Multiplicity Free Jacquet Modules

Avraham Aizenbud and Dmitry Gourevitch

Abstract. Let F be a non-Archimedean local field or a finite field. Let n be a natural number and k be 1 or 2. Consider $G := \mathrm{GL}_{n+k}(F)$ and let $M := \mathrm{GL}_n(F) \times \mathrm{GL}_k(F) < G$ be a maximal Levi subgroup. Let $U < G$ be the corresponding unipotent subgroup and let $P = MU$ be the corresponding parabolic subgroup. Let $J := J_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ be the Jacquet functor, *i.e.,* the functor of coinvariants with respect to U. In this paper we prove that J is a multiplicity free functor, *i.e.,* $\dim \mathrm{Hom}_M(J(\pi), \rho) \leq 1$, for any irreducible representations π of G and ρ of M. We adapt the classical method of Gelfand and Kazhdan, which proves the "multiplicity free" property of certain representations to prove the "multiplicity free" property of certain functors. At the end we discuss whether other Jacquet functors are multiplicity free.

1 Introduction

Let F be a non-Archimedean local field or a finite field. Let n be a natural number and k be 1 or 2. Consider $G := \mathrm{GL}_{n+k}(F)$ and let $M := \mathrm{GL}_n(F) \times \mathrm{GL}_k(F) < G$ be a maximal Levi subgroup. Let $U < G$ be the corresponding unipotent subgroup and let $P = MU$ be the corresponding parabolic subgroup. Let $J := J_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ be the Jacquet functor, *i.e.,* the functor of coinvariants with respect to U. We will fix the notations F, n, G, M, and U throughout the paper.

In this paper we will prove the following theorem.

Theorem A Let π be an irreducible representation of G and ρ be an irreducible representation of M. Then $\dim \mathrm{Hom}_M(J(\pi), \rho) \leq 1$.

As we will show in Section 3, this theorem is equivalent to the following one.

Theorem B Let $G \times M$ act on G/U by $(g, m)([g']) = [gg'm^{-1}]$. This action is well defined since M normalizes U. Consider the space of Schwartz measures $\mathcal{H}(G/U)$, *i.e.,* compactly supported measures which are locally constant with respect to the action of G, as a representation of $G \times M$. Then this representation is multiplicity free, *i.e.,* for any irreducible representation π of $G \times M$ we have

$$\dim \mathrm{Hom}_{G \times M}(\mathcal{H}(G/U), \pi) \leq 1.$$

By Frobenius reciprocity, this theorem is in turn equivalent to the following one.

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Theorem C Let $\mathrm{pr}_M\colon P \to M$ denote the natural projection. Consider P to be embedded in $G \times M$ by $p \mapsto (p, \mathrm{pr}_M(p))$. Then the pair $(G \times M, P)$ is a Gelfand pair, *i.e.,* for any irreducible representation π of $G \times M$ we have $\dim \mathrm{Hom}_P(\pi, \mathbb{C}) \leq 1$.

Theorem A implies the following theorem also.

Theorem D Suppose $k = 1$ and let $H = GL_n(F)$ be standardly embedded inside G. Let π be an irreducible representation of G and ρ be an irreducible representation of H. Then $\dim \mathrm{Hom}_H(J(\pi)|_H, \rho) \leq 1$.

We will prove the above mentioned implications among Theorems A, B, C, and D in Section 3.

1.1   A Sketch of the Proof

Using a version of the Gelfand–Kazhdan criterion, we deduce Theorem B from the following one.

Theorem E Any distribution on $(U^t \backslash G) \times (G/U)$ which is invariant with respect to the action of $G \times M$ given by $(g, m)([x], [y]) := ([mxg^{-1}], [gym^{-1}])$ is also invariant with respect to the involution $([x], [y]) \mapsto ([y^t], [x^t])$.

By the method of Bernstein–Gelfand–Kazhdan–Zelevinski (2.12), it is enough to prove that the involution preserves all $G \times M$ orbits. This we deduce from the following geometric statement.

Proposition F Let

$$X := X_{n,k} := \big\{ A, B \in \mathrm{Mat}_{n+k} \mid AB = BA = 0, \mathrm{rank}(A) = n, \mathrm{rank}(B) = k \big\}.$$

Let G act on $X_{n,k}$ by conjugations. Define the transposition map $\theta := \theta_{n,k}\colon X_{n,k} \to X_{n,k}$ by $\theta(A, B) := (A^t, B^t)$. Then any G-orbit in $X_{n,k}$ is θ-invariant.

We deduce this geometric statement from Lemma 5.1, which states that every M-orbit in $U^t \backslash GL_k(F)/U$ is transposition invariant, where $M < GL_k(F)$ is a Levi subgroup and U is the corresponding unipotent subgroup. This lemma is a straightforward computation since $k \leq 2$, but for bigger k it is not true.

1.2   Related Problems

Case $k = 1$:   When $k = 1$ and F is a local field, a stronger theorem holds. Namely, the functor of restriction from $GL_{n+1}(F)$ to $GL_n(F)$ is multiplicity free. This is proved in [4] for F of characteristic 0 and in [1] for F of positive characteristic. It is also proved for Archimedean F in [3, 15].

This stronger statement does not hold for finite fields, not even for $n = 1$. Theorem D may be viewed as a weaker form of this statement that works uniformly for local and finite fields.

Note that when $k = 1$ and F is a finite field, there is an alternative proof of Theorem D that is based on the classification of irreducible representations of $GL_n(F)$; see [9, 11, 18]. Also, the proof given in the current paper can be simplified for this private case; see [12, 13]. Theorem D is crucial for the study of the representation theory of limits of $GL_n(\mathbb{F}_q)$ when $n \to \infty$; see [16, 17].

The Archimedean case: We believe that the analog of Theorem A for Archimedean F holds. For $k = 1$, it holds as explained above. For $k = 2$, we believe that the proof given in this paper can be adapted to the Archimedean case. However, this will require additional analysis.

Higher rank cases: One can ask whether an analog of Theorem A holds when M is an arbitrary Levi subgroup of G. If F is a local field, we do not know the answer for this question. If F is a finite field, such analog of Theorem A holds only in the cases at hand. This is related to the fact that the restriction of any irreducible representation of the permutation group $S_{n_1 + \cdots + n_l}$ to $S_{n_1} \times \cdots \times S_{n_l}$ is multiplicity free if and only if $l \leq 2$ and $\min(n_1, n_2) \leq 2$. We discuss those questions in Section 6.

