Canad. J. Math. 2025, pp. 1–26
http://dx.doi.org/10.4153/S0008414X24000634

L^p regularity of the Bergman projection on the symmetrized polydisc

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Abstract. We study the L^p regularity of the Bergman projection P over the symmetrized polydisc in $\mathbb{C}^n$. We give a decomposition of the Bergman projection on the polydisc and obtain an operator equivalent to the Bergman projection over antisymmetric function spaces. Using it, we obtain the L^p irregularity of P for $p = \frac{2n}{n-1}$ which also implies that P is L^p bounded if and only if $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$.

1 Introduction

Let Ω be a domain in the complex Euclidean space $\mathbb{C}^n$. Let dV denote the Lebesgue measure. The Bergman projection P_Ω is the orthogonal projection from $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$, the space of all square-integrable holomorphic functions. Associated with P_Ω, there is a unique function K_Ω on $\Omega \times \Omega$ such that for any $f \in L^2(\Omega)$:

$$P_\Omega(f)(z) = \int_\Omega K_\Omega(z;\bar{w})f(w)dV(w). \tag{1.1}$$

The positive Bergman operator P^+_Ω is given by

$$P^+_\Omega(f)(z) = \int_\Omega |K_\Omega(z;\bar{w})|f(w)dV(w). \tag{1.2}$$

By its definition, the Bergman projection is L^2 bounded. An active area of research in several complex variables and harmonic analysis considers the L^p regularity of P_Ω for $p \neq 2$. In particular, people are interested in the connection between the boundary geometry of pseudoconvex domains and the L^p behavior of the projection. On a wide class of domains, the Bergman projection is L^p regular for all $1 < p < \infty$ (see, for instance, [BŞ12, CD06, EL08, Fef74, McN89, McN94a, McN94b, MS94, NRSW88, PS77]). On some other domains, the projection has only a finite range of mapping regularity (see, for example, [BCEM22, Che17, CJY20, CKY20, CZ16, EM16, EM17, Zey13]). We also refer to [Zey20] for a survey on the problem.

Received by the editors October 19, 2023; revised June 8, 2024; accepted June 18, 2024.
B.D.W.'s research is partially supported by the National Science Foundation grants DMS # 1800057 and # 2054863, and # 2000510 and Australian Research Council grant DP 220100285. Z.H.'s research is partially supported by the National Science Foundation of China (NSFC) grant # 12201265.
AMS subject classification: **32A25, 32A36, 32A50.**
Keywords: Bergman projection, Bergman kernel, symmetrized polydisc.

In this article, we focus on the Bergman projection on the symmetrized polydisc $\mathbb{G}^n$. Let $\mathbb{D}^n$ denote the polydisc in $\mathbb{C}^n$. Let Φ_n be the rational holomorphic mapping on $\mathbb{C}^n$ given by $\Phi_n(w_1,\dots,w_n) = (p_1(w),\dots,p_n(w))$, where $p_j(w)$ is the symmetric polynomial in w of degree j:

$$p_j(w_1,w_2,\dots,w_n) = \sum_{k_1<k_2<\dots<k_j} w_{k_1}w_{k_2}\dots w_{k_j}.$$

The symmetrized polydisc $\mathbb{G}^n$ is the image of $\mathbb{D}^n$ under Φ_n:

$$\mathbb{G}^n := \{(p_1(w),\dots,p_n(w)) : w \in \mathbb{D}^n\}. \tag{1.3}$$

When $n = 2$, the symmetrized bidisc

$$\mathbb{G} := \mathbb{G}^2 = \{(w_1+w_2, w_1w_2) : (w_1,w_2) \in \mathbb{D}^2\} \tag{1.4}$$

serves as an interesting example in several complex variables. It is a first known example of many phenomena. We list some of them here below:

- The Lempert theorem may hold on bounded pseudoconvex domains that are not biholomorphically equivalent to any convex domains [AY04].
- Bounded $\mathbb{C}$-convex domains are not necessarily biholomorphically equivalent to convex ones [NPZ08].

See also [ALY18, AY00, Sar15] for some recent work on $\mathbb{G}$.

In addition, the symmetrized polydisc $\mathbb{G}^n$ also serves as an example of a quotient domain and is biholomorphically equivalent to $\mathbb{D}^n/\mathcal{S}_n$, where $\mathcal{S}_n$ is the group of permutations of n coordinate variables in $\mathbb{C}^n$. See [DM23, Gho21] for some recent studies regarding Bergman projections over quotient domains of the form Ω/G.

Partially due to $\mathbb{G}^2$'s interesting properties, the L^p regularity of $P_{\mathbb{G}^2}$ and $P_{\mathbb{G}^n}$ has also attracted attention in recent years. In [CKY20], Chen, Krantz, and Yuan showed that $P_{\mathbb{G}^n}$ is L^p bounded for $p \in (1 + \frac{n-1}{\sqrt{n^2-1}}, 1 + \frac{\sqrt{n^2-1}}{n-1})$. Later, Chen, Jin, and Yuan [CJY20] improved the L^p regular range of $P_{\mathbb{G}}$ to $(4/3, 4)$ and established the Sobolev estimates for $P_{\mathbb{G}}$. While preparing this article, the authors were informed of a discrepancy between the arXiv version of [CJY20] and the version those authors submitted to a journal for publication. In a recent update of [CJY20] posted to the arXiv, the range of L^p regularity for the symmetrized polydisc is at least $(\frac{2n}{n+1}, \frac{2n}{n-1})$, see [CJY23, Remark 1.5]. The main idea in the proof of these results is to use Bell's transformation formula [Bel81] to reformulate the L^p regularity problem of $P_{\mathbb{G}^n}$ into a weighted L^p regularity problem of $P_{\mathbb{D}^n}$ over a weighted L^p space of antisymmetric functions. Yet, the precise L^p regular range for $P_{\mathbb{G}^n}$ was not previously known.

There are mainly two challenges on obtaining the sharp L^p estimates of $P_{\mathbb{G}^n}$: 1. the complexity of the Jacobian of $(p_1,\dots,p_n)$ for large n dimension makes estimations complicated. 2. The cancellation caused by integrating antisymmetric functions creates obstacles to precisely analyze the (un)boundedness of the operator. To us, the second issue is more crucial and distinguishes the problem on $\mathbb{G}^n$ from other settings like the Hartogs triangle. Actually, this issue leads to an interesting yet nontrivial weighted inequality problem in harmonic analysis. We elaborate below using a simple analogical example:

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Let T be a singular integral operator on $L^p(\mathbb{R}^2)$. Set

$$L^p_{\text{anti}}(\mathbb{R}^2, |x_1 - x_2|^a) := \{f \in L^p(\mathbb{R}^2, |x_1 - x_2|^a) : f(x_1, x_2) = -f(x_2, x_1)\}.$$

For which p is the operator T bounded on $L^p_{\text{anti}}(\mathbb{R}^2, |x_1 - x_2|^a)$?

From the classical weighted theory, the singularity of the weight function $|x_1 - x_2|^a$ over the line $\{x_1 = x_2\}$ may cause unboundedness issue for T over $L^p(\mathbb{R}^2, |x_1 - x_2|^a)$. On the other hand, the antisymmetry property $f(x_1, x_2) = -f(x_2, x_1)$ implies that for any $U \subseteq \mathbb{R}$,

$$\int_{U\times U} f dV = 0,$$

suggesting possible better behavior of T on the subspace $L^p_{\text{anti}}(\mathbb{R}^2, |x_1 - x_2|^a)$ than on the entire weighted L^p space. Nevertheless, the usual harmonic analysis methods for weighted L^p cannot be directly applied to this subspace case.

