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BICYCLIC SEMIRINGS

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Abstract

Let B_P be the bicyclic semigroup over $P = G \cap [1, \infty)$ where G is a subgroup of the multiplicative group of positive real numbers. If $+$ is an addition which makes B_P, with its usual multiplication, into a semiring, then $+$ is idempotent, and P is embedded as a sub-semiring in B_P, and for each x in P, $1 \leqslant x+1 \leqslant x$ and $1 \leqslant 1+x \leqslant x$. We show that any idempotent addition on P with these inequalities holding is max, min or trivial. The trivial addition on P extends trivially. If addition on P is min, then let

$$U = \{(x, y) \in B_P\colon (x, y)+(1, 1) = (1, 1)\},$$

$$U' = \{(x, y) \in B_P\colon (1, 1)+(x, y) = (1, 1)\},$$

and

$$R_1 = \{(x, y) \in B_P\colon x>y \text{ or } x = 1 = y\}$$

We characterize all additions on B_P in terms of U and U'; and, in particular, if $U = U'$ is a proper subset of R_1, we demonstrate a correspondence between all such additions and certain homomorphisms of G to $(0, \infty)$.

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1. Introduction

Let G be a subgroup of the positive real numbers under ordinary multiplication, and let $P = P(G) = G \cap [1, \infty)$. Let $B_P = P \times P$ together with this multiplication:

$$(x, y)(z, w) = \left(\frac{xz}{y \wedge z}, \frac{yw}{y \wedge z}\right),$$

where $y \wedge z = \min(y, z)$. If $P = \{1, x, x^2, \ldots\}$, where $x > 1$, then B_P is the bicyclic semigroup, whose structure is well known (see, for example, Clifford and Preston (1961)).

An inverse semigroup is a semigroup S with the property that for any x in S there is a unique element x^{-1} in S such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The bicyclic semigroup is such a semigroup, and so is B_P for any P defined as above.

A (topological) semiring is a non-empty Hausdorff space T together with two continuous associative binary operations, $+$ and $\cdot$, such that for any x, y and z in T,

$$z\cdot(x+y) = (z\cdot x)+(z\cdot y) \quad \text{and} \quad (x+y)\cdot z = (x\cdot z)+(y\cdot z).$$

If T has the additional property that T is multiplicatively a topological inverse semigroup (one in which the inversion operation as well as the multiplication is continuous), then we define T to be an *inverse semiring*.

In this paper we describe the additions which may be placed on B_P so that, together with the given multiplication and the product topology, B_P becomes an inverse semiring, which we will call a *bicyclic semiring*.

In Section 2, we show that any semiring addition on B_P is idempotent; that is, for any (x,y) in B_P, $(x,y)+(x,y) = (x,y)$. We also show that P is embedded in B_P as a subsemiring, and furthermore that on P, for any x, $1 \leqslant x+1 \leqslant x$ and $1 \leqslant 1+x \leqslant x$. For this reason we study in Section 1 those idempotent additions on P which have this property, and show that there are only four possibilities:

(i) $x+y = x$ for each x,y in P (left trivial addition);
(ii) $x+y = y$ for each x,y in P (right trivial addition);
(iii) $x+y = x\wedge y$ for each x,y in P (min addition);
(iv) $x+y = x\vee y$ for each x,y in P (max addition).

This generalizes the result of Pearson (1966) for the case $P = [1,\infty)$.

Section 3 is devoted to a characterization of those additions on B_P which, when restricted to $P\times\{1\}$, are min, and have the property that the set

$$U = \{(x,y)\in B_P\colon (x,y)+(1,1) = (1,1)\}$$

is properly contained in

$$R_1 = \{(x,y)\in B_P\colon x>y \text{ or } x = 1 = y\}.$$

We show that each such addition corresponds to a homomorphism $f\colon G\to(0,\infty)$ such that graph $(f_P)\subseteq R_1$ and for each (x,y) in B_P, graph (f) meets

$$D(x,y) = \{(ax,ay)\colon a>0\}$$

in a unique point of $G\times G$. We point out that if addition is max on $P\times\{1\}$, the situation is symmetrical. All other cases, including the trivial, are discussed in Section 2.

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1. Idempotent additions on P

In this section we examine idempotent additions on P. We first remark that since G is either a cyclic subgroup of $(0,\infty)$ or dense in $(0,\infty)$, it follows that P is either a cyclic subsemigroup of $[1,\infty)$ or dense in $[1,\infty)$.

LEMMA 1.1. *Let $(R,\cdot,+)$ be a semiring with an additively idempotent multiplicative identity* 1. *Then*

(a) *$(R,+)$ is idempotent.*

(b) *The sets $S=\{y\in R: y+1=y\}$, $S'=\{y\in R: 1+y=y\}$, $U=\{y\in R: y+1=1\}$, and $U'=\{y\in R: 1+y=1\}$ are closed subsemirings of R.*

PROOF. (a) If $x\in R$, $x+x=x(1+1)=x(1)=x$.

(b) It is almost immediate that all these sets are closed additive subsemigroups of R, and that 1 is an element of each. Now let x and y be elements of S. Then $xy+1=x(y+1)+1=xy+x+1=xy+x=x(y+1)=xy$, so $xy\in S$. Similarly, S' is multiplicatively closed. If x and y are in U, then

$$xy+1=xy+y+1=(x+1)y+1=y+1=1,$$

so $xy\in U$; similarly U' is multiplicatively closed.

We now wish to describe the idempotent additions on P with the property that $1\leqslant x+1\leqslant x$ and $1\leqslant 1+x\leqslant x$ for all x. First we need a lemma.

LEMMA 1.2. *Let T be a subsemigroup of $([0,\infty),+)$, and let Q be a subsemigroup of T such that 0 is a limit point of Q. Then $T=Q^*$, the closure of Q in T.*

PROOF. Let x be an element of T and let $0<\varepsilon<x$. Then there is a positive integer N such that $n\geqslant N$ implies that $(2n+1)\varepsilon>x$. Hence, $n\varepsilon>x-(n+1)\varepsilon$ and so $n(x+\varepsilon)>(n+1)(x-\varepsilon)$, so that $(x+\varepsilon)/(n+1)>(x-\varepsilon)/n$. Thus,

$$\bigcup_{n\geqslant N}\left(\frac{x-\varepsilon}{n},\frac{x+\varepsilon}{n}\right)=\frac{(0,x+\varepsilon)}{N}.$$

Now since 0 is a limit point of Q, there exists some s in $Q\cap(0,(x+\varepsilon)/N)$ and so $s\in((x-\varepsilon)/n,(x+\varepsilon)/n)$ for some $n\geqslant N$. Hence, $ns\in Q\cap(x-\varepsilon,x+\varepsilon)$ and so $Q\cap(x-\varepsilon,x+\varepsilon)\neq\emptyset$. This shows that $x\in Q^*$ and so $T=Q^*$.

Now since $([0,\infty),+)$ is isomorphic to $([1,\infty),\cdot)$ this lemma shows that for any subsemigroup P of $([1,\infty),\cdot)$ and subsemigroup Q of P having 1 as a limit point, we have $Q^* = P$.

LEMMA 1.3. *Let P be dense in $[1,\infty)$, and let $+$ be a semiring addition on P with the property that for any x in P, $1 \leqslant x+1 \leqslant x$ and $1 \leqslant 1+x \leqslant x$. Then either $P = S$ or $P = U$. Also, either $P = S'$ or $P = U'$.*

PROOF. If there is some x in P with $[1,x] \subseteq U$, then $P = \bigcup_{n \geqslant 0} [x^n, x^{n+1}] \subseteq U$ so $U = P$. Hence, if $U \neq P$, then for each $x > 1$, there is a y in $(1,x)$ with $y \notin U$. If $y \in S$ then $(1,x) \cap S \neq \emptyset$. If $y \notin S$, then $1 < y+1 < y$ so $y+1 \in (1,x)$. Moreover, $(y+1)+1 = y+(1+1) = y+1$ and so $y+1 \in S$. Hence, in this case also, $(1,x) \cap S \neq \emptyset$. Thus $(1,x) \cap S \neq \emptyset$ for any $x > 1$ so 1 is a limit point of S. The above lemma shows $S^* = P$, and since S is closed, $S = P$.

We now prove a similar result for the case when P is cyclic. In this case, we are able to drop the hypothesis that $1 \leqslant x+1 \leqslant x$ and $1 \leqslant 1+x \leqslant x$. We first need the following technical lemma.

