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ANOTHER POSSIBLE ABNORMALITY OF COMPACT SPACE-TIME

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ABSTRACT. It is shown that the Lorentz structure of a compact pre-space-time M can be so chosen such that M can not satisfy the strong energy condition. Thus, combining both the causal and the strong energy conditions, a stronger case against the compact space-times as proper arenas of physics can be made.

Compact space-times are not thought to be proper arenas of physics because they necessarily violate the causal condition; i.e., they always contain closed time-like curves [1]. In the present paper, it is shown that compact space-times can possess another abnormal property.

Besides the causal condition, astrophysicists and cosmologists are accustomed to assume the so-called "energy condition" in their investigations of the macro-cosmos. For example, whenever they employ a perfect fluid as a model for some macrocosmic entity, they require that an observer comoving with the fluid finds its energy density to be positive and its pressure greater than one third of the negative of the energy density. In general, the assumption is made that the stress-energy tensor $T_{\mu\nu}$ associated with "normal" macrocosmic matter satisfies the condition:

$$T_{\mu\nu}u^{\mu}u^{\nu}-u^{2}T/2 \geq 0, \tag{1}$$

where u^{μ} is an arbitrary time-like vector field. Moreover, the stress-energy tensor of the electromagnetic field satisfies it. But Einstein's gravitational field equations transform Eq. (1) into the energy condition on the metric structure of space-time:

$$R_{\mu\nu}u^{\mu}u^{\nu} \geq 0. \tag{2}$$

As has just been indicated, the energy condition can be construed as a condition on the curvature of space-time. For Riemannian manifolds, the inter-dependence of the homological and curvature properties of differentiable manifolds has been known for quite a long time. For example, the following theorem of Bochner and Myers [2] relates the Ricci curvature to the homology group of the manifold:

THEOREM. (*Bochner and Myers*). *In a compact Riemannian manifold* V^{n}, *there exists no harmonic vector field* v_{μ} *which satisfies* $R_{\mu\nu}v^{\mu}v^{\nu}\geq 0$ *unless* $v_{\mu;\nu}=0$. *But, in the*

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latter case, $R_{\mu\nu}v^{\mu}v^{\nu}=0$. (*In particular, if the manifold has positive definite curvature throughout, i.e.,* $R_{\mu\nu}v^{\mu}v^{\nu}>0$, *then there exists no non-zero harmonic vector field in* V^n *and, consequently, if* V^n *is orientable, then* $b_1(V^n)=0$.)

In the following, it will be shown that a Riemannian metric can be associated with the pseudo-Riemannian metric of certain compact space-times in such a way that the theorem just cited implies that these space-times violate the strong energy condition (i.e., inequality holds in Eq. (2)). Before we proceed, however, we remind the reader that a space-time M is an orientable, connected, four-dimensional differentiable manifold with a given metric of Lorentz signature; i.e., $(-, +, +, +)$. Hereafter an orientable, connected, differentiable four-manifold which *admits* a metric of Lorentz signature will be called a *pre-space-time.*

It is clear that if a compact, connected, orientable, differentiable 4-manifold M admits one Lorentz metric, it also admits infinitely many other Lorentz metric. Thus, the following proposition might be anticipated.

PROPOSITION 1. *The Lorentz structure of any compact pre-space-time M can always be so chosen such that M has a time-like harmonic vector field.*

Proof. It has been established that every compact pre-space-time has a non-vanishing first betti number [3]. Thus, by Hodge's theorem, there exists at least one non-zero harmonic vector field on M. But the Lorentz structure can always be chosen so that the non-zero harmonic vector field is everywhere time-like, i.e., the non-zero harmonic vector field is chosen to be the vector field which induces the Lorentz metric.

PROPOSITION 2. *The Lorentz structure of a compact pre-space-time M can be so chosen such that M can not satisfy the strong energy condition.*

Proof. It is well established that from any Lorentz structure $g_{\mu\nu}$ and an arbitrary time-like vector field ξ^{μ} we can induce a Riemannian structure $\bar{g}_{\mu\nu}$ on the underlying manifold in the following way: let $\bar{g}_{\mu\nu}=g_{\mu\nu}-2\xi_{\mu}\xi_{\nu}/\xi^2$*. As ξ^{μ} is globally defined, the induced Riemannian structure $\bar{g}_{\mu\nu}$ is also globally defined.

Locally, in every coordinate neighborhood U_i, we can choose the coordinate system in such a way that ξ^{μ} is a time-like unit vector field. Thus in every U_i, the induced Riemannian metric is given by $\bar{g}_{\mu\nu}=g_{\mu\nu}+2\xi_{\mu}\xi_{\nu}$. Consequently, in every U_i, $\bar{R}_{\mu\nu}\xi^{\mu}\xi^{\nu}=R_{\mu\nu}\xi^{\mu}\xi^{\nu}$, where $\bar{R}_{\mu\nu}$ is the Ricci tensor of the induced Riemannian structure $\bar{g}_{\mu\nu}$. If M satisfies the strong energy condition, i.e., if $R_{\mu\nu}\xi^{\mu}\xi^{\nu}>0$ throughout M, then $\bar{R}_{\mu\nu}\xi^{\mu}\xi^{\nu}>0$ everywhere on M. But the theorem of Bochner and Myers says that if the compact pre-space-time M satisfies the strong energy condition, thus $\bar{R}_{\mu\nu}\xi^{\mu}\xi^{\nu}>0$, then there does not exist any harmonic vector field in M. Such a result contradicts Proposition 1.

* This is trivially true in a coordinate system in which $\xi^{\mu}=\delta^{\mu}_{0}$.

A pre-space-time M is a "usual" differentiable 4-manifold with an extra requirement, namely, that it admits a vector field. So long as a differentiable 4-manifold admits a Lorentz structure, it therefore admits infinitely many Lorentz structures. That is, there is no "preferred" Lorentz structure.

In considering the Lorentz structure on a pre-space-time to be arbitrary, if we grant that the strong energy condition is a "physically reasonable" condition (at least in the cosmological sense), then Proposition 2 explicitly rules out compact pre-space-time as a proper arena of physics. Thus, combining both the causal and the strong energy conditions, we can make a stronger case against the compact space-times to be proper arenas of physics.

Here we have been using the strong energy condition for compact space-times, but, conceivably, this condition can be weakened somewhat. The global restrictions of the strong energy condition and the like for non-compact space-times will be considered.

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