In this article, we overcome these issues on $\mathbb{G}^n$ and give the precise L^p regular range for $P_{\mathbb{G}^n}$ and $P^+_{\mathbb{G}^n}$:

Theorem 1.1 *$P_{\mathbb{G}^n}$ and $P^+_{\mathbb{G}^n}$ are L^p bounded if and only if $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$.*

When $n = 2$, $P_{\mathbb{G}}$ is L^p bounded if and only if $p \in (\frac{4}{3}, 4)$. In contrast to this result, Dall'Ara and Monguzzi [DM23] recently showed that, if one replaces $\mathbb{D}^2$ by unit ball $\mathbb{B}_2$ in (1.4), the Bergman projection over the newly formed domain $\{(w_1 + w_2, w_1 w_2) : (w_1, w_2) \in \mathbb{B}_2\}$ will possess completely different L^p mapping properties. In particular, they proved the following:

Set $D_{2^k} := \{(w_1^{2^k} + w_2^{2^k}, w_1 w_2) : (w_1, w_2) \in \mathbb{B}_2\}$ with $k \in \mathbb{N} \cup \{0\}$. Then the Bergman projection on D_{2^k} is L^p bounded for all $p \in (1, \infty)$.

Our computations suggest that the distinction between results on $\mathbb{G}$ and D_{2^k} is caused by the product structure of $\mathbb{D}^2$. It is yet to be investigated on what exact geometric property of these domains will determine the L^p mapping behaviors of the projection over them.

Our proof strategy of Theorems 1.1 can be summarized as follows:

(1) Similar to [CJY20, CKY20], we reformulate Theorem 1.1 into a weighted L^p regularity result of $P_{\mathbb{D}^n}$ for antisymmetric functions on the polydisc $\mathbb{D}^n$ (see Theorems 2.4 and 2.3).

(2) We prove in detail the L^p boundedness results for $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$ using known weighted estimates on the polydisc (see Theorem 2.3, Section 3, and [CJY23, Remark 1.5]).

(3) To obtain the unboundedness result for the case $p = \frac{2n}{n-1}$, we decompose $P_{\mathbb{D}^n}$ into the sum of two operators T_1^n and T_2^n (see (4.5) and (4.6)), where $T_1^n = 0$ and $T_2^n = P_{\mathbb{D}^n}$ over spaces of antisymmetric functions (see Lemmas 4.1 and 4.3).

(4) By using T_2^n, we further reduce the (un)boundedness problem of $P_{\mathbb{D}^n}$ over a space of antisymmetric functions into a problem about an operator $\tilde{T}^n$ over a different space of symmetric functions. Finally, we provide examples for the unboundedness of $\tilde{T}^n$ there (see Theorems 4.2 and 4.4 and their proofs).

We remark that the decomposition $P_{\mathbb{D}^n} = T_1^n + T_2^n$ is crucial in our proof. Using the kernel function of T_2^n, we are able to "cancel out" part of the weight of the space,

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transform the problem from an antisymmetric function space to a symmetric one, and reduce norm computation difficulty in n-dimensional case all at once.

Our article is organized as follows: In Section 2, we provide known lemmas and reduce L^p estimates of $P_{\mathbb{G}^n}$ and $P^+_{\mathbb{G}^n}$ into weighted L^p estimates of $P_{\mathbb{D}^n}$ for (anti)symmetric functions. In Section 3, we recall the known weighted L^p norm estimates of $P_{\mathbb{D}}$ and give a detailed proof for the L^p boundedness result for $P_{\mathbb{G}^n}$ and $P^+_{\mathbb{G}^n}$. In Section 4, we present the decomposition of $P_{\mathbb{D}^n}$ and examples for the L^p irregularity of $P_{\mathbb{G}^n}$ for $p = \frac{2n}{n-1}$. In Section 5, we point out some directions for future research.

Given functions of several variables f and g, we use $f \lesssim g$ to denote that $f \leq Cg$ for a constant C. If $f \lesssim g$ and $g \lesssim f$, then we say f is comparable to g and write $f \approx g$.

2 Pull back from $\mathbb{G}^n$ to $\mathbb{D}^n$

This section focuses on reformulating the L^p regularity of $P_{\mathbb{G}^n}$ into a problem on the polydisc $\mathbb{D}^n$. Most of the lemmas and results were included in [CJY20, CKY20]. We provide proofs here for completeness of our article.

2.1 From $\mathbb{G}^n$ to $\mathbb{D}^n$

Recall that $\Phi_n(w) = (p_1(w), p_2(w), \dots, p_n(w))$, where

$$p_j(w_1, w_2, \dots, w_n) = \sum_{k_1<k_2<\cdots<k_j} w_{k_1} w_{k_2} \dots w_{k_j}.$$

Then Φ_n is a ramified rational proper covering map of order $n!$ with complex holomorphic Jacobian

$$J_{\mathbb{C}}\Phi_n = \prod_{j<k} (w_j - w_k)$$

(see, for example, [CKY20]). Let $h \in L^p(\mathbb{G}^n)$. Via a change of variables, the estimate

$$\|P_{\mathbb{G}^n}(h)\|_{L^p(\mathbb{G}^n)} \lesssim \|h\|_{L^p(\mathbb{G}^n)}$$

is equivalent to

$$\|P_{\mathbb{G}^n}(h) \circ \Phi_n\|_{L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^2)} \lesssim \|h \circ \Phi_n\|_{L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^2)}. \tag{2.1}$$

Using the Bell's transformation formula [Bel81],

$$P_{\mathbb{D}^n}(J_{\mathbb{C}}\Phi_n \cdot (h \circ \Phi_n)) = J_{\mathbb{C}}\Phi \cdot (P_{\mathbb{G}^n}(h) \circ \Phi_n),$$

(2.1) becomes the following weighted estimate:

$$\|P_{\mathbb{D}^n}(J_{\mathbb{C}}\Phi_n \cdot (h \circ \Phi_n))\|_{L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})} \lesssim \|J_{\mathbb{C}}\Phi_n \cdot h \circ \Phi_n\|_{L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})}. \tag{2.2}$$

By Bell's transformation formula for the Bergman kernel,

$$\sum_{j=1}^{n!} K_{\mathbb{D}^n}(z; \overline{\phi_j(w)}) \overline{J_{\mathbb{C}}(\phi_j)(w)} = J_{\mathbb{C}}\Phi_n(z) K_{\mathbb{G}_n}(\Phi_n(z), w),$$

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where ϕ_j are the $n!$ local inverses of Φ. Therefore, to show the estimate

$$\|P^+_{\mathbb{G}^n}(h)\|_{L^p(\mathbb{G}^n)} \lesssim \|h\|_{L^p(\mathbb{G}^n)},$$

it is sufficient to prove that

$$\|P^+_{\mathbb{D}^n}(|J_{\mathbb{C}}\Phi_n| \cdot (h \circ \Phi_n))\|_{L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})} \lesssim \||J_{\mathbb{C}}\Phi_n| \cdot h \circ \Phi\|_{L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})}. \tag{2.3}$$

Let $\mathcal{S}_n$ denote the family of all permutations of $\{z_1, \dots, z_n\}$. Since Φ_n is invariant under any permutation, the function $h \circ \Phi_n$ also inherits symmetry properties. To clearly describe them, we give several definitions below. For $j, k \in \{1, \dots, n\}$ with $j < k$, we let $\tau_{j,k}$ denote the 2-cycle in $\mathcal{S}_n$ that interchanges z_j and z_k. For $j = 1, \dots, n$, we will also abuse the notation for $\tau \in \mathcal{S}_n$ and let $\tau(j)$ denote the index such that $\tau(z_j) = z_{\tau(j)}$.

Definition 2.1 Let f be a function on $\mathbb{D}^n$.

(1) f is called (j,k) symmetric if $f(z_1, \dots, z_n) = f \circ \tau_{j,k}(z_1, \dots, z_n)$, and is called symmetric if $f(z_1, \dots, z_n) = f \circ \tau_{j,k}(z_1, \dots, z_n)$ for any $j \neq k$.
(2) f is called (j,k) antisymmetric if $f(z_1, \dots, z_n) = -f \circ \tau_{j,k}(z_1, \dots, z_n)$ and is called antisymmetric if $f(z_1, \dots, z_n) = -f \circ \tau_{j,k}(z_1, \dots, z_n)$ for any $j \neq k$.