LEMMA 1.4. *Let $P = G \cap [1,\infty)$ for any subgroup G of $(0,\infty)$, and let $+$ be an idempotent semiring addition on P. Let $x \in P$, and let $y = 1+x$. If $y > x$, then*

(a) *$y/x+1 = y/x$, and hence $y^n/x^n+1 = y^n/x^n$ for every positive integer n;*
(b) *$y^n/x^n+x = y^{n+1}/x^n$ for every positive integer n;*
(c) *for every positive integer n and for each $p \leqslant n$ such that $y^p > x^n$, $y^p/x^n+1 = y^n/x^n$.*

PROOF. (a) $x(y/x+1) = y+x = 1+x+x = 1+x = y$ and so by cancellation, $y/x+1 = y/x$. Since S is a multiplicative semigroup, $(y/x)^n+1 = (y/x)^n$ for every n.

(b) For $n = 1$, $y/x+x = y/x+1+x = y/x+y = (y/x)(1+x) = (y/x)y = y^2/x$. Now suppose that $y^{n-1}/x^{n-1}+x = y^n/x^{n-1}$. Then

$$y^n/x^n+x = y^n/x^n+1+x = y^n/x^n+y = (y/x)(y^{n-1}/x^{n-1}+x)$$

$$= (y/x)(y^n/x^{n-1}) = y^{n+1}/x^n.$$

(c) Let $p = n-i$. The statement that $y^{n-i}/x^n+1 = y^n/x^n$ is true for $i = 0$ by (a). Now suppose that $y^{n-(i-1)} > x^n$ and that $(y^{n-(i-1)}/x^n)+1 = y^n/x^n$. Then $y^{n-i} > x^n$, we have

$$y[(y^{n-i}/x^n)+1] = (y^{n-(i-1)}/x^n)+y = (y^{n-(i-1)}/x^n)+1+y$$

$$= y^n/x^n+1+x = y^n/x^n+x = y^{n+1}/x^n = y(y^n/x^n)$$

and so

$$y^{n-i}/x^n + 1 = y^n/x^n.$$

LEMMA 1.5. *Let P be cyclic, and let $+$ be an idempotent semiring addition on P. If S, U, S' and U' are defined as in Lemma 1.1, then $P = S$ or $P = U$; and similarly, $P = S'$ or $P = U'$.*

PROOF. We show the lemma for S' and U'. Since S' and U' are multiplicative semigroups, it is enough to show that $1+x = 1$ or $1+x = x$ where x is the generator of P. Suppose that $1+x = y = x^m$ for some integer $m > 1$, and that $x+1 = x^n$ for some integer $n \geqslant 0$. Then

$$x^n = x+1 = (x^m/x^{m-1})+1 = (y/x^{m-1})+1 = (y^{m-1}/x^{m-1})+1 \quad \text{(by Lemma 1.4(c))}$$
$$= y^{m-1}/x^{m-1} \quad \text{(by Lemma 1.4(a))} = x^{m(m-1)-(m-1)} = x^{(m-1)^2},$$

and hence $n = (m-1)^2$. Now since

$$(y^2/x^2)(1+x^2) = y^2/x^2 + y^2 = y(y/x^2 + y) = y(y/x^2 + 1 + y)$$
$$= y(y^2/x^2 + y) = y^2(y/x^2 + 1) = y^2(y^2/x^2),$$

we have $1+x^2 = y^2$, and similarly, $x^2+1 = (x^n)^2 = x^{2n}$. But

$$x^2+1 = y/x^{m-2}+1 = y^{m-2}/x^{m-2} = x^{m(m-2)}/x^{m-2} = x^{(m-1)(m-2)}$$

and hence $2n = (m-1)(m-2)$ and so $2(m-1)^2 = (m-1)(m-2)$. Solving this quadratic equation gives $m = 0$ or $m = 1$, contradicting the assumption that $m > 1$. Thus m is either 0 or 1. Similarly, $n = 0$ or $n = 1$.

THEOREM 1.6. *If P is $G \cap [1, \infty)$ where G is any subgroup of $[0, \infty)$, and if $+$ is a semiring addition with the property that for every x in P, $1 \leqslant x+1 \leqslant x$ and $1 \leqslant 1+x \leqslant x$, then one of the following describes the addition:*

(a) *for each x, y in P, $x+y = x$;*
(b) *for each x, y in P, $x+y = y$;*
(c) *for each x, y in P, $x+y = x \vee y$;*
(d) *for each x, y in P, $x+y = x \wedge y$.*

PROOF. Lemmas 1.3 and 1.5 show that exactly one of the following is true for every x in P:

(i) $x+1 = x$ and $1+x = 1$;
(ii) $x+1 = 1$ and $1+x = x$;
(iii) $x+1 = 1+x = x$;
(iv) $x+1 = 1+x = 1$.

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If (i) is true, then for each x and y in P,

$$x+y = (x+1)+y = x+(1+y) = x+1 = x,$$

and similarly, if (ii) is true, then for each x, y in P,

$$x+y = y.$$

If (iii) is true, and if $x<y$, then

$$x+y = x(1+y/x) = x(y/x) = y,$$

while if $x>y$, then

$$x+y = (x/y+1)y = (x/y)y = x$$

and so in either case $x+y = x\vee y$. Similarly, if (iv) is true then

$$x+y = x\wedge y.$$

REMARK 1.7. We conjecture that the hypothesis $1\leqslant x+1\leqslant x$ and $1\leqslant 1+x\leqslant x$ may be omitted from the dense case for P. The work of Pearson (1966) and Lemma 1.5 above show that it may be omitted if $P=[1,\infty)$ or if P is cyclic.

2. Additions on B_P with $U\supseteq R_1$

In this section we first show that all semiring additions on B_P are idempotent and that the subset $P\times\{1\}$ is a subsemiring isomorphic to P and that for each x in P, $1\leqslant 1+x\leqslant x$ and $1\leqslant x+1\leqslant x$. Thus, Theorem 1.6 applies and $P\times\{1\}$ is additively max, min or trivial. We show immediately that the trivial addition on $P\times\{1\}$ can only extend trivially and assume that $P\times\{1\}$ has the min addition. In this case, we show that the set $\{(x,y)\in B_P\colon x<y\}$ is contained in both S and S', where these are defined for B_P as in Lemma 1.1, and we describe the additions in which U contains the set $\{(x,y)\in B_P\colon x>y$ or $x=y=1\}$.

We remark for the reader that $(1,1)$ is a multiplicative identity for B_P, and that for each element (x,y) of B_P, $(x,y)^{-1}=(y,x)$. The multiplicative idempotents are precisely the diagonal elements $\{(x,x)\}$.

LEMMA 2.1. *If + is a semiring addition on B_P, then B_P is additively idempotent.*

PROOF. Since $(1,1)$ is a multiplicative identity for B_P, then Lemma 1.1 implies that it is sufficient to show $(1,1)$ is an additive idempotent. Let $(e,f)=(1,1)+(1,1)$. If $x>1$, we have

$$\begin{aligned}(xe/x\wedge e, xf/x\wedge e) &= (x,x)(e,f) = (x,x)[(1,1)+(1,1)] = (x,x)+(x,x)\\ &= (x,1)[(1,1)+(1,1)](1,x) = (x,1)(e,f)(1,x)\\ &= (xe,f)(1,x) = (xe,fx).\end{aligned}$$

Thus, $x \wedge e = 1$. Similarly, $(xe, fx) = (e, f)(x, x) = (ex/f \wedge x, fx/f \wedge x)$ and so

$$f \wedge x = 1 = e \wedge x.$$

Now since $x > 1$, $e = 1 = f$ and so $(1,1)+(1,1) = (1,1)$.

LEMMA 2.2. *Let $x > 1$ be an element of P. Then there exists $a \in P$ such that $a \leqslant x$ and $(x,1)+(1,1) = (a,1)$, and there exists $b \in P$ with $b \leqslant x$ such that*

$$(1,1)+(x,1) = (b,1).$$

Furthermore, $(1,1)+(1,x) = (1,x/a)$ *and* $(1,x)+(1,1) = (1,x/b)$.

PROOF. We prove the assertion for a and x/a; the proof for b and x/b is similar. Let $(x,1)+(1,1) = (a,c)$. Then

$$\begin{aligned}(xa, xc) &= (xa, c)(1,x) = (x,1)(a,c)(1,x) = (x,1)[(x,1)+(1,1)](1,x)\\ &= [(x^2,1)+(x,1)](1,x) = (x^2,x)+(x,x) = [(x,1)+(1,1)](x,x)\\ &= (a,c)(x,x) = (ax/c \wedge x, cx/c \wedge x).\end{aligned}$$

Thus, $c \wedge x = 1$; but $x > 1$ and so $c = 1$. Now let $(1,1)+(1,x) = (1,s)$. Then since $(a/a \wedge x, x/a \wedge x) = (1,x)(a,1) = (1,x)[(x,1)+(1,1)] = (1,1)+(1,x) = (1,s)$, we have $a = a \wedge x$ and hence $s = x/a$.

