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ON THE LOGARITHMIC COEFFICIENTS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract

Logarithmic coefficient bounds for some univalent functions are given in this paper.

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Let S denote the usual class of univalent functions $f(z)$ normalized so that $f(0) = f'(0) - 1$ in $|z| < 1$. Let C denote the set of functions $f(z)$ normalized as above and which satisfy the condition $\operatorname{Re} zf'(z)/g(z) \geq 0$ in $|z| < 1$, where $g(z)$ itself is subject to the conditions $\operatorname{Re} zg'(z)/g(z) \geq 0$, $g(0) = 0$ and $\operatorname{Re} g'(0) > 0$ in $|z| < 1$. Then $f(z)$ is called *close-to-convex* relative to the starlike function $g(z)$. We denote this set of functions $g(z)$ by S^*. Now let K denote the set of functions $Q(z)$ which satisfy the conditions: $Q(0) = 0$, $\operatorname{Re} Q'(0) > 0$ and $\operatorname{Re}(zQ''/Q' + 1) \geq 0$ in $|z| < 1$. Such $Q(z)$ are called convex. It is well known that each $zQ'(z) \in S^*$ and $K \subset S^* \subset C \subset S$ (see for example [6, pp. 40–46]; [11, pp. 11–18]; [14, p. 361]).

Now, as in [6, p. 151], each $f(z) \in S$ has a logarithmic expansion

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n \tag{1}$$

in $|z| < 1$ where γ_n are known as the logarithmic coefficients. The problem of the best upper bounds for $|\gamma_n|$ is still open. In fact even the proper order of magnitude is still not known. It is known, however, for the starlike functions that the best bound

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is $|\gamma_n| \leq 1/n \quad (n \geq 1)$ and that this is not true in general [6, p. 151]; [5, p. 898]; [1, p. 140] and [7].

The importance of this problem in relation to the Bieberbach conjecture was pointed out by Milin in his conjecture (see [1, p. 141]; [6, pp. 155–156]; [5, p. 899]) that

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0,$$

which led De Branges, by proving this conjecture, to the proof of the Bieberbach conjecture [2]. Milin has shown [6, p. 151] that $\sum_{k=1}^{n}\left(k|\gamma_k|^2 - 1/k\right) \leq \delta$ where $\delta < 0.312$.

It is known that δ cannot be reduced to zero in general [6, p. 155]. However, its exact value remains unknown. In this note we show that $|\gamma_n| \leq 1/n \quad (n \geq 1)$, for $f \in C$, and consequently $\delta = 0$ as for the starlike functions. We shall use $\sum a_n z^n \ll \sum b_n z^n$ to mean $|a_n| \leq b_n$ for $n \geq 1$ [12, p. 52] and $f \prec F$ to mean that $f(0) = F(0)$ and $f(z : |z| < 1) \subset F(z : |z| < 1)$ or equivalently $f(z) = F(\phi(z))$, where $\phi(0) = 0$ and $|\phi(z)| < 1$ [6, p. 190].

THEOREM 1. *Let $f \in C$ so that* (1) *holds. Then for $n \geq 1$ we have*

$$|\gamma_n| \leq \frac{1}{n}. \tag{2}$$

PROOF. For $f \in C$ let $\{f_k\}$ be a sequence in C which converges uniformly on compact subsets to $f \in C$. Let also for a fixed n, $J(g) = |\gamma_n|$ where $g(z) = \log(f(z)/z) = 2\sum_{n=1}^{\infty}\gamma_n z^n$ and let $\log(f_k(z)/z) = 2\sum_{n=1}^{\infty}\gamma_n^{(k)} z^n$. Then, using the coefficient formula we deduce for $z = re^{i\theta}$, $0 < r < 1$, that

$$\begin{aligned} \left||\gamma_n^{(k)}| - |\gamma_n|\right| &\leq |\gamma_n^{(k)} - \gamma_n| \\ &= \left|\frac{1}{2\pi i}\int_{|z|=r}\left(\log\frac{f_k(z)}{z} - \log\frac{f(z)}{z}\right)\frac{dz}{z^{n+1}}\right| \\ &\leq r^{-n}\max_{|z|=r}|\log(f_k(z)/f(z))| \\ &\to 0, \end{aligned}$$

since $f_k \to f$ uniformly on $|z| = r$ as $k \to \infty$ [11, p. 40]. Thus we see that $|\gamma_n^{(k)}|^2 \to |\gamma_n|^2$ so that $J(g)$ is continuous.