By the above definition, $h \circ \Phi_n$ is symmetric while $J_{\mathbb{C}}\Phi_n$ is antisymmetric. Therefore, the function $J_{\mathbb{C}}\Phi_n \cdot h \circ \Phi_n$ is antisymmetric and $|J_{\mathbb{C}}\Phi_n| \cdot h \circ \Phi_n$ is symmetric. It's also not hard to see that $P_{\mathbb{D}^n}(J_{\mathbb{C}}\Phi_n \cdot (h \circ \Phi_n))$ and $P^+_{\mathbb{D}^n}(J_{\mathbb{C}}\Phi_n \cdot (h \circ \Phi_n))$ are antisymmetric and $P^+_{\mathbb{D}^n}(|J_{\mathbb{C}}\Phi_n| \cdot (h \circ \Phi_n))$ is symmetric. Set

$$L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p}) := \{f \in L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p}) : f \text{ is antisymmetric}\}, \tag{2.4}$$

$$L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p}) := \{f \in L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p}) : f \text{ is symmetric}\}. \tag{2.5}$$

$L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ and $L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ turn out to be equivalent to $L^p(\mathbb{G}^n)$.

The next lemma gives the norm equivalence of $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$, $L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$, and $L^p(\mathbb{G}^n)$. When $p = 2$, this lemma can be viewed as a special case of [Try13, Theorem 1].

Lemma 2.2 *The following statements are true:*

(1) $L^p_{anti}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ *is norm equivalent to* $L^p(\mathbb{G}^n)$ *via the mapping:*

$$f \mapsto \sum_{j=1}^{n!} \left(\frac{f}{J_{\mathbb{C}}\Phi}\right) \circ \phi_j. \tag{2.6}$$

(2) $L^p_{sym}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ *is norm equivalent to* $L^p(\mathbb{G}^n)$ *via the mapping:*

$$f \mapsto \sum_{j=1}^{n!} \left(\frac{f}{|J_{\mathbb{C}}\Phi_n|}\right) \circ \phi_j. \tag{2.7}$$

Proof We prove the statement for $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$. The proof for $L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ is similar. We begin by showing that the mapping in (2.6)

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is norm preserving. Since f is antisymmetric, the function $\frac{f}{J_{\mathbb{C}}\Phi_n}$ is symmetric. Thus, $\left(\frac{f}{J_{\mathbb{C}}\Phi_n}\right)\circ\phi_j = \left(\frac{f}{J_{\mathbb{C}}\Phi_n}\right)\circ\phi_k$ for any j, k and

$$\begin{aligned}\int_{\mathbb{D}^n} |f|^p |J_{\mathbb{C}}\Phi_n|^{2-p} dV &= \int_{\mathbb{D}^n} \left|\frac{f}{J_{\mathbb{C}}\Phi_n}\right|^p |J_{\mathbb{C}}\Phi|^2 dV \\ &= \sum_{j=1}^{n!} \int_{\phi_j(\mathbb{G}^n)} \left|\frac{f}{J_{\mathbb{C}}\Phi_n}\right|^p |J_{\mathbb{C}}\Phi_n|^2 dV \\ &= \sum_{j=1}^{n!} \int_{\mathbb{G}^n} \left|\left(\frac{f}{J_{\mathbb{C}}\Phi_n}\right)\circ\phi_j\right|^p dV \\ &= (n!)^{1-p} \int_{\mathbb{G}^n} \left|\sum_{j=1}^{n!}\left(\frac{f}{J_{\mathbb{C}}\Phi_n}\right)\circ\phi_j\right|^p dV.\end{aligned}$$

Note also that $h \mapsto \frac{1}{n!} J_{\mathbb{C}}\Phi_n \cdot h \circ \Phi_n$ is the inverse of (2.6), the mapping in (2.6) is onto which completes the proof. ∎

By Lemma 2.2 and the fact that $|P_{\mathbb{G}^n}(f)(z)| \le P^+_{\mathbb{G}^n}(|f|)(z)$, the next two theorems are sufficient to yield Theorem 1.1.

Theorem 2.3 *$P_{\mathbb{D}^n}$ and $P^+_{\mathbb{D}^n}$ are bounded on $L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ for $p \in \left(\frac{2n}{n+1}, \frac{2n}{n-1}\right)$.*

Theorem 2.3 appears as [CJY23, Remark 1.5] with the same range of p.

Theorem 2.4 *$P_{\mathbb{D}^n}$ is unbounded on $L^p_{anti}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ for $p = \frac{2n}{n-1}$.*

Last, we reference below the Forelli–Rudin estimates on $\mathbb{D}$ which will be used in the proof of Theorem 2.4 (see, for example, [Zhu05] for its proof).

Lemma 2.5 (Forelli–Rudin) *For $\varepsilon < 1$ and $z \in \mathbb{D}$, let*

$$a_{\varepsilon,s}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\bar{w}|^{2-\varepsilon-s}} dV(w), \tag{2.8}$$

Then:

(1) *for $s > 0$, $a_{\varepsilon,s}(z)$ is bounded on $\mathbb{D}$;*
(2) *for $s = 0$, $a_{\varepsilon,s}(z)$ is comparable to the function $-\log(1-|z|^2)$;*
(3) *for $s < 0$, $a_{\varepsilon,s}(z)$ is comparable to the function $(1-|z|^2)^s$.*

3 Proof of Theorem 2.3

While [CJY23, Remark 1.5] sketches the proof of Theorem 2.3, here we provide all the relevant details to make the article self-contained (see also [ZY, Corollary 6.1]). The main ingredient of the weighted norm estimates of the positive Bergman operator $P^+_{\mathbb{D}}$ over weighted L^p spaces. On the unit disc $\mathbb{D}$, the boundedness of $P_{\mathbb{D}}$ and $P^+_{\mathbb{D}}$

on weighted L^p spaces is closely related to the Bekollé–Bonami constant of weight functions. Let T_z denote the Carleson tent over z in the unit disc $\mathbb{D}$ defined as below:

- $T_z := \left\{ w \in \mathbb{D} : \left| 1 - \bar{w}\frac{z}{|z|} \right| < 1 - |z| \right\}$ for $z \neq 0$, and
- $T_z := \mathbb{D}$ for $z = 0$.

Bekollé and Bonami [BB78] characterized weighted L^p spaces where $P_{\mathbb{D}}$ and $P^+_{\mathbb{D}}$ are bounded:

Theorem 3.1 (Bekollé–Bonami [BB78]) *Let the weight $u(w)$ be a positive, locally integrable function on the unit disc $\mathbb{D}$. Let $1 < p < \infty$. Then the following conditions are equivalent:*

(1) *$P : L^p(\mathbb{D}, u) \mapsto L^p(\mathbb{D}, u)$ is bounded.*
(2) *$P^+ : L^p(\mathbb{D}, u) \mapsto L^p(\mathbb{D}, u)$ is bounded.*
(3) *The Bekollé–Bonami constant*

$$B_p(u) := \sup_{z \in \mathbb{D}} \frac{\int_{T_z} u(w)dV(w)}{\int_{T_z} dV(w)} \left(\frac{\int_{T_z} u^{-\frac{1}{p-1}}(w)dV(w)}{\int_{T_z} dV(w)} \right)^{p-1}$$

is finite.

Using dyadic harmonic analysis technique, various authors established quantitative weighted L^p norm estimates of the Bergman projection (see [HW20b, HWW21, PR13, RTW17]).

Theorem 3.2 [RTW17, Lemma 15] *Let the weight function u be positive, locally integrable on $\mathbb{D}$. Then for $p \in (1, \infty)$,*

$$\|P_{\mathbb{D}}\|_{L^p(\mathbb{D},u)} \leq \|P^+_{\mathbb{D}}\|_{L^p(\mathbb{D},u)} \lesssim \left(B_p(u)\right)^{\max\{1,(p-1)^{-1}\}}.$$

Lemma 3.3 *For a fixed point $a \in \mathbb{D}$, let $u_p(w) = |a - w|^{2-p}$. Then for any $p \in (4/3, 4)$, $B_p(u_p) \lesssim 1$, where the upper bound is independent of a. Moreover, if we choose arbitrary m points $a_1, \ldots, a_m$ in $\mathbb{D}$, and set*

$$v_p(w) = \prod_{j=1}^{m} |a_j - w|^{2-p},$$

then for any $p \in (\frac{2m+2}{m+2}, \frac{2m+2}{m})$, $B_p(v_p) \lesssim 1$. Here, the upper bounds may depend on constants m and p but are independent of a_j.

