# A SPECIAL FORMULA FOR THE LIE CHARACTER

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In 1942 Thrall **(3)** introduced what he called the *Lie character* to study the structure of the free Lie ring. These characters are defined by associative representations of the full linear group $GL_q(C)$. Their importance is seen in the fact that, for each $n$, the splitting of the submodule of Lie forms of degree $n$ into irreducible invariant subspaces is completely specified by the Lie character $\zeta_n$.

Thrall's original determination of the Lie character depended on a very difficult recursive procedure which is hardly feasible for $n > 10$. Two years later, however, Brandt **(1)** used his results to get an explicit expression for the general Lie character in terms of characters of the symmetric group. But even using Brandt's formula would be impractical without tables of characters for the symmetric group; thus for $n \geqslant 15$ it is of mainly theoretical interest.

However, there is an important special case where we can so transform Brandt's formula as to permit computation for very much higher $n$: namely, the case in which the number of generators of the Lie ring is 2. This note is to derive such an improved formula and several of its consequences.

**1. Preliminaries.** Let $R$ be the free associative formal power-series ring over the complex numbers in non-commutative indeterminates $x_1, \ldots, x_q$. If we introduce within $R$ a non-associative operation given by $[f, g] = fg - gf$ we can single out a subspace $L$ of $R$ which is isomorphic to the free Lie ring on $q$ generators. $L$ is defined recursively as the $C$-space generated by the $x_i$'s and all those "bracket products" $[f, g]$ for which $f$ and $g$ are themselves in $L$.

Let $L_n$ and $R_n$ be respectively the submodules of Lie forms of degree $n$ and of all homogeneous polynomials of degree $n$. Then $L$ and $R$ can be written as direct sums:

$$L = L_1 + L_2 + \ldots,$$
$$R = R_1 + R_2 + \ldots .$$

Next let $f_i = f_i\,(x_1, \ldots, x_q)$ $(i = 1, \ldots, \psi_n)$ be a basis for $L_n$, $\psi_n$ being the dimension of $L_n$ as given by Witt **(5)**. If $A: x_i \to \sum a_{ij}\, x_j$ is any element of $GL_q(C)$ then $A$ defines an automorphism of $L$ in which each $L_n$ is mapped into itself. Let $\mathscr{L}_n(A)$ be the matrix, with respect to the basis of $f_i$'s, of the trans-

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formation so induced in $L_n$. Then the mapping $\mathscr{L}_n: A \to \mathscr{L}_n(A)$ is a representation of $GL_q(C)$ which Thrall called the *Lie representation*. Now all the irreducible representations of $GL_q$ within $R_n$ are well known **(4)**; there is one of these, say $R_n^\lambda$, corresponding to each partition $(\lambda)$ of $n$. Hence the Lie representation must have the form

$$\mathscr{L}_n = \sum_{(\lambda)} c_\lambda R_n^\lambda \tag{1}$$

where the summation is over all partitions of $n$. To determine $\mathscr{L}_n$ it is only necessary to find the multiplicities $c_\lambda$. But to find these we may as well work with the *Lie character*

$$\zeta_n = \operatorname{Tr} \mathscr{L}_n = \sum_{(\lambda)} c_\lambda \{\lambda\}. \tag{2}$$

Here $\{\lambda\}$ is the S-function (Schur function) defined by the partition $(\lambda)$.

Starting from Thrall's recursive procedure for computing $\zeta_n$, Brandt was able to derive the explicit formula

$$\zeta_n = \frac{1}{n} \sum_{d|n} \mu(d)\, s_d^{n/d}, \tag{3}$$

where $\mu$ is the Möbius function and $s_d$ Newton's sum function: $s_d(A) = \operatorname{Tr} A^d$ (which is $\epsilon_1{}^d + \ldots + \epsilon_q{}^d$ if $A$ has eigenvalues $\epsilon_1, \ldots, \epsilon_q$). Since the expressions $s_d{}^{n/d}$ are linear combinations of S-functions whose coefficients are characters of the symmetric group $\mathfrak{S}_n$, this formula permits computation of $\zeta_n$ for $n \leqslant 14$ (those values of $n$ for which there are tables of characters of $\mathfrak{S}_n$). To go higher one would have to compute sizable parts of the appropriate table of characters before beginning.