1.3 Contents of the Paper

In Section 2 we give the necessary preliminaries. In Subsection 2.1 we introduce notation that we will use throughout the paper. In Subsection 2.2 we give some preliminaries and notation on l-spaces, l-groups and their representations based on [7]. In Subsection 2.3 we define multiplicity free functors and formulate two theorems that enable us to reduce the "multiplicity free" property of a strongly right exact functor between the categories of smooth representations of two l-groups to the "multiplicity free" property of a certain representation of the product of those groups. We prove those theorems in Appendix A. In Subsection 2.4 we formulate a version of the Gelfand–Kazhdan criterion for the "multiplicity free" property of representations of the form $\mathcal{S}(X)$. We prove this version in Appendix B. In Subsection 2.5 we recall a criterion for vanishing of equivariant distributions in terms of stabilizers of points. In Subsection 2.6 we recall the Deligne (weight) filtration attached to a nilpotent operator on a vector space.

In Section 3 we prove equivalence of Theorems A, B, and C and deduce Theorem D from them. In Section 4 we reduce Theorem B to the geometric statement (Proposition F). In Section 5 we prove the geometric statement.

In Section 6 we discuss whether an analog of Theorem A holds when M is an arbitrary Levi subgroup. In Subsection 6.1 we answer an analogous question for permutation groups. In Subsection 6.2 we discuss the connection between the questions for permutation groups and general linear groups over finite fields. In Subsection 6.3 we discuss the local field case.

In Appendix A we prove theorems from Subsection 2.3 on strongly right exact functors between the categories of smooth representations of two reductive groups. In Appendix B we prove a version from Subsection 2.4 of the Gelfand–Kazhdan criterion for the "multiplicity free" property of geometric representations.

2 Preliminaries

2.1 General Notation

- For a group H acting on a set X and a point $x \in X$ we denote by Hx or by $H(x)$ the orbit of x and by H_x the stabilizer of x. We also denote by X^H the set of fixed points in X.

- For a representation V of a group H we denote by V^H the space of invariants and by V_H the space of coinvariants, *i.e.,* $V_H := V/(\mathrm{Span}\{v - gv \mid g \in H, v \in V\})$.

- For a Lie algebra $\mathfrak{g}$ acting on a vector space V we denote by $V^{\mathfrak{g}}$ the space of invariants. Similarly, for any element $X \in \mathfrak{g}$ we denote by V^X the kernel of the action of X.

- For a linear operator $A\colon V \to W$ we denote the cokernel of A by $\mathrm{Coker}\,A := W/\mathrm{Im}\,A$.

- For a linear operator $A\colon V \to V$ and an A-invariant subspace $U \subset V$ we denote by $A|_U\colon U \to U$ and $A|_{V/U}\colon V/U \to V/U$ the natural induced operators.

2.2 *l*-Spaces and *l*-Groups

Let us recall the standard terminology of *l*-spaces introduced in [7].

- An *l*-space is a Hausdorff locally compact totally disconnected topological space.

- For an *l*-space X we denote by $\mathcal{S}(X)$ the space of Schwartz functions on X, *i.e.,* locally constant compactly supported functions on X. We denote by $\mathcal{S}^*(X)$ the dual space and call its elements distributions.

- In [7] there was introduced the notion of "*l*-sheaf". As was later realized (see [6, §§1.3]), this notion is equivalent to the usual notion of sheaf on an *l*-space, so we will use the results of [7] for sheaves.

- For a sheaf $\mathcal{F}$ on an *l*-space X we denote by $\mathcal{S}(X, \mathcal{F})$ the space of compactly supported sections of $\mathcal{F}$ and by $\mathcal{S}^*(X, \mathcal{F})$ its dual space.

- Note that $\mathcal{S}(X_1, \mathcal{F}_1) \otimes \mathcal{S}(X_2, \mathcal{F}_2) \cong \mathcal{S}(X_1 \times X_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2)$ for any *l*-spaces X_i and sheaves $\mathcal{F}_i$ on them.

- An *l*-group is a topological group which has a basis of topology at 1 consisting of open compact subgroups. In fact, any topological group which is an *l*-space is an *l*-group.

- Let an *l*-group G act (continuously) on an *l*-space X. Let $a\colon G \times X \to X$ be the action map and $p\colon G \times X \to X$ be the projection. A G-equivariant sheaf on X is a sheaf $\mathcal{F}$ on X together with an isomorphism $a^*\mathcal{F} \to p^*\mathcal{F}$ that satisfy the natural conditions.

- For a representation V of an *l*-group H we denote by V^∞ the space of smooth vectors, *i.e.,* vectors whose stabilizers are open.

- We denote $(V^*)^\infty$ by $\widetilde{V}$.

- For an *l*-group H we denote by $\mathcal{H}(H)$ the convolution algebra of smooth, *i.e.,* locally constant with respect to the action of H, compactly supported measures on H.

- Similarly for a transitive H-space X we denote by $\mathcal{H}(X)$ the space of smooth compactly supported measures on X.

- For an *l*-group H we denote by $\mathcal{M}(H)$ the category of smooth representations of H.

• Recall that if an *l*-group H acts (continuously) on an *l*-space X and $\mathcal{F}$ is an H-equivariant sheaf on X, then $\mathcal{S}(X, \mathcal{F})$ is a smooth representation of H.

Definition 2.1   A representation V of an *l*-group H is called *admissible* if one of the following equivalent conditions holds.

• For any open compact subgroup $K < H$ we have $\dim V^K < \infty$.
• There exists an open compact subgroup $K < H$ such that $\dim V^K < \infty$.
• For any open compact subgroup $K < H$, $V|_K = \bigoplus_{\rho \in \mathrm{Irr}\, K} n_\rho \rho$, where n_ρ are finite numbers and $\mathrm{Irr}\, K$ denotes the collection of isomorphism classes of irreducible representations of K.
• The natural morphism $V \xrightarrow{\sim} \widetilde{\widetilde{V}}$ is an isomorphism.

Theorem 2.2 (Harish-Chandra)   *Let H be a reductive (not necessarily connected) group defined over F. Then every smooth irreducible representation of $H(F)$ is admissible.*

Definition 2.3   Let H be an *l*-group. An $\mathcal{H}(H)$-module M is called *unital* if $\mathcal{H}(H)M = M$.