The following is now immediate, using Lemma 2.2 and Theorem 1.6.

THEOREM 2.3. *$P \times \{1\}$ is a subsemiring of B_P which is multiplicatively isomorphic to P, and hence the addition on $P \times \{1\}$ is either trivial, max or min.*

We dispose of the trivial addition at once.

THEOREM 2.4. *If $+$ is a semiring addition on B_P which is trivial when restricted to $P \times \{1\}$, then $+$ is trivial on B_P.*

PROOF. Suppose $+$ is left trivial on $P \times \{1\}$. Then for any x in P,

$$(x,1)+(1,1) = (x,1) \quad \text{and} \quad (1,1)+(x,1) = (1,1)$$

and so the a of Lemma 2.2 is x and the b is 1, and hence $(1,1)+(1,x) = (1,1)$ and $(1,x)+(1,1) = (1,x)$. Thus, for any (x,y) and (z,w) in B_P,

$$\begin{aligned}(x,y)+(z,w) &= [(x,y)+(z,y)]+(z,w) = (x,y)+[(z,y)+(z,w)]\\ &= (x,y)+(z,y) = (x,y).\end{aligned}$$

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Hence, the addition on B_P is left trivial. The situation is symmetrical for the right trivial addition.

In the remainder of this section, we assume that addition on $P \times \{1\}$ is min; in this case we will see that addition on $\{1\} \times P$ is max; it is easy to show that the case where addition on $P \times \{1\}$ is max is completely symmetrical.

LEMMA 2.5. *Suppose addition restricted to $P \times \{1\}$ is min, and let $(x, y) \in B_P$.*
(a) *If $(z, w) \in B_P$ with $x \leqslant z$ and $y \geqslant w$, then $(x, y)+(z, w) = (x, y) = (z, w)+(x, y)$.*
(b) *If $x < y$, then $(x, y)+(1, 1) = (x, y) = (1, 1)+(x, y)$.*
(c) *If $x \geqslant y$, then there exist a and b in P with $a \leqslant y$ and $b \leqslant y$ such that*

$$(x, y)+(1, 1) = (a, a) \quad \textit{and} \quad (1, 1)+(x, y) = (b, b).$$

PROOF. (a) Since $(x, 1)+(1, 1) = (1, 1) = (1, 1)+(x, 1)$, the a and b of Lemma 2.2 are 1, so that $(1, x)+(1, 1) = (1, 1)+(1, x) = (1, x)$ and hence for every x and y in P,

$$(1, x)+(1, y) = (1, x \vee y) \text{ and}$$

$$(x, 1)+(1, y) = (x, 1)+[(1, 1)+(1, y)] = [(x, 1)+(1, 1)]+(1, y)$$
$$= (1, 1)+(1, y) = (1, y).$$

Thus, if $x \leqslant z$ and $y \geqslant w$,

$$(x, y)+(z, w) = [(x, y)+(z, y)]+(z, w) = (x, y)+[(z, y)+(z, w)]$$
$$= (x, y)+(z, y) = (x, y).$$

(b) and (c) are proved as follows. Let $(x, y)+(1, 1) = (a, c)$. Then

$$(ax/c \wedge x, cx/c \wedge x) = (a, c)(x, x) = [(x, y)+(1, 1)](x, x) = (x^2/x \wedge y, xy/x \wedge y)+(x, x),$$

which equals (x, y) if $x < y$ and (x, x) if $x \geqslant y$. Thus, if $x < y$, then $ax/c \wedge x = x$ and $cx/c \wedge x = y$ and so $a = c \wedge x$. If $a = c$, then $x = y$; but $x < y$ and so $a = x$ and hence $c = y$. If $x \geqslant y$, then $ax/c \wedge x = x$ and $cx/c \wedge x = x$ and so $a = c \leqslant x$. Premultiplying (a, c) by (y, y), we find that if $x \geqslant y$, then $a \leqslant y$. This completes the proof of the lemma.

We now introduce some notation which will be referred to throughout the rest of this paper. Let

$$L = \{(x, y) \in B_P \colon x \leqslant y\}, \quad R = \{(x, y) \in B_P \colon x \geqslant y\} \quad \text{and} \quad D = L \cap R = \{(x, x) \colon x \in P\}.$$

As in Lemma 1.1,

$$U=\{(x,y)\in B_P\colon (x,y)+(1,1)=(1,1)\},$$

$$U'=\{(x,y)\in B_P\colon (1,1)+(x,y)=(1,1)\},$$

$$S=\{(x,y)\in B_P\colon (x,y)+(1,1)=(x,y)\}$$

and

$$S'=\{(x,y)\in B_P\colon (1,1)+(x,y)=(x,y)\}.$$

Finally, for (x,y) in B_P, let $D(x,y)=\{(ax,ay)\colon a>0\}$, and let $R_1=(R\setminus D)\cup\{(1,1)\}$.

REMARK 2.6. If (x,y) and (z,w) are two elements of B_P, assume $x\leqslant z$. Then one and only one of the following statements is true:

(a) $y\geqslant w$;
(b) $y<w$ and $z/x<w/y$;
(c) $y<w$ and $z/x\geqslant w/y$.

In case (a), $(x,y)+(z,w)=(x,y)=(z,w)+(x,y)$ by Lemma 2.5. If either (b) or (c) is true, then $(x,y)+(z,w)=(x,1)[(1,1)+(z/x,w/y)](1,y)$. Hence, in case (b), $(x,y)+(z,w)=(z,w)=(z,w)+(x,y)$, by Lemma 2.5, and it is evident that a complete description of the addition on B_P depends on a description of addition by $(1,1)$ on the subset R of B_P. We have the following partial result: if (x,y) and (z,w) are elements of B_P with neither $(x/z,y/w)$ nor $(z/x,w/y)$ in R, then

$$(x,y)+(z,w)=\left(\frac{xw\wedge yz}{y\wedge w},\, y\vee w\right).$$

We now examine the diagonal D of B_P. If P is dense in $[1,\infty)$, then since $L\setminus D\subseteq S\cap S'$ by Lemma 2.5 and $D\subseteq L^*$, we have $D\subseteq S\cap S'$ by Lemma 1.1, and hence $L=S=S'$. Section 3 will be devoted to characterizing semiring additions on B_P such that $D\subseteq S\cap S'$ and U is a proper subset of R_1.

If $P=\{1,x,x^2,\ldots\}$ for $x>1$, then by Lemma 2.5, either $(1,1)+(x,x)=(1,1)$ or $(1,1)+(x,x)=(x,x)$, and similarly for $(x,x)+(1,1)$. Now if $(1,1)+(x,x)=(1,1)$, suppose that for $1\leqslant k<n$, $(1,1)+(x^k,x^k)=(1,1)$. Then

$$\begin{aligned}(1,1)+(x^n,x^n)&=[(1,1)+(x^{n-1},x^{n-1})]+(x^n,x^n)\\&=(1,1)+[(x^{n-1},x^{n-1})+(x^n,x^n)]\\&=(1,1)+(x^{n-1},1)[(1,1)+(x,x)](1,x^{n-1})\\&=(1,1)+(x^{n-1},x^{n-1})=(1,1).\end{aligned}$$

Hence $D\subseteq U'$. If $(1,1)+(x,x)=(x,x)$ then suppose that for $1\leqslant k<n$,

$$(1,1)+(x^k,x^k)=(x^k,x^k).$$

Then

$$(1,1)+(x^n,x^n) = (1,1)+(x^{n-1},1)(x,x)(1,x^{n-1})$$
$$= (1,1)+(x^{n-1},1)[(1,1)+(x,x)](1,x^{n-1})$$
$$= (1,1)+(x^{n-1},x^{n-1})+(x^n,x^n)$$
$$= (x^{n-1},x^{n-1})+(x^n,x^n)$$
$$= (x^{n-1},1)[(1,1)+(x,x)](1,x^{n-1}) = (x^n,x^n),$$

and hence by induction $D \subseteq S'$. Similar manipulations hold for U and S. We summarize this discussion as follows.

THEOREM 2.7. *Suppose + is a semiring addition on* B_P.
(a) *Either* $D \subseteq U$ *or* $D \subseteq S$; *also* $D \subseteq U'$ *or* $D \subseteq S'$.
(b) *If* P *is dense in* $[1,\infty)$, *then* $D \subseteq S \cap S'$.