**2. Restriction to two generators.** Nothing has yet been said about the relative sizes of $q$, the number of generators, and $n$, the "power" of this $n$th power representation. The chief force of the restriction to 2 generators comes in the fact that whenever $q < n$ those irreducible representations which correspond to partitions of $n$ into more than $q$ parts do not appear in the $n$th power tensor representation, and hence not in $\mathscr{L}_n$ either. Another advantage is that when $q = 2$ we can readily derive a simple expression for the symmetric group characters that occur as coefficients in the expression for $s_d{}^{n/d}$ in terms of S-functions.

To go from Brandt's formula to one in S-functions is to change basis in the vector space of all homogeneous symmetric polynomials of degree $n$. One basis is that of the S-functions, $\{\lambda\}$. Brandt's formula uses a basis of products of Newton sums. If we write

$$S(\mu) = s_{\mu_1} \ldots s_\mu$$

for the product of the Newton sums

$$s_{\mu_i} = \textstyle\sum_j y_j^{\mu_i}$$

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(with $s_0$ set equal to 1) then the $S(\mu)$'s provide another basis for this same space of homogeneous functions, as $(\mu)$ runs over all partitions of $n$. The change of basis is given by

$$S(\mu) = s_{\mu_1} \ldots s_{\mu_n} = \sum_{(\lambda)} \chi_\mu^\lambda \{\lambda\}, \tag{4}$$

where $\chi_\mu{}^\lambda$ is the value of the symmetric-group character $\chi^\lambda$ for any argument with $\mu_1$ 1-cycles, $\mu_2$ 2-cycles, and so forth.

The only $S(\mu)$'s in Brandt's formula are those of the form $s_d \ldots s_d$ ($n/d$ factors) for each divisor $d$ of $n$. Therefore, in view of the fact that the $\{\lambda\}$'s are a basis we can substitute from (4) in (3) to get

$$\zeta_n = \sum_{(\lambda)} c_\lambda \{\lambda\} = \frac{1}{n} \sum_{d|n} \mu(d) \left( \sum_{(\lambda)} \chi^\lambda_{d^{n/d}} \{\lambda\} \right),$$

so that

$$c_\lambda = \frac{1}{n} \sum_{d|n} \mu(d)\, \chi^\lambda_{d^{n/d}}, \tag{5}$$

the sum being taken over values of $\chi^\lambda$ for any permutation with $n/d$ $d$-cycles.

Since the one-part partition gives tensors which map into zero in $L$, the only $(\lambda)$'s in our formula when $q = 2$ are the two-part partitions $(n-k, k)$, $1 \leqslant k \leqslant [\frac{1}{2}n]$. If we now write $c_{nk}$ for the coefficient $c_\lambda$, all these reductions of the problem can be summarized in

LEMMA 1. *When $q = 2$ the Lie character is given by*

$$\zeta_n = \sum_{k=1}^{[\frac{1}{2}n]} c_{nk} \{n-k, k\}, \tag{6}$$

*where the coefficient $c_{nk}$ is*

$$c_{nk} = \frac{1}{n} \sum_{d|n} \mu(d)\, \chi^{n-k,k}_{d^{n/d}}. \tag{7}$$

These character values fall out at once from a consequence of Frobenius's formula. To state this handily it helps to change partition notation and denote the class of a typical permutation by $(\alpha) = (1^{\alpha_1} \ldots s^{\alpha_s})$, where $s$ is now the largest part in the partition and $\alpha_k$ is the number of $k$-cycles in any member of the class $(\alpha)$. The value of a character $\chi^{n-k,k}$ for such a permutation is then (**2**, p. 143)

$$\chi_\alpha^{n-k,k} = \sum_{(\beta)} \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_s}{\beta_s} - \sum_{(\gamma)} \binom{\alpha_1}{\gamma_1} \ldots \binom{\alpha_s}{\gamma_s}, \tag{8}$$

the sums being taken over all partitions $(\beta)$ and $(\gamma)$ of $k$ and $k-1$ respectively: that is, over all solutions in non-negative integers of the equations

$$\sum_{i=1}^{s} i\beta_i = k \text{ and } \sum_{i=1}^{s} i\gamma_i = k-1.$$

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LEMMA 2. *The character values are given by*

$$\chi_{d^{n/d}}^{n-k,k} = \begin{cases} \binom{n}{k} - \binom{n}{k-1} & \text{if } d = 1, & \text{(a)} \\ \binom{n/d}{k/d} & \text{if } d|k, & \text{(b)} \\ -\binom{n/d}{(k-1)/d} & \text{if } d|k-1, & \text{(c)} \\ 0 & \text{if } d \nmid k, d \nmid k-1. & \text{(d)} \end{cases}$$

In the present "$\alpha$-notation" we are writing $(d, \ldots, d)$ as $(1^0 2^0 \ldots d^{n/d})$ so that in this case $\alpha_d = n/d$ and the other $\alpha_i = 0$. The result follows at once from formula (8).