Theorem 2.4 (Bernstein–Zelevinsky)   *Let H be an l-group.*

(i)   *The natural functor between $\mathcal{M}(H)$ and the category of unital $\mathcal{H}(H)$-modules is an equivalence of categories.*
(ii)   *The category $\mathcal{M}(H)$ is abelian.*

2.3   Multiplicity Free Functors

Definition 2.5   Let H be an *l*-group. We call a representation $\pi \in \mathcal{M}(H)$ *multiplicity free* if for any irreducible admissible representation $\tau \in \mathcal{M}(H)$ we have $\dim_{\mathbb{C}} \mathrm{Hom}(\pi, \tau) \leq 1$.

Let H' be an *l*-group. We call a functor $\mathcal{F} \colon \mathcal{M}(H) \to \mathcal{M}(H')$ a *multiplicity free functor* if for any irreducible admissible representation $\pi \in \mathcal{M}(H)$ the representation $\mathcal{F}(\pi)$ is multiplicity free.

Remark   Note that if H is not reductive, then the multiplicity free property might be rather weak since there might be too few admissible representations.

Theorem 2.6   *Let H and H' be l-groups. Let $\mathcal{F} \colon \mathcal{M}(H) \to \mathcal{M}(H')$ be a $\mathbb{C}$-linear functor that commutes with arbitrary direct limits (or, equivalently, is right exact and commutes with arbitrary direct sums). Let $\Pi := \mathcal{F}(\mathcal{H}(H))$. Consider the action of H on $\mathcal{H}(H)$ given by $g\mu := \mu * \delta_{g^{-1}}$. It defines an action of H on Π that commutes with the action of H'. In this way Π becomes a representation of $H \times H'$. Then*

(i) Π *is a smooth representation;*
(ii) $\mathcal{F}$ *is canonically isomorphic to the functor given by $\pi \mapsto (\Pi \otimes \pi)_H$.*

This theorem is known. For the sake of completeness we include its proof in Appendix A.1.

Theorem 2.7 *Let H and H' be l-groups. Let $\mathcal{F}\colon \mathcal{M}(H) \to \mathcal{M}(H')$ be a $\mathbb{C}$-linear functor that commutes with arbitrary direct limits. Then $\mathcal{F}$ is a multiplicity free functor if and only if $\mathcal{F}(\mathcal{H}(H))$ is a multiplicity free representation of $H \times H'$.*

For the proof, see Appendix A.2.

2.4   Gelfand–Kazhdan Criterion for Multiplicity Free Property of Geometric Representations

Theorem 2.8 *Let H be an l-group. Let X and Y be H-spaces and $\mathcal{F}$ and $\mathcal{G}$ be H-equivariant sheaves on X and Y, respectively. Let $\tau\colon X \to Y$ be a homeomorphism (not necessarily H-invariant). Suppose that we are given an isomorphism $\tau_* \mathcal{F} \simeq \mathcal{G}$. Define $T\colon X \times Y \to X \times Y$ by $T(x, y) := (\tau^{-1}(y), \tau(x))$. It gives an involution T on the space $\mathcal{S}^*(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$.*

Suppose that any $\xi \in \mathcal{S}^*(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$ that is invariant with respect to the diagonal action of H is invariant with respect to T. Then for any irreducible admissible representation $\pi \in \mathcal{M}(H)$ we have

$$\dim \operatorname{Hom}(\mathcal{S}(X, \mathcal{F}), \pi) \cdot \dim \operatorname{Hom}(\mathcal{S}(Y, \mathcal{G})), \widetilde{\pi}) \leq 1.$$

In the case when X and Y are transitive and correspond to each other in a certain way, this theorem is a classical theorem by Gelfand and Kazhdan (see [10]). For the general case the proof is the same and we repeat it in Appendix B. In fact, in this paper we could use the classical formulation of this theorem, but we believe that it is useful in the general formulation.

Definition 2.9 Let H be an l-group. Let $\theta\colon H \to H$ be an involution. Let X be an H-space.

(i)   Denote by $\theta(X)$ the H-space which coincides with X as an l-space but with the action of H twisted by θ.

(ii)   Similarly, for a representation π of H we denote by $\theta(\pi)$ the representation $\pi \circ \theta$.

(iii)   Let $\mathcal{F}$ be an H-equivariant sheaf on X. Let us define an equivariant sheaf $\theta(\mathcal{F})$ on $\theta(X)$. As a sheaf, $\theta(\mathcal{F})$ coincides with $\mathcal{F}$ and the equivariant structure is defined in the following way. Let $a\colon H \times X \to X$ denote the action map and $p_2\colon H \times X \to X$ denote the projection. Let $\alpha\colon a^*(\mathcal{F}) \to p_2^*(\mathcal{F})$ denote the equivariant structure of $\mathcal{F}$. We have to define an equivariant structure $\theta(\alpha)\colon (\theta(a))^*(\theta(\mathcal{F})) \to p_2^*(\theta(\mathcal{F}))$, where $\theta(a)\colon H \times \theta(X) \to \theta(X)$ is the action map. Note that $(\theta(a))^*(\theta(\mathcal{F})) \cong (\theta \times \operatorname{Id})^*(a^*(\mathcal{F}))$. Since $\theta \times \operatorname{Id}$ is an involution, it is enough to define a map between $a^*(\mathcal{F})$ and $(\theta \times \operatorname{Id})^*(p_2^*(\mathcal{F}))$. Let β denote the canonical isomorphism between $(\theta \times \operatorname{Id})^*(p_2^*(\mathcal{F}))$ and $(p_2 \circ (\theta \times Id))^*(\mathcal{F}) = p_2^*(\mathcal{F})$. Now the desired map is given by $\beta^{-1} \circ \alpha$.

Remark   Clearly, $\mathcal{S}(\theta(X), \theta(\mathcal{F})) \cong \theta(\mathcal{S}(X, \mathcal{F}))$.

Notation   Let $H := \operatorname{GL}_{n_1} \times \cdots \times \operatorname{GL}_{n_k}$. We denote by κ the Cartan involution $\kappa(g) := (g^t)^{-1}$.