Proof We first consider the case of the weight u_p. Note that u_p and $u_p^{-1/(p-1)}$ are integrable on $\mathbb{D}$ if and only if $p \in (\frac{4}{3}, 4)$. Then, it is enough to show that $B_p(|a - w|^b) \lesssim 1$ with an upper bound independent of a if both u_p and $u_p^{-1/(p-1)}$ are integrable on $\mathbb{D}$. We consider the integral of u_p and $u_p^{-1/(p-1)}$ over T_z for arbitrary $z \in \mathbb{D}$. Notice that $T_z = \mathbb{D} \cap \{w : |w - \frac{z}{|z|}| < 1 - |z|\}$ is the intersection set of the unit disc $\mathbb{D}$ and the disc centered at the point $z/|z|$ with Euclidean radius $1 - |z|$. A geometric consideration then yields that the Lebesgue measure $V(T_z)$ of T_z is comparable to $(1 - |z|)^2$.

If $|a - z| < 3(1 - |z|)$, then T_z is contained in a ball B_a given by

$$B_a = \{w \in \mathbb{C} : |w - a| < 5(1 - |z|)\}.$$

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Thus,

$$\begin{aligned}
&\frac{\int_{T_z} u_p(w)dV(w)}{\int_{T_z} dV(w)}\left(\frac{\int_{T_z} u_p^{-\frac{1}{p-1}}(w)dV(w)}{\int_{T_z} dV(w)}\right)^{p-1} \\
&\lesssim \frac{\int_{B_a}|w-a|^{2-p}dV(w)\left(\int_{B_a}|w-a|^{(p-2)/(p-1)}dV(w)\right)^{p-1}}{(1-|z|)^{2p}} \\
&= \frac{(5(1-|z|))^{4-p}\cdot\left((p-1)(3p-4)^{-1}(5(1-|z|))^{(3p-4)/(p-1)}\right)^{p-1}}{(4-p)5^{2p}(1-|z|)^{2p}} \\
&= \frac{(p-1)^{p-1}}{(4-p)(3p-4)^{p-1}},
\end{aligned}$$

provided u_p and $u_p^{-1/(p-1)}$ are integrable. If $|a-z| \geq 3(1-|z|)$, then $|a-w| \approx |a-z|$ for all $w \in T_z$ and hence

$$\begin{aligned}
&\frac{\int_{T_z} u_p(w)dV(w)}{\int_{T_z} dV(w)}\left(\frac{\int_{T_z} u_p^{-\frac{1}{p-1}}(w)dV(w)}{\int_{T_z} dV(w)}\right)^{p-1} \\
&\lesssim \frac{|a-z|^{2-p}\int_{T_z} dV(w)}{\int_{T_z} dV(w)}\left(\frac{|a-z|^{(p-2)/(p-1)}\int_{T_z} dV(w)}{\int_{T_z} dV(w)}\right)^{p-1} \\
&= 1.
\end{aligned}$$

Since the upper bound obtained in both cases are independent of the choice of a and T_z, we conclude that $B_p(u_p)$ is bounded above by a constant if and only if $p \in (4/3, 4)$ and the upper bound is independent of a_j.

Now we turn to the case of weight $v_p(w) = \prod_{j=1}^m |a_j - w|^{2-p}$. By a similar proof as above, $B_p(|a-w|^{(2-p)m}) \lesssim 1$ for any $p \in (\frac{2m+2}{m+2}, \frac{2m+2}{m})$ where the upper bound is independent of a. Using Hölder's inequality, we obtain for any $z \in \mathbb{D}$

$$\begin{aligned}
&\frac{\int_{T_z} v_p(w)dV(w)}{\int_{T_z} dV(w)}\left(\frac{\int_{T_z} v_p^{-\frac{1}{p-1}}(w)dV(w)}{\int_{T_z} dV(w)}\right)^{p-1} \\
&\lesssim \left(\prod_{j=1}^m \left(\frac{\int_{T_z}|a_j-w|^{m(2-p)}dV(w)}{\int_{T_z} dV(w)}\right)^{\frac{1}{mp}}\left(\frac{\int_{T_z}|a_j-w|^{m(p-2)/(p-1)}dV(w)}{\int_{T_z} dV(w)}\right)^{\frac{1}{m}-\frac{1}{mp}}\right)^p \\
&\lesssim \left(\prod_{j=1}^m B_p\left(|a_j-w|^{m(2-p)}\right)\right)^{\frac{p}{m}} \lesssim 1.
\end{aligned}$$

Therefore, $B_p(v_p) \lesssim 1$ with upper bound independent of points a_j. ■

With Lemma 3.3, we are ready to show Theorem 2.3:

Proof of Theorem 2.3 Since $|P_{\mathbb{D}^n}(h)(z)| \leq P^+_{\mathbb{D}^n}(|h|)(z)$ for any $h \in L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$, it suffices to show the boundedness for $P^+_{\mathbb{D}^n}$. Note that $J_{\mathbb{C}}\Phi_n(w)$ consists

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of $n-1$ many factors of each variable w_j. When integrating with respect to the single variable w_j, only these $n-1$ factors matter in $J_{\mathbb{C}}\Phi_n(w)$. Thus the boundedness of $P^+_{\mathbb{D}^n}$ on $L^p(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ for $p \in \left(\frac{2n}{n+1}, \frac{2n}{n-1}\right)$ follows from Fubini and Lemma 3.3 with $m = n-1$. ∎

4 Proof of Theorem 2.4

We will first prove the theorem for the case $n = 2$, clearly illustrating the decomposition we use for $P_{\mathbb{D}^2}$. Then we dive into the case for general n where the decomposition procedure and estimations are more complicated yet the same strategy applies.

4.1 The case for $n = 2$

Note that $J_{\mathbb{C}}\Phi_2 = w_1 - w_2$. To prove Theorem 2.4, we consider the decomposition $P_{\mathbb{D}^2} = T_1^2 + T_2^2$ where

$$(4.1) \qquad T_1^2(f)(z_1,z_2) = \int_{\mathbb{D}^2} \frac{f(w_1,w_2)dV}{\pi^2(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)(1-z_1\bar{w}_2)(1-z_2\bar{w}_1)},$$

$$(4.2) \qquad T_2^2(f)(z_1,z_2) = \int_{\mathbb{D}^2} \frac{(z_1-z_2)(\bar{w}_1-\bar{w}_2)f(w_1,w_2)dV}{\pi^2(1-z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2(1-z_1\bar{w}_2)(1-z_2\bar{w}_1)}.$$

Lemma 4.1 *T_1^2 is a zero operator on $L^p_{anti}(\mathbb{D}^2, |w_1-w_2|^{2-p})$.*

Proof Note that $T_1^2(f)(z_1,z_2)$ is symmetric by its definition. For any $f \in L^p_{\text{anti}}(\mathbb{D}^2, |w_1-w_2|^{2-p})$,

$$T_1^2(f)(z_1,z_2) = T_1^2(-f)(z_2,z_1) = -T_1^2(f)(z_1,z_2),$$

which implies $T_1^2(f) = 0$. ∎

By Lemma 4.1, $P_{\mathbb{D}^2} = T_2^2$ on $L^p_{\text{anti}}(\mathbb{D}^2, |w_1-w_2|^{2-p})$. So, Theorem 2.4 can be further reduced into the following statement in the case $n = 2$.

