ON CONFORMALLY FLAT SPACES WITH COMMUTING CURVATURE AND RICCI TRANSFORMATIONS

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Let \((M, g)\) be a \(C^\infty\) Riemannian manifold and \(A\) be the field of symmetric endomorphisms corresponding to the Ricci tensor \(S\); that is,

\[ S(X, Y) = g(AX, Y). \]

We consider a condition weaker than the requirement that \(A\) be parallel (\(\nabla A = 0\)), namely, that the "second exterior covariant derivative" vanish (\(\nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A = 0\)), which by the classical interchange formula reduces to the property

\[ (P) \quad R(X, Y) \circ A = A \circ R(X, Y), \]

where \(R(X, Y)\) is the curvature transformation determined by the vector fields \(X\) and \(Y\).

The property \((P)\) is equivalent to

\[ (Q) \quad R(AX, X) = 0. \]

To see this we observe first that a skew symmetric and a symmetric endomorphism commute if and only if their product is skew symmetric. Thus we have

\[ (P) \iff R(Z, W)A\text{ is skew symmetric} \]

\[ \iff g(R(Z, W)AX, X) = 0 \]

\[ \iff g(R(AX, X)Z, W) = 0 \]

\[ \iff (Q). \]

Let \(M\) be a connected conformally flat manifold of dimension \(n, n \geq 3\). Then the Ricci endomorphisms determine the curvature according to the formula

\[ (1) \quad R(X, Y) = \frac{1}{n-2} (AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y, \]

where \(r = \text{trace } A\) and \(X \wedge Y\) denotes the endomorphism

\[ Z \mapsto g(Y, Z)X - g(X, Z)Y. \]

In this paper the connected conformally flat spaces satisfying \((P)\) are classified.

**Lemma 1.** Let \(M\) be an \(n\)-dimensional conformally flat space satisfying \((P)\).
Then
\[ A^2 - \frac{r}{n-1} A = \rho I, \]
where \( \rho \) is a \( C^\infty \) function on \( M \) and \( I \) is the identity field.

\textbf{Proof.} Setting \( Y = AX \) in (1) and then applying \((Q)\) gives
\[ BX \wedge X = 0, \]
where
\[ B = A^2 - \frac{r}{n-1} A. \]
Since (3) may be interpreted as an exterior product, we conclude that every \( X \)
is an eigenvector of \( B \), so \( B = \rho I \) for some scalar field \( \rho \).

\textbf{Lemma 2.} Under the conditions in Lemma 1, \( A \) has at most the two eigenvalues
\[ \frac{r \pm \sqrt{r^2 + 4(n-1)\rho}}{2(n-1)}. \]
Let \( M' \) be the open subset of \( M \) on which \( r^2 + 4(n-1)\rho \neq 0 \). Then the eigenspaces of \( A \) form smooth complementary orthogonal distributions on each connected component of \( M' \).

The eigenvalues are the roots of
\[ \mu^2 - \frac{r}{n-1} \mu - \rho = 0; \]
the rest is also routine.

Let us fix notation as follows: The eigenvalues of \( A \) are \( \mu_1 \) and \( \mu_2 \). They are defined and continuous on all of \( M \) and distinct on \( M' \). The eigenspaces on \( M' \) are \( D_1 \) and \( D_2 \), of dimensions \( k \) and \( n-k \). We shall use adapted orthonormal frames and coframes \( \{X_a, X_b\} \) and \( \{\omega_a, \omega_b\}, a, b = 1, \ldots, k \) and \( i, j = k+1, \ldots, n \); moreover, \( i, j = 1, \ldots, n \). The corresponding connection and curvature forms are \( \omega_{ab} \), etc. and \( \Omega_{ab} \), etc.

\textbf{Lemma 3.} Let \( K = (\mu_1 - \mu_2)/(n-2) \). On \( D_1 \) the sectional curvature is \( K \), on \( D_2 \) it is \(-K\), and on mixed sections it vanishes; that is,
\[ \Omega_{ab} = K \omega_a \wedge \omega_b, \]
\[ \Omega_{ab} = -K \omega_a \wedge \omega_b, \]
\[ \Omega_{ab} = 0. \]

\textbf{Proof.} Noting that \( r/(n-1) = \mu_1 + \mu_2 \), formula (1) becomes
\[ R(X, Y) = \frac{1}{n-2} \{AX \wedge Y + X \wedge AY - (\mu_1 + \mu_2)X \wedge Y \}. \]
The rest follows by taking orthonormal eigenvectors for \( X \) and \( Y \).
Note that $M'$ is just the set on which $K \neq 0$.

**Theorem.** Let $M$ be an $n$-dimensional connected conformally flat space satisfying $(P)$, $n \geq 3$. Then $M$ is one of four types:

(a) $M$ is flat ($M'$ empty).
In the remaining cases $M = M'$; that is, $M$ is either flat everywhere or has no flat points; moreover, $k$ is constant.

(b) $M$ has constant curvature ($k = 0$ or $n$).