Theorem 2.10 ([10])   *Let $H := \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$. Let π be an irreducible smooth representation of $H(F)$. Then $\widetilde{\pi} \simeq \kappa(\pi)$.*

Corollary 2.11   *Let $H := \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$. Let X be an $H(F)$-space. Let $\mathcal{F}$ be an $H(F)$-equivariant sheaf on X. Suppose that any $\xi \in \mathcal{S}(X \times \kappa(X), \mathcal{F} \boxtimes \kappa(\mathcal{F}))$ that is invariant with respect to the diagonal action of $H(F)$ is invariant with respect to swap of the coordinates. Then the representation $\mathcal{S}(X, \mathcal{F})$ is multiplicity free.*

2.5   Bernstein–Gelfand–Kazhdan–Zelevinski Criterion for Vanishing of Invariant Distributions

Theorem 2.12 (Bernstein–Gelfand–Kazhdan–Zelevinsky)   *Let an algebraic group H act on an algebraic variety X, both defined over F. Let H' be an open subgroup of $H(F)$. Let $\mathcal{F}$ be a sheaf over $X(F)$. Suppose that for any $x \in X(F)$ we have*

$$(\mathcal{F}_x \otimes \Delta_{H'}|_{H'_x} \otimes \Delta_{H'_x}^{-1})^{H'_x} = 0,$$

where $\Delta_{H'}$ and $\Delta_{H'_x}$ denote the modular functions of the groups H' and H'_x. Then $\mathcal{S}^*(X(F), \mathcal{F})^{H'} = 0$.

This theorem follows from [7, §6] and [5, §§1.5].

Corollary 2.13   *Let an algebraic group H act on an algebraic variety X, both defined over F. Let $\sigma\colon X \to X$ be an involution defined over F. Suppose that σ normalizes the action of H. Then each $H(F)$-invariant distribution on $X(F)$ is invariant under σ.*

Proof   Let H' denote the group of homeomorphisms of $X(F)$ generated by the actions of $H(F)$ and σ and H'' denote the subgroup generated by the action of $H(F)$. Define a character χ of H' by $\chi(H'') = 1$, $\chi(H' - H'') = -1$. By the previous theorem, $\mathcal{S}^*(X, \chi)^{H'} = 0$. On the other hand, every $H(F)$-invariant distribution on $X(F)$ is a sum of a σ-invariant distribution and an (H', χ)-equivariant distribution. ∎

2.6   Deligne Filtration

Theorem 2.14 (Deligne)   *Let A be a nilpotent operator on a vector space V. Then there exists a unique and finite decreasing filtration $V^{\geq i}$ such that*

(i)   *A is of degree 2 with respect to this filtration;*
(ii)   *A^l gives an isomorphism $V^{\geq l}/V^{\geq l+1} \simeq V^{\geq -l}/V^{\geq -l+1}$.*

For proof see [8, Proposition I.6.I]

Definition 2.15   We will denote this filtration by $\mathcal{D}_A^{\geq i}(V)$ and call it the Deligne filtration.

Remark   In the case $\mathrm{char}\, F = 0$, this filtration coincides with the weight filtration defined by extending A to an sl_2-triple using the Jacobson–Morosov theorem.

Notation   The filtration $\mathcal{D}_A^{\geq i}(V)$ induces filtrations on $\mathrm{Ker}\, A$ and $\mathrm{Coker}\, A$ in the following way:

$$\mathcal{D}_{A,+}^{\geq i}(\mathrm{Ker}\, A) := \mathcal{D}_A^{\geq i}(V) \cap \mathrm{Ker}\, A,$$

$$\mathcal{D}_{A,-}^{\leq i}(\mathrm{Coker}\, A) := \mathcal{D}_A^{\leq -i}(V)/(\mathrm{Im}\, A \cap \mathcal{D}_A^{\leq -i}(V)).$$

Denote by $\mu_A \colon \mathrm{Gr}_{A,-}^i(\mathrm{Coker}\, A) \to \mathrm{Gr}_{A,+}^i(\mathrm{Ker}\, A)$ the isomorphism between the associated graded parts given by A^i.

3   Implications among the Main Results

In this section we prove that Theorems A, B, and C are equivalent and imply Theorem D.

Theorem A $\Leftrightarrow$ Theorem B   Note that $J_M^G(\mathcal{H}(G)) \cong \mathcal{H}(U \backslash G)$, where the action of M is from the left and the action of G is from the right. Clearly this representation of $G \times M$ is isomorphic to the representation $\mathcal{H}(G/U)$ that was described in Theorem B. The equivalence follows now from Theorem 2.7.   ∎

Theorem B $\Leftrightarrow$ Theorem C   Note that $(G \times M)/P = G/U$. Hence $\mathcal{H}(G/U) = \mathcal{H}((G \times M)/P)$. Now

$$\mathrm{Hom}_{G \times M}(\mathcal{H}(G/U), \pi) = \mathrm{Hom}_{G \times M}(\mathcal{H}((G \times M)/P), \pi)$$
$$= \mathrm{Hom}_{G \times M}(\tilde{\pi}, C^\infty((G \times M)/P))$$
$$= \mathrm{Hom}_{G \times M}(\tilde{\pi}, \mathrm{Ind}_P^{G \times M}(\mathbb{C})) = \mathrm{Hom}_P(\tilde{\pi}, \mathbb{C}). \qquad \blacksquare$$

Theorem A $\Rightarrow$ Theorem D   Note that the center $Z(G)$ of G lies in M, and that $M \cong Z(G) \times H$. Now let π be an irreducible representation of G. Then $Z(G)$ acts on it by a character χ. Let ρ be an irreducible representation of H. Extend it to a representation of M by letting $Z(G)$ act by χ. Then $\mathrm{Hom}_H(J(\pi), \rho) = \mathrm{Hom}_M(J(\pi), \rho)$, which is at most one-dimensional by Theorem A.   ∎

4   Reduction to the Geometric Statement

Definition 4.1   Let

$$X := X_{n,k} := \{A, B \in Mat_{n+k}(F) \mid AB = BA = 0, \mathrm{rank}(A) = n, \mathrm{rank}(B) = k\}.$$

Let G act on $X_{n,k}$ by conjugations. We define the transposition map $\theta := \theta_{n,k} \colon X_{n,k} \to X_{n,k}$ by $\theta(A, B) := (A^t, B^t)$.

In this section we deduce Theorem B from the following geometric statement.

Proposition 4.2 (Geometric Statement)   *Any G-orbit in $X_{n,k}$ is θ-invariant.*

Definition 4.3 • We denote by $E_{n,k}$ the *l*-space of exact sequences of the form
$$0 \to F^n \xrightarrow{\phi} F^{n+k} \xrightarrow{\psi} F^k \to 0.$$ We consider the natural action of $G \times M$ on $E_{n,k}$ given by $(g, (h_1, h_2))(\phi, \psi) := (g\phi h_1^{-1}, h_2 \psi g^{-1})$.
• Denote by $\tau \colon E_{n,k} \to E_{k,n}$ the map given by $\tau(\phi, \psi) := (\psi^t, \phi^t)$.
• Denote by $T \colon E_{n,k} \times E_{k,n} \to E_{n,k} \times E_{k,n}$ the map given by $T(e_1, e_2) := (\tau(e_2), \tau(e_1))$.