Theorem 4.2 *T_2^2 is unbounded on $L^p_{anti}(\mathbb{D}^2, |w_1-w_2|^{2-p})$ for $p = 4 = \frac{2\times 2}{2-1}$.*

Proof Let $\tilde{T}^2$ denote the operator given as follows:

$$\tilde{T}^2(h)(z) := (J_{\mathbb{C}}\Phi_2(z))^{-1}T_2(h\bar{J}_{\mathbb{C}}\Phi_2)(z).$$

Then

$$(4.3) \qquad \tilde{T}^2(h)(z) = \int_{\mathbb{D}^2} \frac{(\bar{w}_1-\bar{w}_2)^2h(w)dV}{\pi^2(1-z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2(1-z_1\bar{w}_2)(1-z_2\bar{w}_1)},$$

and $\|T_2^2\|_{L^p_{\text{anti}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^{2-p})} = \|\tilde{T}^2\|_{L^p_{\text{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)}$ provided one of the norms is finite. Thus it suffices to show that $\tilde{T}^2$ is unbounded on $L^p_{\text{sym}}(\mathbb{D}^2, |J_{\mathbb{C}}\Phi_2|^2)$ for $p = 4$. For $s \in [\frac{1}{2}, 1)$, we set

$$h_s(w) = \frac{1}{\pi(1-sw_1)^2} + \frac{1}{\pi(1-sw_2)^2}.$$

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Then

$$\begin{aligned}\|h_s\|^4_{L^4_{\mathrm{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)} &= \int_{\mathbb{D}^2}\left|\frac{1}{\pi(1-sw_1)^2}+\frac{1}{\pi(1-sw_2)^2}\right|^4|w_1-w_2|^2dV(w)\\ &\lesssim \int_{\mathbb{D}}\frac{1}{\pi^4|1-sw_1|^8}\int_{\mathbb{D}}|w_1-w_2|^2dV(w_2)dV(w_1)\\ &\approx (1-s)^{-6},\end{aligned}$$

where the last equality follows from the Forelli–Rudin estimates (2.8). Note that the kernel function of $\tilde{T}^2$ is anti-holomorphic in w variables and h_s can be expressed in terms the conjugate of the Bergman kernels:

$$\sum_{j=1}^{2}\frac{1}{\pi(1-sw_j)^2}=\pi\left(\overline{K_{\mathbb{D}^2}((s,0);(\bar{w}_1,0))}+\overline{K_{\mathbb{D}^2}((0,s);(0,\bar{w}_2))}\right).$$

The reproducing property of the Bergman projection implies:

$$\begin{aligned}&\tilde{T}^2(h_s)(z)\\ &=\int_{\mathbb{D}^2}\frac{(\bar{w}_1-\bar{w}_2)^2}{\pi^2(1-z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2(1-z_1\bar{w}_2)(1-z_2\bar{w}_1)}\sum_{j=1}^{2}\frac{1}{\pi(1-sw_j)^2}dV(w)\\ &=\frac{s^2}{\pi(1-z_1s)^2(1-z_2s)}+\frac{s^2}{\pi(1-z_2s)^2(1-z_1s)}.\end{aligned}$$

Thus

$$\begin{aligned}&\|\tilde{T}^2(h_s)\|^4_{L^4_{\mathrm{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)}\\ &\quad=\int_{\mathbb{D}^2}\left|\frac{s^2}{\pi(1-z_1s)^2(1-z_2s)}+\frac{s^2}{\pi(1-z_2s)^2(1-z_1s)}\right|^4|z_1-z_2|^2dV(z)\\ &\quad=\int_{\mathbb{D}^2}\left|\frac{1}{1-z_1s}+\frac{1}{1-z_2s}\right|^4\frac{s^8|z_1-z_2|^2}{\pi^4|1-z_1s|^4|1-z_2s|^4}dV(z). \qquad (4.4)\end{aligned}$$

For fixed $s<1$, set $U(s)=\{z\in\mathbb{D}:\mathrm{Arg}(1-zs)\in\left(-\frac{\pi}{6},\frac{\pi}{6}\right)\}$. Then for $z_1,z_2\in U(s)$,

$$\left|\frac{1}{1-z_1s}+\frac{1}{1-z_2s}\right|\geq\frac{1}{2|1-z_1s|}.$$

Applying this inequality to (4.4) gives

$$\int_{\mathbb{D}^2}\left|\frac{1}{1-z_1s}+\frac{1}{1-z_2s}\right|^4\frac{s^8|z_1-z_2|^2}{\pi^4|1-z_1s|^4|1-z_2s|^4}dV(z)\gtrsim\int_{U^2(s)}\frac{|z_1-z_2|^2}{|1-z_1s|^8|1-z_2s|^4}dV(z).$$

Since

$$\frac{z_1-z_2}{(1-z_1s)(1-z_2s)}=\frac{1}{s(1-z_1s)}-\frac{1}{s(1-z_2s)},$$

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we have

$$\begin{aligned}
&\int_{U^2(s)} \frac{|z_1-z_2|^2}{|1-z_1s|^8|1-z_2s|^4}\,dV(z)\\
&=\int_{U^2(s)} \frac{1}{|1-z_1s|^6|1-z_2s|^2}\left|\frac{1}{s(1-z_1s)}-\frac{1}{s(1-z_2s)}\right|^2 dV(z)\\
&=\int_{U^2(s)} \frac{1}{s^2|1-z_1s|^6|1-z_2s|^2}\left(\frac{1}{|1-z_1s|^2}+\frac{1}{|1-z_2s|^2}-2\mathrm{Re}\frac{1}{(1-z_1s)(1-\bar z_2 s)}\right)dV(z)\\
&\geq \int_{U^2(s)} \frac{1}{|1-z_1s|^8|1-z_2s|^2}+\frac{1}{|1-z_1s|^6|1-z_2s|^4}-2\frac{1}{|1-z_1s|^7|1-z_2s|^3}\,dV(z).
\end{aligned}$$

By realizing that $|1-zs| = s|\frac{1}{s}-z|$ and applying polar coordinates, one can obtain the following Forelli–Rudin estimates (2.8) on $U(s)$.

$$\int_{U(s)} \frac{1}{|1-zs|^a}\,dV(z) \approx \begin{cases}(1-s)^{2-a}, & a>2,\\ -\log(1-s), & a=2,\\ 1, & a<2.\end{cases}$$

We leave the details of its proof to readers as an exercise. Using these estimates,

$$\int_{U^2(s)} \frac{1}{|1-z_1s|^8|1-z_2s|^2}\,dV(z) \approx -(1-s)^{-6}\log(1-s)$$

$$\int_{U^2(s)} \frac{1}{|1-z_1s|^6|1-z_2s|^4}\,dV(z) \approx \int_{U^2(s)} \frac{1}{|1-z_1s|^7|1-z_2s|^3}\,dV(z) \approx (1-s)^{-6},$$

which implies that $\|\tilde T^2(h_s)\|^4_{L^4_{\mathrm{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)} \approx -(1-s)^{-6}\log(1-s)$.

As $s\to 1$,

$$\frac{\|\tilde T^2(h_s)\|^4_{L^4_{\mathrm{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)}}{\|h_s\|^4_{L^4_{\mathrm{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)}} \gtrsim -\log(1-s)\to\infty,$$

proving that $\tilde T^2$ is unbounded on $L^4_{\mathrm{sym}}(\mathbb{D}^2,|J_{\mathbb{C}}\Phi_2|^2)$. ■

4.2 The case for general *n*

Like the case $n=2$, our proof for general n also involves a decomposition of $P_{\mathbb{D}^n}$ into operators T_1^n and T_2^n.

$$T_1^n(h)(z) = \int_{\mathbb{D}^n} \frac{\prod_{1\le j<k\le n}(1-z_k\bar w_j)(1-z_j\bar w_k)-\prod_{1\le j<k\le n}(z_j-z_k)(\bar w_j-\bar w_k)}{\pi^n\prod_{1\le j\le k\le n}(1-z_k\bar w_j)(1-z_k\bar w_j)}$$

$$\times\, h(w)dV(w). \tag{4.5}$$

$$T_2^n(h)(z) = (P_{\mathbb{D}^n}-T_1^n)(h)(z) = \int_{\mathbb{D}^n} \frac{\prod_{1\le j<k\le n}(z_j-z_k)(\bar w_j-\bar w_k)}{\pi^n\prod_{1\le j\le k\le n}(1-z_k\bar w_j)(1-z_j\bar w_k)}h(w)dV(w). \tag{4.6}$$

Lemma 4.3 *T_1^n is a zero operator on $L^p_{anti}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^{2-p})$.*

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Proof Recall that $\tau_{j,k}$ is the permutation that interchanges variables w_j and w_k, and a kernel function $K(z;\bar{w})$ on $\mathbb{D}^n \times \mathbb{D}^n$ is called (j,k)-symmetric in w if $K(z;\bar{w}) = K(z;\bar{\tau}_{j,k}(w))$. If $K(z;\bar{w})$ is (j,k)-symmetric in $\bar{w}$, then for any antisymmetric $f \in L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$, we have

$$\begin{aligned}\int_{\mathbb{D}^n} K(z;\bar{w}) f(w)dV(w) &= -\int_{\mathbb{D}^n} K(z;\bar{\tau}_{j,k}(w)) f(\tau_{j,k}(w))dV(w)\\ &= -\int_{\mathbb{D}^n} K(z;\bar{w}) f(w)dV(w).\end{aligned}$$

Thus operators with (j,k)-symmetric kernel functions in w annihilate $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$.