(c) $M$ is locally the Riemannian product of a $k$-dimensional space of constant curvature $K$ and an $(n - k)$-dimensional space of constant curvature $-K(1 \leq k \leq n - 1)$.

(d) There is an open $C^\infty$ map $t: M \to \mathbb{R}^+$ (positive reals) such that $K = K_0/t^2$ for some constant $K_0$. The map $t$ is a Riemannian submersion having fibres which are totally umbilical hypersurfaces of constant (intrinsic) curvature $(1 + K_0)/t^2$ ($k = 1$ or $n - 1$).

**Proof.** Define the vector valued 1-form $F = F^i \otimes X_i$ by

$$F^i = A^i \omega^j - \frac{r}{2(n - 1)} \omega^i,$$

where $A^i_j$ are the components of $A$. (The summation convention is employed here and in the sequel.) The $X_i$ and $\omega^i$ are any local vector field basis and the dual basis of 1-forms, respectively. If $\omega^i_j$ are the connection forms for this basis we define the exterior covariant derivative $DF$ of $F$ as the vector-valued 2-form $(DF)^i \otimes X_i$, where

$$(DF)^i = dF^i + \omega^i_j \wedge F^j.$$

It is easily checked that $DF$ is independent of the choice of basis. Using the first structural equation viz., $d\omega^i = -\omega^i_j \wedge \omega^j$, and the coefficients $\Gamma^i_{kj}$ of $\omega^i_j (\omega^i_j = \Gamma^i_{kj} \omega^k)$, we obtain

$$(DF)^i = \left( X_k A^i_j + A^i_j \Gamma^k_{kj} + A^i_k \Gamma^j_{kh} - \frac{1}{2(n - 1)} \delta^i_j X_k r \right) \omega^k \wedge \omega^j$$

$$= \left( \nabla_k A^i_j - \frac{1}{2(n - 1)} \delta^i_j \nabla_k r \right) \omega^k \wedge \omega^j,$$

where $\delta^i_j$ is the Kronecker delta. As a tensor, this has the components

$$(n - 2) C^i_{jk} = \nabla_k A^i_j - \nabla_j A^i_k - \frac{1}{2(n - 1)} (\delta^i_j \nabla_k r - \delta^i_k \nabla_j r),$$

where $C^i_{jk}$ is Weyl’s 3-index tensor. For a conformally flat space it is known that $C^i_{jk} = 0$. We use this by calculating $DF$ in terms of an orthonormal basis adapted to the distributions $D_i$. In particular we can lower all superscripts.
Thus,

\[ F_\alpha = A_\alpha \omega_\alpha - \frac{r}{2(n-1)} \omega_\alpha \]

\[ = \mu_1 \omega_\alpha - \frac{1}{2} (\mu_1 + \mu_2) \omega_\alpha \]

\[ = L \omega_\alpha, \]

where \( L = (n-2)K/2 \), and

\[ F_\alpha = A_\alpha \omega_\alpha - \frac{r}{2(n-1)} \omega_\alpha \]

\[ = \mu_2 \omega_\alpha - \frac{1}{2} (\mu_1 + \mu_2) \omega_\alpha \]

\[ = -L \omega_\alpha, \]

from which

\[ dF_\alpha = dL \wedge \omega_\alpha + L d\omega_\alpha + \omega_{ab} \wedge \omega_{\beta} - \omega_{\alpha \beta} \wedge \omega_{\beta} \]

\[ = dL \wedge \omega_\alpha - L \omega_\alpha \wedge \omega_\alpha + L (\omega_{ab} \wedge \omega_{\beta} - \omega_{\alpha \beta} \wedge \omega_{\beta}) \]

\[ = dL \wedge \omega_\alpha - 2L \omega_{\alpha \beta} \wedge \omega_{\beta} \]

\[ = 0, \]

\[ dF_\alpha = -dL \wedge \omega_\alpha + 2L \omega_{ab} \wedge \omega_\beta \]

\[ = 0. \]

When \( k = n \), \( K \) is constant and \( M' = M \) follows immediately from Schur's theorem (or (4)).

Otherwise, by Cartan's lemma, (4) says that for each \( a \), \( dL \) and the \( \omega_{ab} \) are dependent at most on \( \omega_a \) and the \( \omega_\alpha \) and (5) says that the same forms are dependent at most on \( \omega_a \) and the \( \omega_\alpha \). Thus if \( 2 \leq k \leq n - 2 \) we can make two choices of \( \alpha \) for each \( a \) and vice-versa, showing that \( dL = 0 \) and \( \omega_{\alpha \beta} = 0 \). Consequently, \( L \) and \( K = 2L/(n-2) \) are constant and \( D_1 \) and \( D_2 \) are parallel (in particular, completely integrable).