The following lemma is straightforward.

Lemma 4.4 (i) $G/U \cong E_{n,k}$ as a $G \times M$-space.
(ii) The transposition map τ defines an isomorphism of $G \times M$-spaces $\tau \colon E_{n,k} \to \kappa(E_{k,n})$.

Notation Denote by $C_{n,k} \colon E_{n,k} \times E_{k,n} \to X_{n,k}$ the composition map given by

$$C_{n,k}((\phi_1, \psi_1), (\phi_2, \psi_2)) := (\psi_2 \circ \phi_1, \psi_1 \circ \phi_2).$$

The following lemma is straightforward.

Lemma 4.5 (i) $C_{n,k}$ defines a bijection between $G \times M$-orbits on $E_{n,k} \times E_{k,n}$ and G-orbits on $X_{n,k}$.
(ii) $C_{n,k} \circ T = \theta \circ C_{n,k}$.

Corollary 4.6 The Geometric Statement implies that all $G \times M$-orbits on $E_{n,k} \times E_{k,n}$ are T-invariant.

Corollary 4.7 The Geometric Statement implies Theorem B.

Proof Let $G \times M$ act on $G/U \times G/U$ by $(g, m) \cdot ([x], [y]) := ([gxm^{-1}], [g^{-t} y m^t])$. By the previous corollary and Lemma 4.4, all $G \times M$-orbits on $G/U \times G/U$ are invariant with respect to the swap of coordinates. By Corollary 2.13 this implies that every distribution on $G/U \times G/U$ that is invariant with respect to the action of $G \times M$, is invariant with respect to the swap of coordinates. Therefore, by Corollary 2.11,

$$\dim \operatorname{Hom}_{G \times M}(\mathcal{H}(G/U), \pi) \leq 1. \qquad \blacksquare$$

5 Proof of the Geometric Statement

The proof is by induction on n. From now on we assume that the Geometric Statement holds for all dimensions smaller than n.

Remark The proof that will be given here is valid for any field F.

We will use the following lemma.

Lemma 5.1 (Key Lemma) Let $G' := \mathrm{GL}_k$. Let P'_+ be its parabolic subgroup. Let P'_- be the opposite parabolic. Let P'' be the subgroup of $P'_+ \times P'_-$ consisting of pairs with the same Levi part. Consider the two-sided action of $P'_+ \times P'_-$ on G' (given by $(p_1, p_2)g := p_1 g p_2^{-1}$) and its restriction to P''. Then any P'' orbit on G' is transposition invariant.

Since $k \leq 2$, this lemma is a straightforward computation.

Remark   The analogous statement for $k \geq 3$ is not true, and therefore the analog of the Geometric Statement for $k \geq 3$ is also not true. In fact, this lemma is the only place where we use the assumption $k \leq 2$.

Notation   Denote $X' := X'_{n,k} := \{(A, B) \in X \mid A \text{ is nilpotent}\}$.

We will show that the Geometric Statement follows from Proposition 5.2.

Proposition 5.2   *Any G-orbit in $X'_{n,k}$ is θ-invariant.*

Proof that Proposition 5.2 implies Proposition 4.2   Let $(A, B) \in X - X'$. We must show that there exists $g \in G$ such that $gAg^{-1} = A^t$ and $gBg^{-1} = B^t$.

Decompose $F^{n+k} := V \oplus W$ such that $A = A' \oplus A''$, where A' is a nilpotent operator on V and A'' is an invertible operator on W. Note that $\dim V < n + k$. Since $AB = BA = 0$, we have $B = B' \oplus 0$, where B' is an operator on V and 0 denotes the zero operator on W. Without loss of generality we may assume that V and W are coordinate spaces.

By the induction hypothesis, there exists $g_1 \in \text{GL}(V)$ such that $g_1 A' g_1^{-1} = A'^t$ and $g_1 B' g_1^{-1} = B'^t$. It is well known that there exists $g_2 \in \text{GL}(W)$ such that $g_2 A'' g_2^{-1} = A''^t$. Take $g := g_1 \oplus g_2$.   ∎

Notation   Let A be a nilpotent operator on a vector space V. Let $\nu_A \colon \text{GL}(V)_A \to \text{GL}(\text{Ker}\,A) \times \text{GL}(\text{Coker}\,A)$ denote the map defined by $\nu_A(g) := (g|_{\text{Ker}\,A}, g|_{\text{Coker}\,A})$. Denote also

$$\mathcal{P}_A := \Big\{ g, h \in \text{GL}(\text{Ker}\,A) \times \text{GL}(\text{Coker}\,A) \mid g \text{ preserves } \mathcal{D}_{A,+}, \, h \text{ preserves } \mathcal{D}_{A,-},$$
$$\text{Gr}_{\mathcal{D}_{A,+}}(g) \text{ corresponds to } \text{Gr}_{\mathcal{D}_{A,-}}(h) \text{ under the identification } \mu_A \Big\}.$$

Here $\text{Gr}_{\mathcal{D}_{A,+}}(g)$ denotes the action of g on the associated graded space.

Lemma 5.3   *Let A be a nilpotent operator on a vector space V. Then $\text{Im}(\nu_A) = \mathcal{P}_A$.*

Proof   Clearly $\text{Im}(\nu_A) \subset \mathcal{P}_A$. Let $\mathfrak{p}$ denote the Lie algebra of $\mathcal{P}_A$. It is enough to show that the map $d\nu_A \colon \mathfrak{gl}(V)_A \to \mathfrak{p}$ is onto. Let $V = \bigoplus V_i$ be the decomposition of V to Jordan blocks with respect to the action of A. We have

$$\mathfrak{gl}(V)_A = (V^* \otimes V)^A = \bigoplus_{i,j} (V_i^* \otimes V_j)^A,$$

$$\mathfrak{gl}(\text{Ker}\,A) = (V^A)^* \otimes V^A = \bigoplus_{i,j} (V_i^A)^* \otimes V_j^A,$$

$$\mathfrak{gl}(\text{Coker}\,A) = (V/AV)^* \otimes (V/AV) = \bigoplus_{i,j} (V_i/AV_i)^* \otimes (V_j/AV_j).$$