For $l = 1, \ldots, n$, we define the operator P_l to be as follows:

$$P_l(h)(z) = \int_{\mathbb{D}^n} \frac{\prod_{1\le j<k\le l}(1-z_j\bar{w}_k)(1-z_k\bar{w}_j)\prod_{1\le j<k\le n, 1\le l<k\le n}(z_j-z_k)(\bar{w}_j-\bar{w}_k)}{\pi^n \prod_{1\le j\le k\le n}(1-z_k\bar{w}_j)(1-z_j\bar{w}_k)} \times h(w)dV(w). \tag{4.7}$$

Then $P_1 = T_2^n$ and $P_n = P_{\mathbb{D}^n}$. We claim that $P_{\mathbb{D}^n} = P_l$ on $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ for all $l = 1, \ldots, n$. Then $T_1^n = P_{\mathbb{D}^n} - P_1 = 0$ on $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$. We prove the claim by induction on l.

Let K_l denote the kernel function of P_l. When $l = 2$,

$$K_2(z;\bar{w}) = \frac{(1-z_1\bar{w}_2)(1-z_2\bar{w}_1)\prod_{1\le j<k\le n,(j,k)\ne(1,2)}(z_j-z_k)(\bar{w}_j-\bar{w}_k)}{\pi^n\prod_{1\le j\le k\le n}(1-z_k\bar{w}_j)(1-z_j\bar{w}_k)}.$$

Then

$$\begin{aligned}&K_2(z;\bar{w}) - K_1(z;\bar{w})\\ &= \frac{((1-z_1\bar{w}_2)(1-z_2\bar{w}_1)-(z_1-z_2)(\bar{w}_1-\bar{w}_2))\prod_{1\le j<k\le n,(j,k)\ne(1,2)}(z_j-z_k)(\bar{w}_j-\bar{w}_k)}{\pi^n\prod_{1\le j\le k\le n}(1-z_k\bar{w}_j)(1-z_j\bar{w}_k)}\\ &= \frac{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)\prod_{1\le j<k\le n,(j,k)\ne(1,2)}(z_j-z_k)(\bar{w}_j-\bar{w}_k)}{\pi^n\prod_{1\le j\le k\le n}(1-z_k\bar{w}_j)(1-z_j\bar{w}_k)}\\ &= \frac{\prod_{1\le j<k\le n,(j,k)\ne(1,2)}(z_j-z_k)(\bar{w}_j-\bar{w}_k)}{\pi^n\prod_{j=3}^n(1-z_j\bar{w}_j)\prod_{j,k=1}^n(1-z_k\bar{w}_j)}.\end{aligned}$$

It is not hard to check that $K_2 - K_1$ is $(1,2)$-symmetric in w which shows that $P_1 = P_2$ on $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$.

Suppose that $P_1 = P_l$ on $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ for $l = m$. We show that $P_{m+1} = P_m$ on $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$. Let $\mathcal{R}_m$ denote the power set

$$\mathcal{R}_m := \{\mathcal{I} : \mathcal{I} \subseteq \{1, 2, \ldots, m\}\}.$$

Given $\mathcal{I} \in \mathcal{R}_m$, let $|\mathcal{I}|$ denote the cardinality of $\mathcal{I}$. For simplicity of notation, we set $a_{j,k} = 1 - z_j\bar{w}_k$ and $b_{j,k} = (z_j - z_k)(\bar{w}_j - \bar{w}_k)$. Then for $j \ne k$, $a_{j,k}a_{k,j} = a_{j,j}a_{kk} + b_{j,k}$. Note that

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$$\prod_{j=1}^{m} a_{j,m+1}a_{m+1,j} = \prod_{j=1}^{m}(a_{j,j}a_{m+1,m+1} + b_{j,m+1})$$
$$= \sum_{\mathcal{J}\in\mathcal{R}_m} a_{m+1,m+1}^{|\mathcal{J}|} \prod_{j\in\mathcal{J}} a_{j,j} \prod_{k\in\mathcal{J}^c} b_{k,m+1}.$$

We set

$$p_{\mathcal{J}}(z;\bar{w}) := a_{m+1,m+1}^{|\mathcal{J}|} \prod_{j\in\mathcal{J}} a_{j,j} \prod_{k\in\mathcal{J}^c} b_{k,m+1}.$$

Then

$$\prod_{j=1}^{m} a_{j,m+1}a_{m+1,j} = \prod_{j=1}^{m}(1 - z_j\bar{w}_{m+1})(1 - z_{m+1}\bar{w}_j) = \sum_{\mathcal{J}\in\mathcal{R}_m} p_{\mathcal{J}}(z;\bar{w}).$$

Let K_m and K_{m+1} be the kernel function of P_m and P_{m+1}, respectively, as in (4.7). Let $K_{m,\mathcal{J}}$ denote the kernel function

$$K_{m,\mathcal{J}}(z;\bar{w}) := \frac{p_{\mathcal{J}}(z;\bar{w}) \prod_{j<k\le m} a_{j,k}a_{k,j} \prod_{j<k,m+1<k} b_{j,k}}{\pi^n \prod_{j\le k} a_{j,k}a_{k,j}}.$$

We can express K_m and K_{m+1} in terms of $K_{m,\mathcal{J}}(z;\bar{w})$:

$$\begin{aligned}
K_m(z;\bar{w}) &= \frac{\prod_{j<k\le m}(1 - z_j\bar{w}_k)(1 - z_k\bar{w}_j) \prod_{j<k,m<k}(z_j - z_k)(\bar{w}_j - \bar{w}_k)}{\pi^n \prod_{j\le k}(1 - z_k\bar{w}_j)(1 - z_j\bar{w}_k)}\\
&= \frac{p_{\varnothing}(z;\bar{w}) \prod_{j<k\le m} a_{j,k}a_{k,j} \prod_{j<k,m+1<k} b_{j,k}}{\pi^n \prod_{j\le k} a_{j,k}a_{k,j}}\\
&= K_{m,\varnothing}(z;\bar{w}),
\end{aligned}$$

and

$$\begin{aligned}
K_{m+1}(z;\bar{w}) &= \frac{\prod_{j<k\le m+1}(1 - z_j\bar{w}_k)(1 - z_k\bar{w}_j) \prod_{j<k,m+1<k}(z_j - z_k)(\bar{w}_j - \bar{w}_k)}{\pi^n \prod_{j\le k}(1 - z_k\bar{w}_j)(1 - z_j\bar{w}_k)}\\
&= \frac{\sum_{\mathcal{J}\in\mathcal{R}_m} p_{\mathcal{J}}(z;\bar{w}) \prod_{j<k\le m}(1 - z_j\bar{w}_k)(1 - z_k\bar{w}_j) \prod_{j<k,m+1<k}(z_j - z_k)(\bar{w}_j - \bar{w}_k)}{\pi^n \prod_{j\le k}(1 - z_k\bar{w}_j)(1 - z_j\bar{w}_k)}\\
&= \sum_{\mathcal{J}\in\mathcal{R}_m} K_{m,\mathcal{J}}(z;\bar{w}) = K_m(z;\bar{w}) + \sum_{\varnothing\ne\mathcal{J}\in\mathcal{R}_m} K_{m,\mathcal{J}}(z;\bar{w}).
\end{aligned}$$