LEMMA 2.8. *Suppose + is min on* $P \times \{1\}$.
(a) *If* $(x,y) \in U$ (*respectively*, U') *and* $z \geqslant x$, *then* $(z,y) \in U$ (*respectively*, U').
(b) *If* $D \subseteq S$ (*respectively*, $D \subseteq S'$) *and* $a \geqslant 1$, *then for every* (x,y) *in* B_P, $(ax,ay)+(x,y) = (ax,ay)$ (*respectively*, $(x,y)+(ax,ay) = (ax,ay)$).
(c) *If* $D \subseteq S$ (*respectively*, $D \subseteq S'$), *and if* $(x,y) \in U'$ (*respectively*, U) *and* $1 \leqslant a \leqslant y$, *then* $(x/a, y/a) \in U'$ (*respectively*, U).
(d) *If* $D \subseteq S \cap S'$ *or* $D \subseteq U \cap U'$, *then + is abelian.*

PROOF. (a) If $(x,y) \in U$ and $z \geqslant x$, then

$$(z,y)+(1,1) = (z,y)+[(x,y)+(1,1)] = [(z,y)+(x,y)]+(1,1)$$
$$= (x,y)+(1,1) = (1,1)$$

and so $(z,y) \in U$.

(b) Let $D \subseteq S$ and $a > 1$. Then

$$(ax,ay)+(x,y) = (x,1)[(a,a)+(1,1)](1,y) = (x,1)(a,a)(1,y) = (ax,ay).$$

(c) Let $D \subseteq S$, $(x,y) \in U'$ and $1 \leqslant a \leqslant y < x$. Then

$$(1,1)+(x/a,y/a) = (1,1)+(x,y)+(x/a,y/a) = (1,1)+(x,y) \quad \text{(by (b))} = (1,1)$$

and so $(x/a, y/a) \in U'$.

(d) To see that + is abelian, it is enough by Remark 2.6 to show that $(1,1)$ commutes additively with each element (x,y) of R. This is obvious if $D \subseteq U \cap U'$. If $D \subseteq S \cap S'$, let $(a,a) = (1,1)+(x,y)$ and $(b,b) = (x,y)+(1,1)$ as in Lemma 2.5. Then

$$(b,b) = (1,1)+(b,b) = (1,1)+[(x,y)+(1,1)] = [(1,1)+(x,y)]+(1,1)$$
$$= (a,a)+(1,1) = (a,a).$$

LEMMA 2.9. *Let + be a semiring addition on B_P which is min on $P\times\{1\}$. If $D\subseteq U'$ (respectively, $D\subseteq U$), then $R\setminus D\subseteq U$ (respectively, $R\setminus D\subseteq U'$).*

PROOF. Suppose $D\subseteq U'$; then $R=U'$ by Lemma 2.8(a). Suppose that $R\setminus D$ is not contained in U. Then $D\subseteq S$ by Theorem 2.7 and there exist x and y such that $x>y$ with $(x,y)+(1,1)=(a,a)$ for some $a>1$. Then

$$\begin{aligned}(ax,ay)+(1,1)&=[(ax,ay)+(x,y)]+(1,1)\quad\text{(by Lemma 2.8(b))}\\&=(ax,ay)+[(x,y)+(1,1)]=(ax,ay)+(a,a)\\&=(a,1)[(x,y)+(1,1)](1,a)=(a^2,a^2).\end{aligned}$$

Hence,

$$\begin{aligned}(a,a)&=(x,y)+(1,1)=(x,1)[(1,1)+(a,a)](1,y)+(1,1)\\&=(x,y)+(ax,ay)+(1,1)=(x,y)+(a^2,a^2),\end{aligned}$$

which by Lemma 2.5 equals (a^2,a^2) if $y\leqslant a^2$ and equals (a^2b,a^2b) for some $b\geqslant 1$ if $y>a^2$. This contradiction implies that $R\setminus D\subseteq U$. Similarly, if $D\subseteq U$, then $R\setminus D\subseteq U'$. If $D\subseteq U\cap U'$, then $R=U=U'$.

THEOREM 2.10. *Let + be a semiring addition on B_P which is min on $P\times\{1\}$.*

(a) *One of the following is true:*

(i) $U=U'=R$.

(ii) $U=R$ *and* $U'=R_1$.

(iii) $U'=R$ *and* $U=R_1$.

(iv) $U=U'=R_1$.

(v) $U=U'$ *is a proper subset of* R_1.

(b) *In cases (i–iv), for (x,y) and (z,w) in B_P,*

$$(x,y)+(z,w)=\begin{cases}(x\wedge z,y\wedge w) & \text{if } (x/z,y/w)\in U \quad\text{or } (z/x,w/y)\in U',\\ \left(\dfrac{xw\wedge yz}{y\wedge w},y\vee w\right) & \text{otherwise.}\end{cases}$$

If + is max on $P\times\{1\}$, then U and U' are subsets of L, and for (x,y) and (z,w) in B_P,

$$(x,y)+(z,w)=\begin{cases}(x\wedge z,y\wedge w) & \text{if } (x/z,y/w)\in U \quad\text{or } (z/x,w/y)\in U',\\ \left(x\vee z,\dfrac{xw\wedge yz}{x\wedge z}\right) & \text{otherwise.}\end{cases}$$

(c) *If P is dense in $[1,\infty)$, then only* (v) *can be true.*

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PROOF. (a) follows from Theorem 2.7, and Lemma 2.9, and (b) is easy to verify. For (c), we note that $D \subseteq S \cap S'$ by Theorem 2.7b and since in cases (i–iv) above, $U = U^* \supseteq D$, (v) is the only possibility. We discuss this case in Section 3.

3. Additions on B_P with $U \subset R_1$

In this section, we examine semiring additions on B_P which have the property that U is a proper subset of $R_1 = (R \setminus D) \cup \{(1,1)\}$, which implies that $P \times \{1\}$ is additively min. We stress that the situation in which $P \times \{1\}$ is max additively is exactly symmetrical. Recall that by Lemma 2.8 (d) addition is abelian; furthermore, Lemma 2.8(a), (c) imply that $U = U'$ is a subset of R_1 bounded above by a non-decreasing curve C. (The least U can be is $P \times \{1\}$. In this case, it follows from Remark 2.6 that for each (x, y) and (z, w) in B_P,

$$(x, y) + (z, w) = \left(\frac{xw \wedge yz}{y \wedge w}, y \vee w\right).$$

In fact, if $(x, y) \in R$, the "a" of Lemma 2.5 is y.)

LEMMA 3.1. *Let + be a semiring addition on B_P which is min on $P \times \{1\}$. Suppose that U is a proper subset of R_1.*

(a) *If $(x, y) \in R \setminus D$ and $(1,1) + (x, y) = (a, a)$ for some $a > 1$, then*

$$(1,1) + (xc, yc) = (1,1)$$

if and only if $c \leqslant 1/a$; and if $b > 1$, then $(1,1) + (xb/a, yb/a) = (b, b)$.

(b) *If $(x, y) \in L \setminus D$ and if there is a $d > 1$ such that $(x, y) + (d, d) = (px, py)$ where $p > 1$, then $(x, y) + (w, w) = (x, y)$ if and only if $y \leqslant w \leqslant d/p$; and if $b \geqslant 1$, then $(x, y) + (bd/p, bd/p) = (bx, by)$.*

PROOF. (a) Let $(x, y) \in R \setminus D$ and suppose $(1,1) + (x, y) = (a, a)$ with $1 < a \leqslant y$. Since $(a, a) + (x, y) = (a, a) + (1,1) + (x, y) = (a, a)$, we have $(1,1) + (x/a, y/a) = (1,1)$ and Lemma 2.8(*c*) implies that $c \leqslant 1/a$ if and only if $(1,1) + (xc, yc) = (1,1)$.

Now suppose that $b \geqslant 1$ and that $(1,1) + (xb/a, yb/a) = (z, z)$ where $z \leqslant yb/a$. If $z < b/a$, then since $(z, z) + (xb/a, yb/a) = (z, z)$, we have

$$(1,1) + (xb/az, yb/az) = (1,1)$$

and so by the preceding paragraph, $b/az \leqslant 1/a$ and so $b \leqslant z < b/a$. This contradiction

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shows that $z \geqslant b/a$. Now

$$\begin{aligned}(b,b) &= (b/a,1)(a,a)(1,b/a) = (b/a,1)[(1,1)+(x,y)](1,b/a)\\ &= (b/a,b/a)+(xb/a,yb/a) = (b/a,b/a)+(1,1)+(xb/a,yb/a)\\ &= (b/a,b/a)+(z,z) = (z,z)\end{aligned}$$

and so $(1,1)+(xb/a,yb/a) = (b,b)$.