The filtration $\mathcal{D}_{A,+}$ on $\text{Ker}\,A$ gives a natural filtration on $\mathfrak{gl}(\text{Ker}\,A)$. It is easy to see that the one-dimensional space $(V_i^A)^* \otimes V_j^A$ is of degree $\dim V_j - \dim V_i$ with respect

to this filtration. Similarly, $(V_i/AV_i)^* \otimes (V_j/AV_j)$ is of degree $\dim V_i - \dim V_j$. Hence $\mathfrak{p} = \bigoplus \mathfrak{p}_{ij}$, where

$$
\mathfrak{p}_{ij} = \begin{cases}
(V_i^A)^* \otimes V_j^A & \dim V_j > \dim V_i, \\
(V_i/AV_i)^* \otimes (V_j/AV_j) & \dim V_j < \dim V_i, \\
\big\{ (X,Y) \in (V_i^A)^* \otimes V_j^A \oplus (V_i/AV_i)^* \otimes (V_j/AV_j) \\
\quad \big| \ X \text{ corresponds to } Y \text{ under the} \\
\qquad \text{identification given by } A^{\dim V_i - 1} \big\} & \dim V_j = \dim V_i.
\end{cases}
$$

This decomposition gives a decomposition $d\nu_A = \bigoplus \nu_{ij}$, where $\nu_{ij} \colon (V_i^* \otimes V_j)^A \to \mathfrak{p}_{ij}$. It is enough to show that ν_{ij} is surjective for any i and j. Choose a gradation on V_i which is compatible with the Deligne filtration. Let $L_{ij} \subset (V_i^* \otimes V_j)^A$ be the one-dimensional subspace of vectors of weight $\dim V_j - \dim V_i$ with respect to this gradation. It is easy to see that $\nu_{ij}|_{L_{ij}}$ is surjective. ∎

The following lemma is a reformulation of Lemma 5.1.

Lemma 5.4 *Let V and W be linear spaces of dimension k. Suppose that we are given a non-degenerate pairing between V and W. Let $\mathcal{F}$ be a descending filtration on V and $\mathcal{G}$ be the dual ascending filtration on W. Suppose that we are given an isomorphism of graded linear spaces $\mu \colon Gr_{\mathcal{F}}(V) \to Gr_{\mathcal{G}}(W)$. Let*

$$
\mathcal{P} := \big\{ g, h \in \mathrm{GL}(V) \times \mathrm{GL}(W) \mid g \text{ preserves } \mathcal{F}, h \text{ preserves } \mathcal{G} \text{ and} \\
Gr_{\mathcal{F}}(g) \text{ corresponds to } Gr_{\mathcal{G}}(h) \text{ under the identification } \mu \big\}.
$$

Let $\mathcal{P}$ act on $\mathrm{Hom}(V, W)$ by $(g, h)(\phi) := h \circ \phi \circ g^{-1}$. Note that the pairing between V and W defines a notion of transposition on $\mathrm{Hom}(V, W)$. Then any $\mathcal{P}$-orbit on $\mathrm{Hom}(V, W)$ is invariant under transposition.

Proof of Proposition 5.2 Let $(A, B) \in X'$. We must show that there exists $g \in G$ such that $gAg^{-1} = A^t$ and $gBg^{-1} = B^t$. Fix a bilinear form Q on F^{n+k} such that $A_Q^t = A$, where A_Q^t denotes transpose with respect to the form Q. Such Q exists since A is conjugate to A^t. It is enough to show that there exists $g \in G_A$ such that $gBg^{-1} = B_Q^t$. Note that $\mathrm{Ker}\, A = \mathrm{Im}\, B$ and $\mathrm{Ker}\, B = \mathrm{Im}\, A$. Denote by $B' \colon \mathrm{Coker}\, A \to \mathrm{Ker}\, A$ the map induced by B. Consider the natural action of $\mathrm{GL}(\mathrm{Coker}\, A) \times \mathrm{GL}(\mathrm{Ker}\, A)$ on $\mathrm{Hom}(\mathrm{Coker}\, A, \mathrm{Ker}\, A)$. Note that $\mathrm{Ker}\, B_Q^t = \mathrm{Im}\, A$ and $\mathrm{Ker}\, A = \mathrm{Im}\, B_Q^t$ and hence B_Q^t also induces a map $\mathrm{Coker}\, A \to \mathrm{Ker}\, A$. Denote this map by B''. Note that B'' is the transposition of the map B' with respect to the non-degenerate pairing between $\mathrm{Coker}\, A$ and $\mathrm{Ker}\, A$ given by Q. The assertion follows now from Lemma 5.3 and Lemma 5.4. ∎

6 Discussion of the Higher Rank Cases

In this section we discuss whether an analog of Theorem A holds when M is an arbitrary Levi subgroup. If F is a finite field, a negative answer to this question can be obtained from a negative answer to an analogous question for permutation groups.

We discuss permutation groups in Subsection 6.1 and the connection between the two questions in Subsection 6.2. The answer we obtain is that such an analog of Theorem A holds only in the cases at hand.

We discuss the case when F is a local field in Subsection 6.3, but we do not reach a conclusion. Since the results here are negative and mostly known, the discussion is rather informal and some details are omitted.

6.1 The Analogous Problems for the Permutation Groups

Let $M' = S_{n_1} \times \cdots \times S_{n_l}$ and $G' := S_{n_1 + \ldots + n_l}$. One can ask when is (G', M') a strong Gelfand pair, *i.e.,* when is the restriction functor from G' to M' multiplicity free. The answer is that (G', M') is a strong Gelfand pair if and only if $l \leq 2$ and $\min(n_1, n_2) \leq 2$. This is well known, but let us indicate the proof.

The fact that the pairs (S_{n+1}, S_n) and $(S_{n+2}, S_n \times S_2)$ are strong Gelfand pairs follows by Theorems 2.7 and 2.8 from the fact that every permutation from G' is conjugate by M' to its inverse.

In order to show that other pairs mentioned above are not strong Gelfand pairs, we must show that the algebra of $\mathrm{Ad}(M')$-invariant functions on G' with respect to convolution is not commutative unless $l \leq 2$ and $\min(n_1, n_2) \leq 2$.

If $l \geq 3$, then consider the transpositions $\sigma_1 = (1, n_1 + 1)$ and $\sigma_2 = (n_1 + 1, n_2 + 1)$. It is easy to see that the characteristic functions of their M'-conjugacy classes do not commute. If $l = 2$ and $n_1, n_2 \geq 3$, then consider the cyclic permutations $\sigma_1 = (1, 2, 3, n_1 + 1, n_1 + 2, n_1 + 3)$ and $\sigma_2 = (1, n_1 + 1, n_1 + 2)$. It is easy to see that the characteristic functions of their M'-conjugacy classes do not commute.