Now let $h(z) = \log(f_1(z)/z) = 2\sum_{n=1}^{\infty}\gamma_n' z^n$, $\phi(z) = \log(f_2(z)/z) = 2\sum_{n=1}^{\infty}\gamma_n'' z^n$ in $|z| < 1$ where $f_1, f_2 \in C$. Let also $g = th + (1-t)\phi$, $0 \leq t \leq 1$. Then by [11, Lemma 5.6] we have

$$\begin{aligned} J(g) = |\gamma_n|^2 &= |t\gamma_n' + (1-t)\gamma_n''|^2 \\ &\leq t|\gamma_n'|^2 + (1-t)|\gamma_n''|^2 \\ &= tJ(h) + (1-t)J(\phi), \end{aligned}$$

which implies that $J(g)$ is convex.

Thus, in view of [11, Theorem 4.6], we need only prove Theorem 1 for the extreme points of the closed convex hull of C, denoted by EHC, since $\max\{J(f) : f \in EHC\} = \max\{J(f) : f \in C\}$ in this case.

Functions $f \in EHC$ are of the form

$$f(z) = (z - \tfrac{1}{2}(x+y)z^2)/(1-yz)^2 \tag{3}$$

where $x \neq y$ and $|x| = |y| = 1$.

With a suitable rotation (see [11, p. 83]), this can be written in the form

$$f(z) = (z - bz^2)/(1-z)^2$$

where $|b - 1/2| = 1/2$.

Writing $\psi(z) = (1-bz)/(1-z)$ we see that

$$\operatorname{Re}\psi(z) = \frac{1}{2} + \frac{1}{2}\left(\frac{1+(1-2b)z}{1-z}\right) \geq \frac{1}{2}.$$

We also see that $\operatorname{Re}(z\psi''/\psi' + 1) = \operatorname{Re}((1+z)/(1-z)) \geq 0$ which implies that ψ is convex and consequently starlike of order $1/2$ (see [4, p. 418];[6, p. 251]). Thus, as in [4, p. 417] and [16, p. 722], using Herglotz's formula (see [6, pp. 22–40]; [11, pp. 27–30])

$$p(z) = \int_0^{2\pi} \left(\frac{1+e^{-it}z}{1-e^{-it}z}\right) d\mu(t)$$

where $p(0) = 1$, $\operatorname{Re} p(z) \geq 0$, $d\mu(t) \geq 0$ and $\int_0^{2\pi} d\mu(t) = 1$, we obtain

$$\begin{aligned}\log\psi(z) &= \int_0^{2\pi} \log\left(\frac{1}{1-ze^{-it}}\right) d\mu(t) \\ &= \sum_{n=1}^{\infty}\left(\int_0^{2\pi} e^{-int}d\mu(t)\right)\frac{z^n}{n} \\ &\ll \sum_{n=1}^{\infty}\frac{z^n}{n} = \log\frac{1}{1-z}.\end{aligned}$$

Hence we see that

$$\begin{aligned}\log\frac{f(z)}{z} &= \log\frac{1}{1-z} + \log\frac{1-bz}{1-z} \\ &\ll 2\log\frac{1}{1-z}.\end{aligned}$$

This gives (2) by the definition of $\ll$ above.

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COROLLARY 1. *For $f \in C$ we have*

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta &= 1+4\sum_{n=1}^{\infty} n^2|\gamma_n|^2 r^{2n}\\ &\le 1+4\sum_{n=1}^{\infty} r^{2n}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zK'(z)}{K(z)}\right|^2 d\theta\end{aligned}$$

where $z = re^{i\theta}$, $0 < r < 1$, and $K(z) = z(1-z)^{-2}$.

THEOREM 2. *For $f \in EHC$ we have*

$$\log\frac{f(z)}{z} \prec \log\frac{K(z)}{z}, \tag{4}$$

or equivalently

$$\frac{f(z)}{z} \prec \frac{K(z)}{z}$$

where $K(z)$ is as defined above.