We show that for any nonempty $\mathcal{J}\in\mathcal{R}_m$, $K_{m,\mathcal{J}}$ is a linear combination of (j,k)-symmetric kernel functions. Then for antisymmetric $f \in L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$,

$$\begin{aligned}
P_{m+1}(f)(z) &= \int_{\mathbb{D}^n} K_{m+1}(z;\bar{w})f(w)dV(w)\\
&= \int_{\mathbb{D}^n} \sum_{\mathcal{J}\in\mathcal{R}_m} K_{m,\mathcal{J}}(z;\bar{w})f(w)dV(w)\\
&= \int_{\mathbb{D}^n} K_m(z;\bar{w})f(w)dV(w)\\
&= P_m(f)(z),
\end{aligned}$$

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which completes the induction and the proof of the lemma. When $|\mathcal{J}| > 1$, there exists $j_1, j_2 \in \mathcal{J}$, and

$$K_{m,\mathcal{J}}(z;\bar{w}) = \frac{p_{\mathcal{J}}(z;\bar{w})\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$
$$= \frac{a_{m+1,m+1}^{|\mathcal{J}|}\prod_{k\in\mathcal{J}} a_{k,k}\prod_{j\in\mathcal{J}^c} b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}.$$

It's easy to see that $K_{m,\mathcal{J}}(z;\bar{w})$ is (j_1, j_2)-symmetric.

Now we turn to consider the case when $\mathcal{J} = \{j_0\}$. Without loss of generality, we let $j_0 = 1$.

$$K_{m,\{1\}}(z;\bar{w})$$
$$= \frac{p_{\{1\}}(z;\bar{w})\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$
$$= \frac{a_{m+1,m+1}a_{1,1}\prod_{k=2}^m b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$
$$= \frac{a_{m+1,m+1}a_{1,1}(a_{2,m+1}a_{m+1,2} - a_{2,2}a_{m+1,m+1})\prod_{k=3}^m b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$
$$= \frac{a_{m+1,m+1}a_{1,1}a_{2,m+1}a_{m+1,2}\prod_{k=3}^m b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}} - K_{m,\{1,2\}}(z;\bar{w}),$$

where $K_{m,\{1,2\}}(z;\bar{w})$ is $(1,2)$-symmetric in w.

Since $b_{3,m+1} = a_{3,m+1}a_{m+1,3} - a_{3,3}a_{m+1,m+1}$, we have

$$\frac{a_{m+1,m+1}a_{1,1}a_{2,m+1}a_{m+1,2}\prod_{k=3}^m b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$
$$= \frac{a_{m+1,m+1}a_{1,1}a_{2,m+1}a_{m+1,2}a_{3,m+1}a_{m+1,3}\prod_{k=4}^m b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$
$$- \frac{a_{m+1,m+1}^2a_{1,1}a_{2,m+1}a_{m+1,2}a_{3,3}\prod_{k=4}^m b_{j,m+1}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}},$$

where the negative term above is $(1,3)$-symmetric in w. Repeating the above process using the identity $b_{j,m+1} = a_{j,m+1}a_{m+1,j} - a_{j,j}a_{m+1,m+1}$ until no $b_{j,m+1}$ term left, we obtain

$$K_{m,\{1\}}(z;\bar{w}) - \frac{a_{m+1,m+1}a_{1,1}\prod_{k=2}^m a_{k,m+1}a_{m+1,k}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$

is a linear combination of functions that are $(1, j)$-symmetric in w. Since the function

$$\frac{a_{m+1,m+1}a_{1,1}\prod_{k=2}^m a_{k,m+1}a_{m+1,k}\prod_{j<k\le m} a_{j,k}a_{k,j}\prod_{j<k,m+1<k} b_{j,k}}{\pi^n\prod_{j\le k} a_{j,k}a_{k,j}}$$

is $(1, m+1)$-symmetric in w, we are done. ∎

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Since $T_2^n = P_{\mathbb{D}^n}$ on $L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p}|)$, the next theorem implies Theorem 2.4 for general n.

Theorem 4.4 *T_2^n is unbounded on $L^p_{anti}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})$ for $p = \frac{2n}{n-1}$.*

Proof The proof for the case $n > 2$ follows from a similar argument as in the proof of Theorem 4.2. Let $\tilde{T}^n$ denote the operator given as follows:

$$\tilde{T}^n(h)(z) := (J_{\mathbb{C}}\Phi_n(z))^{-1} T_2^n(h\bar{J}_{\mathbb{C}}\Phi_n)(z).$$

Then

$$\tilde{T}^n(h)(z) = \int_{\mathbb{D}^n} \frac{\prod_{j<k}(\bar{w}_j - \bar{w}_k)^2 h(w)dV}{\pi^n \prod_{j\le k}(1 - z_k\bar{w}_j)(1 - z_j\bar{w}_k)}, \tag{4.8}$$

and $\|T_2^n\|_{L^p_{\text{anti}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^{2-p})} = \|\tilde{T}^n\|_{L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^2)}$ provided one of the norms is finite. Thus it suffices to show that $\tilde{T}^n$ is unbounded on $L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^2)$ for $p = \frac{2n}{n-1}$.

Recall that $\mathcal{S}_n$ is the set of all permutations of $\{z_1, \dots, z_n\}$. For $s \in (0,1)$, we set

$$h_s(z) = \sum_{\tau\in\mathcal{S}_n} \frac{1}{\prod_{j=1}^{n-1}(1 - \tau(z_j)s)^n}.$$

Then h_s is a symmetric function with

$$\begin{aligned}
\|h_s\|^p_{L^p_{\text{sym}}(\mathbb{D}^n, |J_{\mathbb{C}}\Phi_n|^2)} &= \int_{\mathbb{D}^n} \left| \sum_{\tau\in\mathcal{S}_n} \frac{1}{\pi^{n-1}\prod_{l=1}^{n-1}(1 - \tau(w_l)s)^n} \right|^p \prod_{1\le j<k\le n} |w_j - w_k|^2 dV(w) \\
&\lesssim \int_{\mathbb{D}^n} \frac{\prod_{1\le j<k\le n}|w_j - w_k|^2}{\prod_{l=1}^{n-1}|1 - w_l s|^{np}} dV(w) \\
&\lesssim \int_{\mathbb{D}^{n-1}} \frac{\prod_{1\le j<k\le n-1}|w_j - w_k|^2}{\prod_{l=1}^{n-1}|1 - w_l s|^{np}} dV(w_1, \dots, w_{n-1}) \\
&\lesssim \int_{\mathbb{D}^{n-1}} \frac{\prod_{1\le j<k\le n-1}|w_j - w_k|^2}{\prod_{l=1}^{n-1}|1 - w_l s|^{2n-4}} \frac{dV(w_1, \dots, w_{n-1})}{\prod_{l=1}^{n-1}|1 - w_l s|^{np+4-2n}}.
\end{aligned} \tag{4.9}$$

To evaluate the integral above, we need an $(n-2)$-step procedure to eliminate the numerator of the integrand, i.e., we rewrite

$$\frac{\prod_{1\le j<k\le n-1}(w_j - w_k)}{\prod_{l=1}^{n-1}(1 - w_l s)^{n-2}}.$$

Step 1. Recall that by partial fractions:

$$\frac{1}{\prod_{j=1}^{n-1}(1 - w_j s)} = \sum_{j=1}^{n-1} \frac{c_j}{(1 - w_j s)},$$

where $c_j = \frac{1}{s^{n-2}\prod_{k=1,k\ne j}^{n-1}(w_j - w_k)}$. Then

$$\frac{\prod_{1\le j<k\le n-1}(w_j - w_k)}{\prod_{l=1}^{n-1}(1 - w_l s)^{n-2}} = \sum_{j_1=1}^{n-1} \frac{\prod_{1\le j<k\le n-1}(w_j - w_k)}{s^{n-2}(1 - w_{j_1}s)\prod_{l=1}^{n-1}(1 - w_l s)^{n-3}\prod_{k=1,k\ne j_1}^{n-1}(w_{j_1} - w_k)}.$$