6.2 Connection with Our Problem for the Finite Fields

Suppose that F is a finite field. Let $M = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_l}(F)$ and $G := \mathrm{GL}_{n_1 + \ldots + n_l}(F)$. Then the multiplicities problem of a Jacquet functor between $\mathcal{M}(G)$ and $\mathcal{M}(M)$ can be considered as a generalization of a deformation of the multiplicities problem of the restriction functor from $\mathcal{M}(G')$ to $\mathcal{M}(M')$.

Indeed, the multiplicities problem of the Jacquet functor is equivalent to the multiplicities problem of the parabolic induction from $\mathcal{M}(M)$ to $\mathcal{M}(G)$. Let $\Sigma := i_{T_M}^M(\mathbb{C})$, where $i_{T_M}^M$ denotes the parabolic induction from the torus of M to M. Let $\Pi := i_{T_G}^G(\mathbb{C})$. Let $\mathcal{A}$ be the subcategory of $\mathcal{M}(M)$ generated by Σ and $\mathcal{B}$ the subcategory of $\mathcal{M}(G)$ generated by Π. Then the multiplicities problem of the parabolic induction from $\mathcal{A}$ to $\mathcal{B}$ is a special case of the multiplicities problem of the parabolic induction from $\mathcal{M}(M)$ to $\mathcal{M}(G)$. Let $A := \mathrm{End}_M(\Sigma)$ and $B := \mathrm{End}_G(\Pi)$. Clearly, $\mathcal{A}$ is equivalent to the category of A-modules and $\mathcal{B}$ is equivalent to the category of B-modules. It is well known that A and B are deformations of the group algebras of M' and G', respectively. Therefore the multiplicities problem of the parabolic induction from $\mathcal{A}$ to $\mathcal{B}$ is a deformation of the multiplicities problem of the induction from M' to G', which in turn is equivalent to the multiplicities problem of the restriction from G' to M'. In fact, one can show that those deformations are trivializable since those algebras are semisimple. One can use this argument in order to show that J_M^G is a multiplicity free functor only if $l \leq 2$ and $\min(n_1, n_2) \leq 2$.

6.3 Higher Rank Cases over Local Fields

First note that the reduction of Theorem B to Lemma 5.1 works without change for arbitrary k. This reduction connects between the Gelfand–Kazhdan criterion for the multiplicity free property of the Jacquet functor from $\mathrm{GL}_{n+k}(F)$ to $\mathrm{GL}_n(F) \times \mathrm{GL}_k(F)$ and the Gelfand–Kazhdan criterion for the multiplicity free property of the Jacquet functor from $\mathrm{GL}_k(F)$ to an arbitrary Levi subgroup. Therefore, we believe that the multiplicity free properties themselves are connected and if one wants to consider the case of arbitrary k, one will also have to consider arbitrary Levi subgroups. At the moment we do not have an opinion when the Jacquet functor from $\mathrm{GL}_n(F)$ to an arbitrary Levi subgroup is multiplicity free.

A Multiplicity Free Functors

A.1 Proof of Theorem 2.6

Proof of Theorem 2.6(i) For any open compact subgroup $K < H$, denote by $\Sigma_K \subset \mathcal{H}(H)$ the subspace of right K-invariant measures. Let $\Pi_K := \mathcal{F}(\Sigma_K)$. Since $\mathcal{F}$ commutes with direct limits, $\varinjlim_K \Pi_K \cong \Pi$, it is easy to see that K acts trivially on the image of Π_K in Π. Hence Π is a smooth representation of H and hence it is a smooth representation of $H \times H'$. ∎

For the proof of (ii) we will need several lemmas.

Notation Denote by $\mathcal{H}(H)_0$ the subalgebra of $\mathcal{H}(H)$ consisting of functions with zero integral.

Lemma A.1 *Let π be a smooth representation of H. Then*

$$\pi_H = \mathrm{Coker}(\mathcal{H}(H)_0 \otimes \pi \to \pi),$$

where by equality we mean equality of quotients of π.

Proof Let V be any vector space. We can consider it as a representation of H with trivial action or as an $\mathcal{H}(H)$-module on which every measure acts by multiplication by its integral. Then

$$\mathrm{Hom}_{\mathbb{C}}(\pi_H, V) = \mathrm{Hom}_H(\pi, V),$$

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{Coker}(\mathcal{H}(H)_0 \otimes \pi \to \pi), V) = \mathrm{Hom}_{\mathcal{H}(H)}(\pi, V).$$

By Theorem 2.4, $\mathrm{Hom}_H(\pi, V) = \mathrm{Hom}_{\mathcal{H}(H)}(\pi, V)$ and therefore

$$\mathrm{Hom}_{\mathbb{C}}(\pi_H, V) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Coker}(\mathcal{H}(H)_0 \otimes \pi \to \pi), V)$$

for any vector space V. The lemma follows now from the Yoneda lemma. ∎

Lemma A.2 *Let π be a smooth representation of H. Let H act on $\mathcal{H}(H) \otimes \pi$ by $g(\mu \otimes v) := (\mu * \delta_{g^{-1}}) \otimes gv$. Then $(\mathcal{H}(H) \otimes \pi)_H = \pi$, where by equality we mean equality of quotients of $\mathcal{H}(H) \otimes \pi$.*

Proof Let us deduce the statement from the Yoneda lemma. Let τ be a smooth representation of H. Then

$$
\begin{aligned}
\operatorname{Hom}_H((\mathcal{H}(H) \otimes \pi)_H, \tau) &= \operatorname{Hom}_{H \times H}(\mathcal{H}(H) \otimes \pi, \tau) \\
&= (\operatorname{Hom}_{\mathbb{C}}(\mathcal{H}(H) \otimes \pi, \tau))^{H \times H} \\
&= (\operatorname{Hom}_{\mathbb{C}}(\mathcal{H}(H), \operatorname{Hom}_{\mathbb{C}}(\pi, \tau)))^{H \times H} \\
&= (\operatorname{Hom}_{\mathcal{H}(H)}(\mathcal{H}(H), \operatorname{Hom}_{\mathbb{C}}(\pi, \tau)))^{H} \\
&= (\operatorname{Hom}_{\mathbb{C}}(\pi, \tau))^{H} = \operatorname{Hom}_H(\pi, \tau). \qquad \blacksquare
\end{aligned}
$$