(b) Let $(x,y)\in L\setminus D$. Recall from Lemma 2.5(b) that $(x,y)+(1,1) = (x,y)$. Suppose that there is $d>1$ such that $(x,y)+(d,d)\neq(x,y)$. Then

$$(x,y)+(d,d) = (x,1)[(1,1)+(d/x,d/y)](1,y) = (xp,yp)$$

where $1<p\leqslant d/y$. By (b), $(1,1)+(w/x,w/y) = (1,1)$ if and only if $y\leqslant w\leqslant d/p$ and if $b\geqslant 1$, then $(1,1)+(bd/px,bd/py) = (b,b)$. Hence, $(x,y)+(w,w) = (x,y)$ if and only if $y\leqslant w\leqslant d/p$; and if $b\geqslant 1$, then $(x,y)+(bd/p,bd/p) = (bx,by)$.

Now suppose that every (x,y) in R possesses an $a = a(x,y)$ as in Lemma 3.1; then $a=1$ if and only if $(x,y)\in U$ and for $b>1$, $(xb,yb)\notin U$; in fact, using Lemma 2.8(c) we see that $(x,y)\in U$ if and only if $a\leqslant 1$. In Example 3.2, we let the curve C be the graph of a non-decreasing homomorphism f, which intercepts

$$D(x,y) = \{(tx,ty)\colon t>0\}$$

in a unique point $(x/a,y/a)$ of $G\times G$. We define an addition $+_f$ in terms of this denominator a, and show that $+_f$ is a semiring addition. In Theorem 3.4 we show that Example 3.2 actually characterizes all additions with U a proper subset of R_1.

EXAMPLE 3.2. *Let f be a continuous non-decreasing homomorphism from G to $((0,\infty),\cdot)$ with the properties that for each (x,y) in B_P, graph (f) meets $D(x,y) = \{(ax,ay)\colon a>0\}$ in a unique point of $G\times G$, and that graph $(f|_P)\subseteq R_1$. Then we define the function $\beta\colon B_P\to G$ so that for $(x,y)\in B_P$, $\beta(x,y)$ is that unique element of G such that $(x/\beta(x,y), y/\beta(x,y))\in$ graph (f). If we define addition by*

$$(x,y)+_f(z,w) = \left(\frac{x\beta(z,w)\wedge z\beta(x,y)}{\beta(z,w)\wedge\beta(x,y)}, \frac{y\beta(z,w)\wedge w\beta(x,y)}{\beta(z,w)\wedge\beta(x,y)}\right)$$

then $+_f$ is a commutative semiring addition on B_P and $U = \{(x,y)\in R\colon y\leqslant f(x)\}$ is a proper subset of R_1.

PROOF. To aid in proving associativity, we establish the following facts: if (x,y) (z,w) are elements of B_P with $\beta(x,y) = p$ and $\beta(z,w) = q$, and if $y/x\geqslant w/z$, then

(i) $xq\leqslant zp$ and $yq\leqslant pw$,

(ii) if $y\geqslant w$ then $p\geqslant q$,

(iii) for any a in G such that $(ax,ay)\in B_P$, $\beta(ax,ay) = ap$.

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To see (i), note that y/x is the slope of $D(x, y)$ and w/z is the slope of $D(z, w)$ and so since f is non-decreasing and $(x/p, y/p)$ and $(z/q, w/q)$ lie on graph(f), we have $x/p \leqslant z/q$ and $y/p \leqslant w/q$. Hence, if $y \geqslant w$, $q \leqslant wp/y \leqslant p$. Finally,

$$(ax/ap, ay/ap) = (x/p, y/p)$$

is the unique intersection of $D(x, y)$ and graph(f). We note that closure follows from these observations.

Now suppose $\beta(x, y) = p$, $\beta(z, w) = q$ and $\beta(a, b) = c$. If $y/x \geqslant z/w \geqslant b/a$, we have

$$\begin{aligned}
&(x, y) + [(z, w) + (a, b)] \\
&\quad = (x, y) + ((zc \wedge aq)/(c \wedge q), \quad (wc \wedge bq)/(c \wedge q)) \\
&\quad = (x, y) + (zc/(c \wedge q), \quad wc/(c \wedge q)) \\
&\quad = \left(\frac{\left(\frac{xqc}{c \wedge q}\right) \wedge \left(\frac{zcp}{c \wedge p}\right)}{p \wedge \left(\frac{qc}{c \wedge q}\right)}, \frac{\left(\frac{yqc}{c \wedge q}\right) \wedge \left(\frac{wcp}{c \wedge q}\right)}{p \wedge \left(\frac{qc}{c \wedge q}\right)} \right) = \left(\frac{\left(\frac{xqc}{c \wedge q}\right)}{p \wedge \left(\frac{qc}{c \wedge q}\right)}, \frac{\left(\frac{yqc}{c \wedge q}\right)}{p \wedge \left(\frac{qc}{c \wedge q}\right)} \right) \\
&\quad = \left(\frac{xqc}{cp \wedge pq \wedge qc}, \frac{yqc}{cp \wedge pq \wedge qc} \right) = \left(\frac{\left(\frac{xqc}{p \wedge q}\right)}{\left(\frac{pq}{p \wedge q}\right) \wedge c}, \frac{\left(\frac{yqc}{p \wedge q}\right)}{\left(\frac{pq}{p \wedge q}\right) \wedge c} \right) \\
&\quad = \left(\frac{\left(\frac{xqc}{p \wedge q}\right) \wedge \left(\frac{apq}{p \wedge q}\right)}{\left(\frac{pq}{p \wedge q}\right) \wedge c}, \frac{\left(\frac{ycq}{p \wedge q}\right) \wedge \left(\frac{bpq}{p \wedge q}\right)}{\left(\frac{pq}{p \wedge q}\right) \wedge c} \right) \\
&\quad = \left(\frac{xq}{p \wedge q}, \frac{yq}{p \wedge q} \right) + (a, b) = \left(\frac{xq \wedge zp}{p \wedge q}, \frac{yq \wedge wp}{p \wedge q} \right) + (a, b) \\
&\quad = [(x, y) + (z, w)] + (a, b).
\end{aligned}$$

Since this addition is clearly commutative, associativity is proven.

We prove distributivity in two parts. First note that if $a \in P$ with $\beta(a, 1) = b$ and if $(x, y) \in B_P$ with $\beta(x, y) = p$ then $\beta(ax, y) = bp$, for

$$y/bp = (y/p)(1/b) = f(x/p)f(a/b) = f(xa/pb).$$

Also if $\beta(1, a) = c$, then $\beta[(1, a)(x, y)] = \beta(x/a \wedge x, ay/a \wedge x) = ap/a \wedge x$; for if $a \leqslant x$, then $ay/cp = f(1xa/cpa)$ and so $\beta(x/a, y) = cp/a$, and if $a \geqslant x$, then

$$(ay/x)/(cp/x) = (a/c)(y/p) = f(1x/cp)$$

and so $\beta(1, ay/x) = cp/x$.

Now if (x,y) and (z,w) are two elements of B_P with $\beta(x,y)=p$, $\beta(z,w)=q$, and if $\beta(a,1)=b$, then

$$\begin{aligned}
&(a,1)\,[(x,y)+(z,w)]\\
&=(a,1)\left(\frac{xq\wedge zp}{p\wedge q},\frac{yq\wedge wp}{p\wedge q}\right)=\left(\frac{a(xq\wedge zp)}{p\wedge q},\frac{yq\wedge wp}{p\wedge q}\right)\\
&=\left(\frac{abxq\wedge abzp}{bp\wedge bq},\frac{ybq\wedge wbp}{bp\wedge bq}\right)=(ax,y)+(az,w)=(a,1)(x,y)+(a,1)(z,w).
\end{aligned}$$

Now if $\beta(1,a)=c$, then

$$\begin{aligned}
&(1,a)\,[(x,y)+(z,w)]\\
&=(1,a)\left(\frac{xq\wedge zp}{q\wedge p},\frac{yq\wedge wp}{q\wedge p}\right)\\
&=\left(\frac{\left(\frac{xq\wedge zp}{q\wedge p}\right)}{a\wedge\left(\frac{xq\wedge zp}{q\wedge p}\right)},\frac{a\left(\frac{xq\wedge zp}{q\wedge p}\right)}{a\wedge\left(\frac{yq\wedge wp}{q\wedge p}\right)}\right)\\
&=\left(\frac{xq\wedge zp}{aq\wedge xq\wedge zp\wedge ap},\frac{ayq\wedge awp}{aq\wedge xq\wedge zp\wedge ap}\right)\\
&=\left(\frac{\left(\frac{xcq}{(a\wedge x)(a\wedge z)}\right)\wedge\left(\frac{zcp}{(a\wedge x)(a\wedge z)}\right)}{\left(\frac{cq}{a\wedge z}\right)\wedge\left(\frac{cp}{a\wedge x}\right)},\frac{\left(\frac{aycq}{(a\wedge x)(a\wedge z)}\right)\wedge\left(\frac{awcp}{(a\wedge x)(a\wedge z)}\right)}{\left(\frac{cq}{a\wedge z}\right)\wedge\left(\frac{cp}{a\wedge x}\right)}\right)\\
&=\left(\frac{x}{a\wedge x},\frac{ay}{a\wedge x}\right)+\left(\frac{z}{a\wedge z},\frac{aw}{a\wedge z}\right)\\
&=(1,a)(x,y)+(1,a)(z,w).
\end{aligned}$$

Combining these two results gives

$$\begin{aligned}
(a,b)\,[(x,y)+(z,w)]&=(a,1)(1,b)\,[(x,y)+(z,w)]\\
&=(a,1)\,[(1,b)(x,y)+(1,b)(z,w)]\\
&=(a,b)(x,y)+(a,b)(z,w).
\end{aligned}$$

Hence, multiplication is distributive over this addition.