When \( k = 1 \) we still have by (5) that \( dL \) and \( \omega_{a1} \) are dependent at most on \( \omega_a \) and \( \omega_1 \). Making two choices of \( \alpha \), we get \( dL = H \omega_1 \) for some \( C^\infty \) function \( H \). Then, (4) reduces to \( \omega_{1\beta} \wedge \omega_{\beta} = 0 \), so the \( \omega_{1\beta} \) cannot depend on \( \omega_1 \). Hence \( \omega_{1a} = C_\alpha \omega_\alpha \) (\( \alpha \) not summed) for some scalar field \( C_\alpha \). But then by (5) again

\[ -H \omega_1 \wedge \omega_\alpha + 2L (-C_\alpha \omega_\alpha) \wedge \omega_1 = 0; \]

that is, \( C = C_\alpha = H/2L \) is the same for all \( \alpha \). The geometrical interpretation of the relation \( \omega_{1a} = C_\alpha \omega_\alpha \) is that \( D_2 \) (the distribution annihilated by \( \omega_1 \)) is completely integrable and has totally umbilical leaves. In fact, \( d\omega_1 = -\omega_{1a} \wedge \omega_a = 0 \), so locally \( \omega_1 \) has a primitive \( u \); that is, \( du = \omega_1 \).
A differential equation for \( C \) may be obtained from the fact that the curvature of the section \( X_1 \wedge X_\alpha \) vanishes:

\[
\Omega_{1\alpha} = d\omega_{1\alpha} + \omega_{1\beta} \wedge \omega_{\beta\alpha} = dC \wedge \omega_\alpha + Cd\omega_\alpha + \omega_{1\beta} \wedge \omega_{\beta\alpha} = dC \wedge \omega_\alpha - C\omega_{\alpha i} \wedge \omega_i + C\omega_\beta \wedge \omega_{\beta\alpha} = \left(\frac{dC}{du} - C^2\right)\omega_1 \wedge \omega_\alpha.
\]

Therefore,

\[
\frac{dC}{du} - C^2 = 0.
\]

Solving this, we obtain either \( C = 0 \) or

\[
C = -\frac{1}{u - u_0} = -\frac{1}{t},
\]

where \( u_0 \) is a constant and hence \( t \) is another primitive for \( \omega_1 \). The signs of \( C \) and \( \omega_1 \) can be changed, if necessary, so as to make \( t > 0 \).

If \( C = 0 \), then it must be so on connected sets. Hence \( H = dL/du = 2LC = 0 \) and \( L \), and hence \( K_\alpha \), are constant. Moreover, \( C = 0 \) says \( D_1 \) and \( D_2 \) are parallel so we are back in case (c).

If \( C \neq 0 \), then we solve \( H = 2LC \) for \( L \), obtaining \( L = L_0/t^2 \), and hence \( K = K_0/t^2 \) for constants \( L_0 \) and \( K_0 \). Thus, \( t = (K_0/K)^{\frac{1}{2}} \) is a primitive for \( \omega_1 \) in each component of \( M' \). We don’t know yet whether there is only one component, so \( K_0 \) might have several values. As a map \( t : M' \to \mathbb{R}^+ \), \( t \) is clearly a Riemannian submersion whose fibres are the leaves of \( D_2 \). As such it is distance-non-increasing. Now suppose that \( M' \neq M \). Let \( \gamma \) be a curve entirely in \( M' \) except for the last point \( \gamma(1) \in M - M' \). The length of \( t\gamma \) is at most that of \( \gamma \) and is therefore bounded. Hence \( t\gamma(1) = \lim_{s \to 1} t\gamma(s) \) exists and is not \( \infty \). It cannot be 0 either, for then there would be a sequence of plane sections converging to a section at \( \gamma(1) \) and having curvatures diverging to \( \lim_{s \to 0} K_0/t^2 \). A similar difficulty is presented at any other finite limit for \( t\gamma(1) \), since we would then have curvatures converging to nonzero values contradicting the fact that \( M - M' \) is flat. Hence, \( M = M' \).

To complete the proof we calculate the intrinsic curvature of the leaves of \( D_2 \). The connection forms \( \omega_{\alpha\beta} \), restricted to a leaf, become the connection forms of the leaf. Thus, denoting the curvature forms of a leaf by \( \Phi_{\alpha\beta} \), the second structural equation for a leaf is

\[
d\omega_{\alpha\beta} = -\omega_{\alpha\gamma} \wedge \omega_{\beta\gamma} + \Phi_{\alpha\beta} = -\omega_{\alpha i} \wedge \omega_{i\beta} + \Omega_{\alpha\beta} = -\omega_{\alpha\gamma} \wedge \omega_{\beta\gamma} + (C^2 + K)\omega_\beta \wedge \omega_\beta.
\]
Evidently the curvature forms of the leaf are
\[ \Phi_{\alpha\beta} = (C^2 + K)\omega_\alpha \wedge \omega_\beta, \]
so the curvature of the leaves of \( D_2 \) is \((1 + K_0)/r^2\).

\textit{Remark.} If \( M \) is complete, then the case (d) cannot occur, since the base of a complete Riemannian submersion must be complete.

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