Corollary A.3 *The following sequence is exact:*

$$
\mathcal{H}(H)_0 \otimes \mathcal{H}(H) \otimes \pi \to \mathcal{H}(H) \otimes \pi \to \pi \to 0.
$$

Proof of Theorem 2.6(ii) Let H act on $\mathcal{H}(H)_0 \otimes \mathcal{H}(H) \otimes \pi$ and $\mathcal{H}(H) \otimes \pi$ by acting on the $\mathcal{H}(H)$ component. Consider the exact sequence of H-representations

$$
\mathcal{H}(H)_0 \otimes \mathcal{H}(H) \otimes \pi \to \mathcal{H}(H) \otimes \pi \to \pi \to 0.
$$

Since $\mathcal{F}$ is right exact, the sequence

$$
\mathcal{F}(\mathcal{H}(H)_0 \otimes \mathcal{H}(H) \otimes \pi) \to \mathcal{F}(\mathcal{H}(H) \otimes \pi) \to \mathcal{F}(\pi) \to 0
$$

is exact.

Since $\mathcal{F}$ commutes with direct sums, the later sequence is isomorphic to

$$
\mathcal{H}(H)_0 \otimes \Pi \otimes \pi \to \Pi \otimes \pi \to \mathcal{F}(\pi) \to 0.
$$

The theorem follows now from Lemma A.1. $\blacksquare$

A.2 Proof of Theorem 2.7

The following lemma is standard.

Lemma A.4 *Let K be a compact l-group and $L < K$ be an open subgroup. Then there is a finite number of isomorphism classes of irreducible representations of K that have an L-invariant vector.*

Corollary A.5 *Let K be a compact l-group. Let $\pi = \prod \pi_\sigma$ be a product of smooth isotypic components of K. Then $\pi^\infty = \bigoplus \pi_\sigma$.*

Corollary A.6 *Let H be an l-group. Let π and ρ be smooth admissible representations of H. Then $\operatorname{Hom}_{\mathbb{C}}(\pi, \rho)^\infty = \widetilde{\pi} \otimes \rho$.*

Corollary A.7 *Let H and H' be l-groups. Let $\mathcal{F} \colon \mathcal{M}(H) \to \mathcal{M}(H')$ be a $\mathbb{C}$-linear right exact functor. Let $\Pi := \mathcal{F}(\mathcal{H}(H))$. Let π and ρ be smooth admissible representations of H and H', respectively. Then $\operatorname{Hom}_{H \times H'}(\Pi, \widetilde{\pi} \otimes \rho) = \operatorname{Hom}_{H'}(\mathcal{F}(\pi), \rho)$.*

Proof

$$\begin{aligned}
\operatorname{Hom}_{H'}(\mathcal{F}(\pi), \rho) &= \operatorname{Hom}_{H'}((\Pi \otimes \pi)_H, \rho) = \operatorname{Hom}_{\mathbb{C}}((\Pi \otimes \pi)_H, \rho)^{H'} \\
&= (\operatorname{Hom}_{\mathbb{C}}(\Pi \otimes \pi, \rho)^H)^{H'} = \operatorname{Hom}_{\mathbb{C}}(\Pi \otimes \pi, \rho)^{H \times H'} \\
&= \operatorname{Hom}_{\mathbb{C}}(\Pi, \operatorname{Hom}_{\mathbb{C}}(\pi, \rho))^{H \times H'} = \operatorname{Hom}_{H \times H'}(\Pi, \operatorname{Hom}_{\mathbb{C}}(\pi, \rho)) \\
&= \operatorname{Hom}_{H \times H'}(\Pi, \operatorname{Hom}_{\mathbb{C}}(\pi, \rho)^{\infty}) = \operatorname{Hom}_{H \times H'}(\Pi, \widetilde{\pi} \otimes \rho). \quad \blacksquare
\end{aligned}$$

Corollary A.8   *Theorem* 2.7 *holds.*

B   Proof of Theorem 2.8

We will use the following classical well-known lemma.

Lemma B.1   *Let H be an l-group and π be an irreducible admissible representation of H.*

(i)   *Let $\rho \in \mathcal{M}(H)$ and $\phi\colon \rho \to \pi$ be a non-zero morphism. Then ϕ is an epimorphism.*

(ii)   *Let $v \in \pi$. If $\psi(v) = 0$ for all $\psi \in \widetilde{\pi}$, then $v = 0$.*

(iii) $\dim \operatorname{Hom}(\pi, \pi) = 1$.

Proof of Theorem 2.8   If $\operatorname{Hom}(\mathcal{S}(X, \mathcal{F}), \pi) = 0$, we are done. Otherwise, let $\phi \in \operatorname{Hom}(\mathcal{S}(X, \mathcal{F}), \pi) - \{0\}$. Let $\psi_1, \psi_2 \in \operatorname{Hom}(\mathcal{S}(Y, \mathcal{G}), \widetilde{\pi})$. Let us show that they are linearly dependent. If one of them is zero, we are done, so we assume the contrary.

Define bilinear forms $\xi_i\colon \mathcal{S}(X, \mathcal{F}) \otimes \mathcal{S}(Y, \mathcal{G}) \to \mathbb{C}$ by $\xi_i(f \otimes h) := \langle \psi_i(h), \phi(f) \rangle$. Let V_i be left kernels of ξ_i, *i.e.,*

$$V_i = \{f \in \mathcal{S}(X, \mathcal{F}) \mid \forall h \in \mathcal{S}(Y, \mathcal{G}). \, \xi_i(f \otimes h) = 0\}.$$

By the previous lemma, $V_i = \operatorname{Ker} \phi$ and hence $V_1 = V_2$. Let W_i be the right kernels of ξ_i. Again, the previous lemma implies that $W_i = \operatorname{Ker} \psi_i$. Now consider ξ_i as elements of $\mathcal{S}^*(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$. Clearly they are H-invariant. Hence, by the assumption of the theorem, ξ_i are invariant with respect to T. Hence $W_i = \tau_* V_i$. Hence $W_1 = W_2$ and by the previous lemma ψ_1 is proportional to ψ_2. This implies that $\dim \operatorname{Hom}(\mathcal{S}(Y, \mathcal{G})), \widetilde{\pi}) \leq 1$. Similarly $\dim \operatorname{Hom}(\mathcal{S}(X, \mathcal{F})), \widetilde{\pi}) \leq 1$. $\blacksquare$

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Massachussetts Institute of Technology, Cambridge, MA 02139, USA
e-mail: aizner@gmail.com

Faculty of Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: dimagur@weizmann.ac.il