We remark that the proof of associativity of $+_f$ does not require that f be a homomorphism; however, our proof of distributivity does. In part (c) of the proof of Theorem 3.4 we will show the necessity of the homomorphism property of f.

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Now we show that β (and hence $+_f$) is continuous. Without loss of generality we can assume that P is dense in $[1,\infty)$. Let $\{(x_n,y_n)\}_{n=1}^{\infty}$ be a sequence from B_P converging to a point (x,y) of B_P, and let $\beta(x_n,y_n)\equiv p_n$. Since $\{(x_n,y_n)\}_{n=1}^{\infty}$ is convergent and hence bounded in P^2, $\{p_n\}_{n=1}^{\infty}$ is also bounded. (In fact $\{p_n\}_{n=1}^{\infty}$ is bounded below by some $\varepsilon>0$), and hence has a subsequence $\{p_{n_i}\}_{i=1}^{\infty}$ which converges to a point a in $(0,\infty)$. Then by definition of p_n and the continuity of f, $y/a=\lim_i f(x_{n_i}/p_{n_i})=f(\lim_i(x_{n_i}/p_{n_i}))=f(x/a)$ and hence $a=\beta(x,y)$. It follows that $\beta(x_n,y_n)\to\beta(x,y)$. This completes the proof that $+_f$ is a semiring addition. Moreover, since $\beta(x,y)\leqslant 1$ if and only if $y\leqslant f(x)$, we have

$$
\begin{aligned}
(x,y)+(1,1) &= \left(\frac{x\beta(1,1)\wedge 1\beta(x,y)}{\beta(1,1)\wedge\beta(x,y)},\frac{y\beta(1,1)\wedge 1\beta(x,y)}{\beta(1,1)\wedge\beta(x,y)}\right)\\
&= \left(\frac{x\wedge\beta(x,y)}{1\wedge\beta(x,y)},\frac{y\wedge\beta(x,y)}{1\wedge\beta(x,y)}\right)=\left(\frac{\beta(x,y)}{1\wedge\beta(x,y)},\frac{\beta(x,y)}{1\wedge\beta(x,y)}\right)\\
&= \begin{cases}(1,1) & \text{if } y\leqslant f(x),\\ (\beta(x,y),\beta(x,y)) & \text{if } y\geqslant f(x).\end{cases}
\end{aligned}
$$

Hence, $U=\{(x,y)\in R\colon y\leqslant f(x)\}$.

REMARK 3.3. The homomorphisms of $[(0,\infty),\cdot]$ to $[(0,\infty),\cdot]$ are the functions $\{f_\alpha\colon \alpha \text{ real}\}$ where $f_\alpha(x)=x^\alpha$ for every x, and the ones which satisfy the conditions of Example 3.2 must have $0\leqslant\alpha<1$. Clearly, any such α satisfies the conditions if $P=[1,\infty)$. However, suppose P is cyclic. Then we can calculate from the relationship $(x/\beta(x,y),y/\beta(x,y))\in\text{graph }(f)\cap G^2$, that if $f(x)=x^\alpha$ for every x in G, then $\beta(x,y)=(y/x^\alpha)^{1/(1-\alpha)}$, and since $P=\{1,a,a^2,\dots\}$ where $a>1$, $\beta(1,a)$ must be a^k for some integer k. That is, $\beta(1,a)=a^{1/(1-\alpha)}=a^k$ and so $k=1/(1-\alpha)$ and hence $\alpha=(k-1)/k$ if $k\neq 0$. We show in part (b) of the proof of Theorem 3.4 that every semiring addition on B_P with P cyclic and U a proper subset of R_1 is $+_\alpha$ where $\alpha=N/(N+1)$ for a non-negative integer N.

THEOREM 3.4. *Let $+$ be a semiring addition on B_P which is min on $P\times\{1\}$. Suppose that U is a proper subset of R_1. Then there exists a non-decreasing homomorphism $f\colon G\to(0,\infty)$ which satisfies the properties of Example* 3.2 *and $+=+_f$ as in Example* 3.2.

PROOF. We prove this theorem in several steps, which we state as follows.

(a) If $h\colon B_P\to G\cup\{0,\infty\}$ is defined so that

$$
h(x,y)=\begin{cases}\sup\{d\colon (x,y)+(d,d)=(x,y)\} & \text{if } (x,y)\in L,\\ \inf\{a\colon (1,1)+(x/a,y/a)=(1,1)\} & \text{if } (x,y)\in R,\end{cases}
$$

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then h is well defined on B_P, the range of h is actually contained in G, and for x and y in P, $h(x,y) = xy/h(y,x)$. Furthermore, for each (x,y) and (z,w) in B_P, if $(z/x, w/y) \in R \setminus D$, then

$$(x,y)+(z,w) = \left(\frac{x(h(z,w) \vee h(x,y))}{h(x,y)}, \frac{y(h(z,w) \vee h(x,y))}{h(x,y)}\right).$$

(b) If P is cyclic, then there exists a non-negative integer N such that $(x,y) \in U$ if and only if $y \leqslant x^{N/(N+1)}$; we let $f(x) = x^{N/(N+1)}$, and in this case, for every (x,y) in B_P, $\beta(x,y) = (y^{(N+1)}/x^N)$ where β is defined for f as in (3.2). If P is dense in $[1,\infty)$ and $f\colon G \to (0,\infty)$ is defined by

$$f(x) = \begin{cases} \sup\{y \in P\colon (x,y) \in U\} & \text{if } x \geqslant 1, \\ 1/f(1/x) & \text{if } x \leqslant 1, \end{cases}$$

then f is a continuous non-decreasing function.

(c) If β is defined for f as in Example 3.2 then $h \equiv \beta$ in the dense case as well as the cyclic. Hence, in either case, $+ = +_f$. Moreover, f is a homomorphism. We now commence the proof.

PROOF. (a) It is a simple observation that $h(x,x) = x$ whether calculated in R or in L, and it follows from Lemma 3.1 that the range of h is contained in $G \cup \{0,\infty\}$. Note that $h(x,y) = 0$ if and only if $(x,y) \in R \setminus D$ and $(1,1)+(ax,ay) = (1,1)$ for every $a \geqslant 1/y$; and $h(x,y) = \infty$ if and only if $(x,y) \in L \setminus D$ and $(x,y)+D = \{(x,y)\}$. We show later that h takes on neither of these values. Suppose that $(x,y) \in L$ and $h(x,y) = c = d/p$ as in Lemma 3.1(b). Then since

$$(x,y)+(c,c) = (x,y)$$

and for $t > c$,

$$(x,y)+(t,t) = (xt/c, yt/c),$$

we have

$$(1,1)+(c/x, c/y) = (1,1)$$

and for $t > c$,

$$(1,1)+(t/x, t/y) = (t/c, t/c).$$

Hence,

$$(1,1)+(y,x) = (1,1)+(xy/x, xy/y) = (xy/c, xy/c)$$

and hence,

$$h(y,x) = xy/c.$$

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We now analyse the addition on B_P. Recall from Remark 2.6 that if (x, y) and (z, w) are elements of B_P such that neither $(z/x, w/y)$ nor $(x/z, y/w)$ is in $R \setminus D$ then

$$(x, y)+(z, w) = \left(\frac{xw \wedge yz}{y \wedge w}, y \vee w\right).$$

We can thus assume that $(z/x, w/y)$ is in $R \setminus D$ and consider three cases: (1) both addends are in R; (2) $(x, y) \in L$ and $(z, w) \in R$; and (3) both addends are in L. We begin by considering elements on which h is finite and non-zero, and then show that either $h(R \setminus D) = \{0\}$ and $h(L \setminus D) = \{\infty\}$, or $h(B_P) \subset G$. Notice that if $x < y$, then $(x, y)+D = \{(x, y)\}$ if and only if $(1, 1)+D(y/x, 1) = \{(1, 1)\}$. We also remark that if $(x, y) \in R$, then if $h(x, y) = a \geqslant 1$, then for every $b \geqslant 1$, $h(bx, by) = ba, by$ (a).

