 2\pi^3 b^2(s)\left[(\pi s)^{-3} - \frac{\cosh(\pi s)}{\sinh^3(\pi s)}\right],
$$

it is sufficient to show that

$$h(t) = \frac{\sinh^3 t}{\cosh t} - t^3 > 0 \qquad \text{for } t > 0.$$

This follows from

$$h(0) = \dot{h}(0) = \ddot{h}(0) = 0,$$

and

$$\dddot{h}(t) = \frac{2 \sinh^4 t}{\cosh^4 t} (4 \cosh^2 t + 3) > 0.$$

5. *Proof of* (18). Letting $f(x) = x/(x^2 + s^2)^2$, we obtain

$$f''(x) = \frac{12x(x^2 - s^2)}{(x^2 + s^2)^4} \left. \begin{matrix} \leqslant 0 \\ \geqslant 0 \end{matrix} \right. \qquad \text{for } \begin{matrix} x \leqslant s, \\ x \geqslant s, \end{matrix}$$

which permits us to use Euler's formula twice:

$$\text{(19)} \quad \sum_{n=1}^{k-1} \frac{n}{(n^2 + k^2)^2} = \sum_{n=1}^{k-1} f(n) = \int_1^k f(t)dt - \tfrac{1}{2} f(t) \Big]_1^k + \frac{1}{12} f'(t) \Big]_1^k + E_1(k),$$

$$\text{(20)} \quad \sum_{n=k}^{\infty} \frac{n}{(n^2 + k^2)^2} = \int_k^{\infty} f(t)dt + \tfrac{1}{2} f(k) - \frac{1}{12} f'(k) + E_2(k),$$

and estimate

$$|E_1(k)| < 2 \left| \frac{B_4}{4!} [f'''(k) - f'''(1)] \right|,$$

$$|E_2(k)| < 2 \left| \frac{B_4}{4!} f'''(k) \right|.$$

We find that

$$f'''(k) = 3k^{-6}/2 \quad \text{and} \quad f'''(1) < 12 \times 16 \times k^{-6},$$

whence

$$|E_1(k) + E_2(k)| < 5k^{-6}/9.$$

Adding (19) and (20) we obtain (18).

6. Lemma. *$c'(s) >$ for $s \in I_0$* $= [0.33, 0.35]$ *and* $s \geqslant 3$.

Proof. We consider the function $p(s)$ defined by

$$\text{(21)} \qquad c'(s) = \frac{8s \sinh(\pi s)}{[\sinh^2(\pi s) - (\pi s)^2]^2} p(s)$$

$$= \frac{8s \sinh(\pi s)}{[\sinh^2(\pi s) - (\pi s)^2]^2}$$

$$\times [\sinh^3(\pi s) - \cosh(\pi s)(\pi s)^3] \Sigma(s)$$

$$- 2s^2 \sinh(\pi s)[\sinh^2(\pi s) - (\pi s)^2] \sum_{n=1}^{\infty} \frac{n}{(n^2 + s^2)^3}.$$

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Calculations yield $p\ (0.35) > p\ (0.33) \geqslant 0.0135$. We approximate $p(s)$ in I_0 by the linear interpolating polynomial

$$P(s) = 0.0135 + (s - 0.0135)\frac{p(0.35) - p(0.33)}{0.35 - 0.33},$$

with an accuracy

$$|P(s) - p(s)| < \tfrac{1}{8}\,(0.35\text{–}0.33)^2\,|p''(s_0)|, \qquad s_0 \in I_0;$$

see **(8)**. Computations show that $|p''(s)| < 4$ in I_0, whence

$$|P(s) - p(s)| < 2 \times 10^{-4}.$$

Since $P(s) \geqslant 0.0135$ in I_0, we obtain $p(s) > 0.0135 > 0$ in I_0.