Since $\alpha_d = n/d$ and all other $\alpha$'s are zero, the product

$$\binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_s}{\beta_s}$$

can be non-zero only if $(\beta)$ is a product of $d$-cycles. This requires that $d|k$, and then the value of the product is

$$\binom{n/d}{k/d}.$$

All four results are now immediate.

COROLLARY. *For every* $n > 2$, $c_{n1} = 1$.

The only divisor of $k = 1$ is 1 and (a) says

$$\chi_{1^n}^{n-1,1} = \binom{n}{1} - \binom{n}{0} = n - 1.$$

On the other hand any $d$ divides $k - 1 = 0$ so we must take separate account of every $d > 1$. For each of these, by (c), we have

$$\chi_{d^{n/d}}^{n-1,1} = -\binom{n/d}{0/d} = -1.$$

Then, using Lemma 1

$$\begin{aligned} c_{n1} &= \frac{1}{n} \sum_{d|n} \mu(d)\, \chi_{d^{n/d}}^{n-1,1} \\ &= \frac{1}{n}\left(\mu(1)\,(n-1) + \sum_{d|n, d \neq 1} \mu(d)\,(-1)\right). \end{aligned}$$

But taking only the $d$'s greater than 1 gives $\sum \mu(d) = -1$, so

$$c_{n1} = \frac{1}{n}\left(n - 1 + (-1)\,(-1)\right) = 1.$$

This fact is clear, too, from direct examination of the Lie ring.

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With Lemma 2 it is straightforward to compute $\zeta_n$ at least within the considerable range of binomial tables. To do so one would first partition the set of divisors $d > 1$ of $n$ into five sets:

$D_0$ — those which do not divide either $k$ or $k-1$ together with any other $d$'s which are not square-free (for which we would have $\mu(d) = 0$);

$D_1$ — divisors of $k$ which are products of an even number of distinct primes (so $\mu(d) = 1$);

$D_2$ — divisors of $k$ which are products of an odd number of distinct primes (so $\mu(d) = -1$);

$D_3$ — divisors of $k-1$ with $\mu(d) = 1$, and

$D_4$ — divisors of $k-1$ with $\mu(d) = -1$.

In these terms we can summarize all the results so far as a theorem. To reduce parentheses we give the value of $nc_{nk}$ rather than $c_{nk}$ itself.

THEOREM. *For any $n > 2$ and any $k$ with $1 \leqslant k \leqslant [\frac{1}{2}n]$, $nc_{nk}$ is*

$$\binom{n}{k} - \binom{n}{k-1} + \sum_{d \in D_1} \binom{n/d}{k/d} - \sum_{d \in D_2} \binom{n/d}{k/d} - \sum_{d \in D_3} \binom{n/d}{(k-1)/d} + \sum_{d \in D_4} \binom{n/d}{(k-1)/d}.$$

COROLLARY 1. *For $n > 3$, $c_{n2} = \left[\frac{n-3}{2}\right]$.*

For if $n$ is odd the sets $D_1, \ldots, D_4$ are all empty so that

$$nc_{n2} = \binom{n}{2} - \binom{n}{1} = n\frac{(n-3)}{2} = n\left[\frac{n-3}{2}\right].$$

If $n$ is even then $D_2 = \{2\}$ is not empty and

$$nc_{n2} = \binom{n}{2} - \binom{n}{1} - \binom{\frac{1}{2}n}{1} = n\frac{(n-4)}{2} = n\left[\frac{n-3}{2}\right].$$

One can likewise write simplified formulas giving $c_{nk}$ for any fixed $k > 2$, for any $n$. But in no such case are the congruence properties so simple as for $k = 2$, nor is there such a handy notation to summarize them as that of the greatest integer function. Thus for $k = 3$ one must distinguish three cases: $n \equiv 0 \pmod 6$, $n \equiv 3 \pmod 6$, and $n$ even but not divisible by 6. On the other hand, by fixing $n$ as a prime and allowing $k$ to vary we can at once state

COROLLARY 2. *When $n = p$ is prime, then for all $k$ with $1 \leqslant k \leqslant [\frac{1}{2}p]$,*

$$pc_{pk} = \binom{p}{k} - \binom{p}{k-1}.$$

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## References


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