Now if (x, y) and (z, w) are two elements of R with $h(x, y) = d \geqslant 1$ and $h(z, w) = c \geqslant 1$, then let $(1, 1)+(z/x, w/y) = (g, g)$. Then

$$\begin{aligned}(gd, gd) &= (1, 1)+(gx, gy) = (1, 1)+[(x, y)+(z, w)] \\ &= [(1, 1)+(x, y)]+(z, w) = (d, d)+(z, w) \\ &= [(d, d)+(1, 1)]+(z, w) = (d, d)+[(1, 1)+(z, w)] \\ &= (d, d)+(c, c) = (d \vee c, d \vee c),\end{aligned}$$

and so $g = d \vee c/d$. Hence,

$$(x, y)+(z, w) = \left(\left(\frac{d \vee c}{d}\right)x, \left(\frac{d \vee c}{d}\right)y\right).$$

If either $d < 1$ or $c < 1$, let $d \wedge c = 1/b$ where $b > 1$; then $h(bx, by) = bd \geqslant 1$ and $h(bz, bw) = bc \geqslant 1$, so that by the preceding formula,

$$(bx, by)+(bz, bw) = \left(\left(\frac{bd \vee bc}{bd}\right)bx, \left(\frac{bd \vee bc}{bd}\right)by\right) = \left(b\left(\frac{d \vee c}{d}\right)x, b\left(\frac{d \vee c}{d}\right)y\right),$$

and hence

$$(x, y)+(z, w) = \left(\left(\frac{d \vee c}{d}\right)x, \left(\frac{d \vee c}{d}\right)y\right)$$

for any (x, y) and (z, w) in R with $h(x, y) > 0$, $h(z, w) > 0$, and $(z/x, w/y)$ in R.

If $(x, y) \in L$ and $(z, w) \in R$ with $h(x, y) = d < \infty$ and $h(z, w) = c > 0$, then

$$\begin{aligned}(x, y)+(z, w) &= (x, y)+(d, d)+(z, w) = (x, y)+(d \vee c, d \vee c) \\ &= \left(\left(\frac{d \vee c}{d}\right)x, \left(\frac{d \vee c}{d}\right)y\right) \quad \text{if } w > d;\end{aligned}$$

and if $w \leqslant d$, then

$$\begin{aligned}(x, y)+(z, w) &= [(x, y)+(w, w)]+(z, w) = (x, y)+[(w, w)+(z, w)] \\ &= (x, y)+(w, w) = (x, y) = \left(\left(\frac{d \vee c}{d}\right)x, \left(\frac{d \vee c}{d}\right)y\right)\end{aligned}$$

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ıd hence the formula of the preceding paragraph holds between elements of and R on which h is neither 0 nor ∞.

Finally, let (x,y) and (z,w) be elements of L with $h(x,y) = d < \infty$, $h(z,w) = c < \infty$, ıd $(z/x, w/y) \in R \setminus D$. Then $(x,y)+(z,w) = (tx, ty)$ for some $t \geqslant 1$. If $d > c$, then

$$\begin{aligned}(tx,ty) &= (x,y)+(z,w) = (x,y)+(d,d)+(z,w) = (x,y)+(dz/c, dw/c)\\ &= (x,1)\,[(1,1)+(dz/cx, dw/cy)]\,(1,y) = (x,1)\,(dt/c, dt/c)\,(1,y)\\ &= (dtx/c, dty/c)\end{aligned}$$

ıd so d would equal c. Hence, $d \leqslant c$. Now choose (a,b) in $R \setminus D$ with

$$(a,b) = q > p \vee dt;$$

ıen $(a/x, b/y)$ and $(a/z, b/w)$ are in $R \setminus D$ and

$$\begin{aligned}(qx/d, qy/d) = (qtx/dt, qty/dt) = (tx,ty)+(a,b) &= [(x,y)+(z,w)]+(a,b)\\ &= (x,y)+[(z,w)+(a,b)] = (x,y)+(qz/c, qw/c)\\ &= (x,1)\,[(1,1)+(qz/cx, qw/cy)]\,(1,y)\\ &= (x,1)\,(qt/c, qt/c)\,(1,y) = (qtx/c, qty/c),\end{aligned}$$

ɔ that $t = c/d$ and hence

$$(x,y)+(z,w) = (cx/d, cy/d) = \left(\left(\frac{d \vee c}{d}\right)x, \left(\frac{d \vee c}{d}\right)y\right).$$

Now suppose that there exists an (a,b) in L/D with $h(a,b) = \infty$. Note that if $\leqslant a$, $h(c,d) = \infty$, and $h(D(a,b) \cap B_P) = \{\infty\}$. Hence, if $h(L \setminus D) \neq \{\infty\}$, we may ssume that there exists (p,x) in $L \setminus D$ such that $h(p,s) = q < \infty$ and $(p/a, s/b) \in R \setminus D$. 'hen $h(q/p, q/s) = 1$ and we may choose $c > q$, and let $(z,w) = (qc/p, qc/s)$; then $(z,w) = c$ and $z > w$. Now let $(a,b)+(p,s) = (ha, hb)$ for some $h \geqslant 1$. Then

$$(ha,hb)+(z,w) = (ha,hb)+(w,w)+(z,w) = (ha,hb)+(w,w) = (ha,hb),$$

ɔ that

$$\begin{aligned}(ha,hb) = (ha,hb)+(z,w) &= [(a,b)+(p,s)]+(z,w) = (a,b)+[(p,s)+(z,w)]\\ &= (a,b)+(cp/q, cs/q) = (hca/q, hcb/q),\end{aligned}$$

ɔ that $c = q$. But c was chosen larger than q, and hence, there is no such (p,s). 'hus, $h(L \setminus D) = \{\infty\}$, and $h(R \setminus D) = \{0\}$, which contradicts the assumption that l be a proper subset of R_1 and, hence, the range of h is contained in G, and the formula

$$(x,y)+(z,w)=\left(x\left(\frac{h(z,w)\vee h(x,y)}{h(x,y)}\right),\, y\left(\frac{h(z,w)\vee h(x,y)}{h(x,y)}\right)\right)$$

is valid for every pair (x,y), (z,w) in B_P with $(z/x, w/y)\in R\setminus D$.

(b) If P is cyclic, we put $P=\{1,x,x^2,\ldots\}$ for $x>1$. Since U is a proper subset of R_1, there is an N such that $(x^{N+1},x^N)\in U$ but $(x^{N+2},x^{N+1})\notin U$. Then

$$(x^{N+2},x^{N+1})+(1,1)=(x,x)$$

by (a). Now

$$\begin{aligned}(x^{2(N+1)},x^{2N})+(1,1)&=(x^{2(N+1)},x^{2N})+(x^{N+1},x^N)+(1,1)\\&=(x^{N+1},1)\,[(x^{N+1},x^N)+(1,1)]\,(1,x^N)+(1,1)\\&=(x^{N+1},1)(1,1)(1,x^N)+(1,1)=(x^{N+1},x^N)+(1,1)\\&=(1,1)\end{aligned}$$

and by induction we can show that for every k, $(x^{k(N+1)},x^{kN})\in U$. On the other hand, we will show that for every k, $(x^{(N+1)k-N},x^{Nk-(N-1)})+(1,1)=(x,x)$. This is true for $k=1$ by the way N was selected. Now suppose that it is true for $k<n$. Then

$$\begin{aligned}&(x^{(N+1)n-N},x^{Nn-(N-1)})+(1,1)\\&\quad=(x^{(N+1)n-N},x^{Nn-(N-1)})+(x^{(N+1)(n-2)},x^{N(n-2)})+(1,1)\\&\quad=(x^{(N+1)(n-2)},1)\,[(x^{N+2},x^{N+1})+(1,1)]\,(1,x^{N(n-2)})+(1,1)\\&\quad=(x^{(N+1)(n-2)},1)(x,x)(1,x^{N(n-2)})+(1,1)\\&\quad=(x^{(N+1)(n-2)+1},x^{N(n-2)+1})+(1,1)\\&\quad=(x^{(N+1)(n-1)-N},x^{N(n-1)-(N-1)})+(1,1)=(x,x).\end{aligned}$$

It follows from this that $(a,b)\in U$ if and only if $a\leqslant b^{N/(N+1)}$. If we define $f\colon G\to(0,\infty)$ by $f(a)=a^{N/(N+1)}$ then since $\alpha=N/(N+1)$, by Example 3.2,

$$\beta(a,b)=\left(\frac{b}{a^\alpha}\right)^{1/(1-\alpha)}=\left(\frac{b}{a^{N/(N+1)}}\right)^{1-(N/N+1)}=\left(\frac{b}{a^{N/(N+1)}}\right)^{N+1}=\frac{b^{N+1}}{a^N},$$

and it is not hard to verify that this formula also gives $h(a,b)$ and hence for every $(a,b)\in B_P$, $h(a,b)=\beta(a,b)$. We will show in (c) that $+=+_f$.