For $s \geqslant 3$, $c'(s)$ is very nearly equal to

$$d(s) = \frac{8s\sinh^2(\pi s)}{\sinh^2(\pi s) - (\pi s)^2}\sum_{n=1}^{\infty}\frac{n(n^2 - s^2)}{(n^2 + s^2)^3}.$$

This can be seen from (21) using $\sinh(\pi s) \approx \cosh(\pi s)$ and $(\pi s)^3 \approx (\pi s)^2$ in comparison to $\sinh^3(\pi s)$. The Euler expansion of the series term enables us to find a constant r such that

$$d(s) > r > 0 \quad \text{and} \quad |c'(s) - d(s)| < r \qquad \text{for } s \geqslant 3.$$

5. Appendix. In the proofs of Theorems 3 and 4 the following non-elementary integrals are needed:

(i) $$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} (\ln\cos t)^n\, e^{\alpha t}\, dt,$$

where n is a positive integer and α an arbitrary real number. (We are grateful to Dr. M. Wyman of Edmonton for help in evaluating this integral).

Consider

$$\begin{aligned} I(x) &= 2\int_0^{\frac{1}{2}\pi} (\cos t)^x \cosh(\alpha t)dt \\ &= 2\int_0^{\frac{1}{2}\pi} (\cos t)^x \cos(i\alpha t)dt \\ &= \frac{\pi\Gamma(x+1)}{2^x\,\Gamma(1 + \frac{1}{2}(x + i\alpha))\cdot\Gamma(1 + \frac{1}{2}(x - i\alpha))}; \end{aligned}$$

see **(4,** Vol. 1, p. 12**)**. Differentiating n times with respect to x and then setting $x = 0$, we obtain

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$$\frac{d^n}{dx^n} I(x)\Big|_{x=0} = 2\int_0^{\frac{1}{2}\pi} (\ln \cos t)^n \cosh(\alpha t) dt$$

$$= \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} (\ln \cos t)^n e^{\alpha t}\, dt$$

$$= \left(\frac{d}{dx}\right)^n \left\{\frac{\pi\Gamma(x+1)}{2^x\, \Gamma(1+\frac{1}{2}(x+i\alpha))\Gamma(1+\frac{1}{2}(x-i\alpha))}\right\}_{x=0}.$$

(ii)
$$\int_0^\infty \ln^2 u\, e^{-2u}\, du.$$

Defining

$$K(t) = \int_0^\infty u^t e^{-2u}\, du = \frac{1}{2}\int_0^\infty \left(\frac{v}{2}\right)^t e^{-v}\, dv$$

$$= 2^{-t-1}\Gamma(t+1) \qquad \text{for } \operatorname{Re} t > -1,$$

we find that

$$\dot{K}(t) = dK/dt = K(t)[\psi(t+1) - \ln 2],$$

and notice that

$$\dot{K}(0) = \int_0^\infty \ln u e^{-2u} du = K(0)[\psi(1) - \ln 2] = -\tfrac{1}{2}(\xi + \ln 2).$$

Differentiating once more, we have

$$\ddot{K}(t) = \dot{K}(t)[\psi(t+1) - \ln 2] + K(t)\dot{\psi}(t+1),$$

and

$$\ddot{K}(0) = \int_0^\infty \ln^2 u e^{-2u} du$$

$$= K(0)[\psi(1) - \ln 2]^2 + K(0)\dot{\psi}(1)$$

$$= \tfrac{1}{2}(\xi + \ln 2)^2 + \frac{\pi^2}{12}$$

(ξ = Euler's constant).

References

1. H. Behnke and F. Sommer, *Theorie der analytischen Funktionen einer komplexen Veränderlichen* (Berlin, 1955).
2. H. Busemann, *The geometry of geodesics* (New York, 1955).
3. C. E. Castonguay and H. G. Helfenstein, *On analytic maps of the pseudosphere*, Monatsh. Math., *68* (1964), 97–101.
4. A. Erdélyi, *Higher transcendental functions* (New York, 1953).
5. H. G. Helfenstein, *Conformal maps with least distortion*, Can. J. Math., *7* (1955), 306–313.
6. R. Nevanlinna, *Eindeutige analytische Funktionen* (Berlin, 1936).
7. ——— *Uniformisierung* (Berlin, 1953).
8. T. F. Steffenson, *Interpolation* (New York, 1927).
9. G. P. Torstov, *Fourier series* (New Jersey, 1962).

University of Ottawa