PROOF. This theorem follows from the fact that $\operatorname{Re}((1-bz)/(1-z)) \ge 1/2$ is equivalent to $(1-bz)/(1-z) \prec 1/(1-z)$, [11, p. 53], and this is equivalent to $\log((1-bz)/(1-z)) \prec \log(1/(1-z))$ [17, p. 23]. Thus we have

$$\begin{aligned}\log\frac{f(z)}{z} &= 2\left(\frac{1}{2}\log\frac{1}{1-z} + (1-\frac{1}{2})\log\frac{1-bz}{1-z}\right)\\ &\prec 2\log\frac{1}{1-z}\end{aligned}$$

as required.

COROLLARY 2. *For $f \in EHC$ we see from* [11, Theorem 3.3]; [6, Theorem 6.1] *and* (4) *that*

$$\int_0^{2\pi}\left|\log\frac{f(z)}{z}\right|^q d\theta \le \int_0^{2\pi}\left|\log\frac{K(z)}{z}\right|^q d\theta$$

where $z = re^{i\theta}$, $0 < r < 1$, and $q > 0$. This extends [10, Theorem 1] *for $f \in EHC$.*

COROLLARY 3. *For $f \in C$ and γ_n as defined in Theorem* 1 *we see from* [6, p. 212 (Exercise 7)], (4) *and* (2) *that*

$$\sum_{k=1}^{n} k|\gamma_k|^2 \le \sum_{k=1}^{n} 1/k.$$

THEOREM 3. *Let $f \in C$ and $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Then for $n \geq 2$ we have*

$$\Big| |a_n| - |a_{n-1}| \Big| \leq 1.$$

PROOF. For some choice of ξ on the boundary, with $|\xi| = 1$, we have

$$(1 - \xi z)\frac{f(z)}{z} = \exp\left\{\sum_{n=1}^{\infty}\left(2\gamma_n - \frac{\xi^n}{n}\right) z^n\right\}.$$

Applying the Lebedev-Milin inequality $|\beta_n|^2 \leq \exp\left\{\sum_{k=1}^{n} k|\alpha_k|^2 - \sum_{k=1}^{n} 1/k\right\}$ for the expansion $\sum_{k=0}^{\infty} \beta_k z^k = \exp\left\{\sum_{k=1}^{n} \alpha_k z^k\right\}$, $\beta_0 = 1$ [6, p. 143]; [5, p. 897], we deduce by using the triangle inequality that

$$\Big| |a_n| - |a_{n-1}| \Big| \leq \exp\left\{2\sum_{k=1}^{n}(k|\gamma_k|^2 - \mathrm{Re}\,(\xi^k \gamma_k))\right\}. \tag{5}$$

We now write $\xi = e^{it}$ and choose t such that $kt + \arg(\gamma_k) = 0$. We see that $e^{kit}\gamma_k = |\gamma_k|$. Using this and (2) in (5) we deduce Theorem 3 since

$$\sum_{k=1}^{n}(k|\gamma_k|^2 - \mathrm{Re}\,(\xi^k \gamma_k)) = \sum_{k=1}^{n}(k|\gamma_k|^2 - |\gamma_k|) \leq 0.$$

REMARK 1. The case $n = 3$ has been proved by Koepf [13]. For the full class S the author [9, p. 13] obtained $||a_3| - |a_2|| < 1.411$.

REMARK 2. It has been shown by Pearce [15] that functions of the form (3) are extreme points of S whenever $0 < |\arg(-x/y)| \leq \pi/4$ and consequently Theorems 2 and 3 hold for these functions. Our results give a partial answer to the questions raised in [8] and [3, Problem 6.71; p. 558].

REMARK 3. The Koeke function $K(z) = z(1-z)^{-2}$ and its rotations show that our results are the best possible.

REMARK 4. Let $(zf'/f)^q = 1 + \sum c_n(q) z^n$ and $[(1+z)/(1-z)]^q = 1 + \sum D_n(q) z^n$ where n, q are positive integers. Then we have $|c_n(q)| \leq D_n(q)$ since $(zf'/f)^q \ll [(1+z)/(1-z)]^q$ in this case. (See [12, Lemma 2.4.1, p. 53].) We now easily see that

$$\int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^{2q} d\theta \leq \int_0^{2\pi} \left|\frac{zK'(z)}{K(z)}\right|^{2q} d\theta$$

and this extends Corollary 1 when q is a positive integer.

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