We now assume P is dense in $[1,\infty)$. Let $x\in P$. Since $(1,1)+(x,z)\neq(1,1)$ for any $z\geqslant x$, the set $U_x=\{y\in P\colon (x,y)\in U\}$ is bounded above. We define $f\colon G\to(0,\infty)$ by

$$f(x)=\begin{cases}\sup U_x & \text{if } x\geqslant 1,\\ 1/f(1/x) & \text{if } x\leqslant 1.\end{cases}$$

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By Lemma 2.8(a), (c), f is a non-decreasing function. Let $x \in P$ and $x_n \to x$. Since f is non-decreasing $f(x_n)$ is bounded and hence has a convergent subsequence $\{f(x_{n_i})\}$, which converges to an element y of $[1, \infty)$. If $y < f(x)$, let $z \in P$ such that $y < z < f(x)$. Since $f(x_{n_i})$ is eventually strictly greater than $z, y \geqslant z$. Hence $y \geqslant f(x)$. Similarly, $y \leqslant f(x)$ and so $y = f(x)$, and it follows that $f(x_n) \to f(x)$, and thus that f is continuous on P. Since f, when restricted to $G \cap (0, 1]$, is the composition of inversions with $f|_P$, f is continuous on $G \cap (0, 1]$ and since $f(1) = 1$, f is continuous on G.

We remark that $\text{graph}(f) \cap B_P \subseteq U$; for suppose $(x, y) \in \text{graph}(f) \cap B_P$. Then $y = \sup\{z \colon (x, z) \in U\}$ and since U is closed, $(x, y) \in U$.

(c) For any element (x, y) of $R \setminus D$, let $a = h(x, y)$. Then since

$$(1, 1) + (x/a, y/a) = (1, 1), \quad y/a \leqslant f(x/a).$$

If $k = f(x/a)$, suppose $y/a < k$; then since P is dense in $[1, \infty)$, there is a $p \in P$ such that $y/a < p < k$ and $(x/a, p) \in U$, which implies that $(px/y, p) \in U \cap D(x, y)$ (since $px/y > x/a$), contradicting (a). Hence, $y/a = f(x/a)$. Suppose that graph(f) contains another point $(x/b, y/b)$ of $D(x, y) \cap B_P$. We may assume that $b > a$. Then if $a < c < b$, since $(x/a, y/a) \in U$, $(x/c, y/c) \in U$ and so $y/c \leqslant f(x/c)$. Suppose $y/c < f(x/c)$; then there exists $d \in P$ such that $(x/c, d) \in U$ and $y/c < d < f(x/c)$. Hence $(x/b, dc/b) \in U$ and so $y/b < dc/b \leqslant f(x/b)$. This contradiction shows that if graph(f) contains two points of $D(x, y)$, it contains all the points on a straight line between those two points. Now we show that $(x/a, y/a) = (x/b, y/b)$. Let $x_n \to x/b$ from the left. Then $(x_n, f(x_n)) \to (x/b, y/b)$ and $(1, 1) + (bx_n/a, by_n/a) \to (1, 1) + (x/a, y/a) = (1, 1)$, but for every n, $(1, 1) + (bx_n/a, by_n/a) = (b/a, b/a)$. Thus, $b/a = 1$, and hence,

$$(x/h(x, y), y/h(x, y))$$

is the unique intersection point of $D(x, y)$ and graph(f). This shows that $h \equiv \beta$ on $R \setminus D$ and since $h(x, x) = x = \beta(x, x)$ for every $x \in P$, $h \equiv \beta$ on R. Now if $(x, y) \in L \setminus \mathrm{D}$ and $h(x, y) = a$, then $h(y/x, 1) = 1/a$ and so $\beta(y/x, 1) = 1/a$ by what was just shown. Hence, $a/y = f(a/x)$ and $(a/x, a/y)$ is the unique intersection point of graph(f) and $D(y/x, 1)$; it follows that $y/a = f(x/a)$ and that $(x/a, y/a)$ is the unique intersection of $D(x, y)$ and graph(f), and so h agrees with β on L as well.

Now we wish to show that $+ = +_f$ as in Example 3.2. We may assume P is either dense or cyclic. Since $h = \beta$ and f is non-decreasing, it follows that h has the property proved for β in Example 3.2 (which only involved the monotonicity of f) that if $w/y < z/x$, then $x\beta(z, w) \leqslant z\beta(x, y)$, $y\beta(z, w) \leqslant w\beta(x, y)$ and $p \geqslant q$ if $y \geqslant w$. If we let

$$k = \left(\frac{x\beta(z, w) \wedge z\beta(x, y)}{\beta(x, y) \wedge \beta(z, w)}, \frac{y\beta(z, w) \wedge w\beta(x, y)}{\beta(x, y) \wedge \beta(z, w)}\right),$$

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then if $x \leqslant z$, $y \leqslant w$ and $w/y < z/x$, we have

$$k = \left(\frac{x(\beta(x,y) \vee \beta(z,w))}{\beta(x,y)}, \frac{y(\beta(x,y) \vee \beta(z,w))}{\beta(x,y)}\right),$$

which equals $(x,y)+(z,w)$ by the formula derived in (c). Similarly, if $x \leqslant z$, $y \leqslant w$ and $w/y \geqslant z/x$, then $\beta(x,y) \leqslant \beta(z,w)$, $x\beta(z,w) \geqslant z\beta(x,y)$ and $y\beta(z,w) \geqslant w\beta(x,y)$ and so $k=(z,w)$ which equals $(x,y)+(z,w)$ by Remark 2.6. Finally, if $x \leqslant z$ and $y \geqslant w$, then

$$\beta(x,y) \geqslant \beta(z,w)$$

and since

$$x/z \leqslant 1 \leqslant y/w, \quad x\beta(z,w) \leqslant z\beta(x,y) \quad \text{and} \quad y\beta(z,w) \leqslant w\beta(x,y).$$

So $k=(x,y)$, which, again by Remark 2.6, is equal to $(x,y)+(z,w)$. This shows that $+ = +_f$ as in Example 3.2.

We now show that f is a homomorphism. We assume first that (x,y) and (z,w) are two elements of $R \setminus D$ with $y=f(x)$ and $w=f(z)$. Without loss of generality, suppose $w/z \geqslant y/x$. Then $\beta(z/w,1)=1/w$, and if p is any element of $P \setminus \{1\}$, $\beta(px,py)=p$. Now

$$(zp/w,p) = (z/w,1)(p,p) = (z/w,1)\,[(1,1)+(px,py)] = (z/w,1)+(zpx/w,py)$$
$$= \left(\frac{zr/w}{r \wedge (1/w)}, \frac{r}{r \wedge (1/w)}\right)$$

(by the calculations used in Example 3.2 for proving associativity), where $r=\beta(zpx/w),py)$. Hence, $r/(r \wedge (1/w)) = p$ and since $p>1$, $r \wedge (1,w) \neq r$ and so $p=rw$. Hence, $yw = py/(p/w) = py/r = f(zpx/wr) = f(xz)$ and hence f is a homomorphism when restricted to $f^{-1}(P)$. Now suppose that $z<1$ and $w=f(z) \in G \cap (0,1]$. Then if (x,y) is as above, $w/z \geqslant y/x$, $\beta(1,w/z)=1/z$, and if $r=\beta(px,pyw/z)$ where $p>1$, we have

$$(p,wp/z) = (1,w/z)+(px,pyw/z) = \left(\frac{r}{r \wedge (1/z)}, \frac{wr}{r \wedge (1/z)}\right).$$

Hence $p=rz$ and $yw = pyw/rz = f(xp/r) = f(zxp/rz) = f(xz)$. Since for $x<1$, $f(x) = 1/f(1/x)$, we easily see that if both x and z are in $G \cap (0,1]$ then $f(xz)=f(x)f(z)$ and so f is a homomorphism on $f^{-1}(G)$. Now suppose $x \in G$ and $y=f(x) \in (0,\infty)$. Let $\{d_n\}_{n=1}^{\infty}$ be a sequence converging to x/y; then if $p_n = \beta(d_n,1)$, it follows that $(d_n/p_n, 1/p) \to (x,y)$. Now if $w=f(z)$ and $y=f(x)$ where x and z are in G, let $x_n \to x$, $z_n \to z$, $y_n = f(x_n)$ and $w_n = f(z_n)$. Then $y_n w_n \to yw$; but $y_n w_n = f(x_n z_n) \to f(xz)$. Hence, f is a homomorphism on all of G. This completes the proof of Theorem 3.4.

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