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Step 2. Now we focus on the j_1th term in the sum above:

$$\frac{\prod_{1\le j<k\le n-1}(w_j-w_k)}{s^{n-2}(1-w_{j_1}s)\prod_{l=1}^{n-1}(1-w_l s)^{n-3}\prod_{k=1,k\neq j_1}^{n-1}(w_{j_1}-w_k)}.$$

Applying the partial fractions yields

$$\frac{1}{\prod_{j=1,j\neq j_1}^{n-1}(1-w_j s)}=\sum_{j=1,j\neq j_1}^{n}\frac{1}{s^{n-3}(1-w_j s)\prod_{k=1,k\neq j_1}^{n-1}(w_{j_1}-w_k)},$$

and

$$\frac{\prod_{1\le j<k\le n-1}(w_j-w_k)}{s^{n-2}(1-w_{j_1}s)\prod_{l=1}^{n-1}(1-w_l s)^{n-3}\prod_{k=1,k\neq j_1}^{n-1}(w_{j_1}-w_k)}=\sum_{j_1=1}^{n-1}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n-1}$$
$$\times\frac{\prod_{1\le j<k\le n-1}(w_j-w_k)}{s^{2n-5}(1-w_{j_1}s)^2(1-w_{j_2}s)\prod_{j=1}^{n-1}(1-w_j s)^{n-4}\prod_{k=1,k\neq j_1}^{n-1}(w_{j_1}-w_k)\prod_{k=1,k\neq j_1,j_2}^{n-1}(w_{j_2}-w_k)}.$$

Step 3. As in Step 2, we turn to the term with sub-indices (j_1, j_2) in the sum above and continue the process by doing partial fractions to

$$\frac{1}{\prod_{j=1,j\notin\{j_1,j_2\}}^{n-1}(1-w_j s)}.$$

Repeat this process. Then after $n-2$ steps, we obtain

$$\begin{aligned}\frac{\prod_{1\le j<k\le n-1}(w_j-w_k)}{\prod_{l=1}^{n-1}(1-w_l s)^{n-2}}&=\sum_{(l_1,l_2,\ldots,l_{n-1})\in\mathcal{S}_{n-1}}\frac{s^{-\frac{1}{2}(n-1)(n-2)}\prod_{1\le j<k\le n-1}(w_j-w_k)}{\prod_{1\le j<k\le n-1}(w_{l_j}-w_{l_k})\prod_{t=1}^{n-1}(1-w_{l_t}s)^{n-1-t}}\\ &=\sum_{(l_1,l_2,\ldots,l_{n-1})\in\mathcal{S}_{n-1}}\frac{\operatorname{sgn}((l_1,\ldots,l_{n-1}))s^{-\frac{1}{2}(n-1)(n-2)}}{\prod_{t=1}^{n-1}(1-w_{l_t}s)^{n-1-t}}.\end{aligned}\tag{4.10}$$

Here, $\operatorname{sgn}((l_1,\ldots,l_{n-1}))$ is the sign of the permutation $(l_1,\ldots,l_{n-1})$.

Applying this identity to (4.9) and using the triangle inequality, we obtain

$$\begin{aligned}&\|h_s\|^p_{L^p_{\mathrm{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}\\ &\lesssim\int_{\mathbb{D}^{n-1}}\frac{\prod_{1\le j<k\le n-1}|w_j-w_k|^2}{\prod_{l=1}^{n-1}|1-w_l s|^{2n-4}}\frac{1}{\prod_{l=1}^{n-1}|1-w_l s|^{np+4-2n}}dV(w_1,\ldots,w_{n-1})\\ &\lesssim\sum_{(l_1,l_2,\ldots,l_{n-1})\in\mathcal{S}_{n-1}}\int_{\mathbb{D}^{n-1}}\frac{s^{-(n-1)(n-2)}}{\prod_{t=1}^{n-1}|1-w_{l_t}s|^{2n-2-2t}}\cdot\frac{1}{\prod_{l=1}^{n-1}|1-w_l s|^{np+4-2n}}dV(w_1,\ldots,w_{n-1})\\ &\lesssim\int_{\mathbb{D}^{n-1}}\frac{1}{\prod_{l=1}^{n-1}|1-w_l s|^{np+2-2l}}dV(w_1,\ldots,w_{n-1}).\end{aligned}\tag{4.11}$$

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For $p = \frac{2n}{n-1}$, $np+2-2l \geq np+2-2(n-1) > 2$. Thus the Forelli–Rudin estimates (2.8) imply

$$\begin{aligned}
&\int_{\mathbb{D}^{n-1}} \frac{1}{\prod_{l=1}^{n-1} |1-w_l s|^{np+2-2l}} dV(w_1,\dots,w_{n-1}) \\
&= \prod_{l=1}^{n-1} \int_{\mathbb{D}} \frac{1}{|1-w_l s|^{np+2-2l}} dV(w_1,\dots,w_{n-1}) \\
&\approx \prod_{l=1}^{n-1} (1-s)^{-np+2l} = (1-s)^{-n^2-n}. \qquad (4.12)
\end{aligned}$$

Hence $\|h_s\|^p_{L^p_{\text{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)} \lesssim (1-s)^{-n^2-n}$.

Now we turn to compute $\tilde{T}^n(h_s)$. Let I denote the identity operator. For the variable w_j, let $\mathcal{D}_{w_j}$ denote the partial differential operator

$$\mathcal{D}_{w_j} = I + w_j \frac{\partial}{\partial w_j}.$$

For any $k \in \mathbb{N}$ and holomorphic function $f(w) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha w^\alpha$ on $\mathbb{D}^n$,

$$(\mathcal{D}_{w_j})^k f(w) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha (\alpha_j+1)^k w^\alpha.$$

For each integer $k > 2$,

$$\frac{1}{(1-w_j s)^k} = \sum_{m=0}^{\infty} (m+1)_{k-1} s^m w_j^m = \sum_{m=0}^{\infty} (m+2)_{k-2}((m+1)s^m w_j^m),$$

where the Pochhammer symbol $(m+2)_{k-2} = (m+2)\dot{(}m+3)\dots(m+k-1)$ is a polynomial in m of degree $k-2$. Thus, there exists a polynomial q_{k-2} of degree $k-2$ such that

$$\frac{1}{(1-w_j s)^k} = q_{k-2}(\mathcal{D}_{w_j})\left(\frac{1}{\pi(1-w_j s)^2}\right).$$

For holomorphic functions f, g on $\mathbb{D}^n$ with $f(w) = \sum_\alpha c_\alpha w^\alpha$ and $g(w) = \sum_\alpha d_\alpha w^\alpha$,

$$\begin{aligned}
\int_{\mathbb{D}^n} f \overline{\prod_{j=1}^n q_{k-2}(\mathcal{D}_{w_j})(g)} dV &= \int_{\mathbb{D}^n} \left(\sum_\alpha c_\alpha w^\alpha\right)\left(\sum_\alpha d_\alpha \prod_{j=1}^n q_{k-2}(\alpha_j+1)\bar{w}^\alpha\right) dV(w) \\
&= \sum_\alpha c_\alpha d_\alpha \prod_{j=1}^n q_{k-2}(\alpha_j+1) \int_{\mathbb{D}^n} |w|^{2\alpha} dV(w) \\
&= \int_{\mathbb{D}^n} \left(\sum_\alpha c_\alpha \prod_{j=1}^n q_{k-2}(\alpha_j+1) w^\alpha\right)\left(\sum_\alpha d_\alpha \bar{w}^\alpha\right) dV(w) \\
&= \int_{\mathbb{D}^n} \prod_{j=1}^n q_{k-2}(\mathcal{D}_{w_j})(f)(w)\bar{g}(w) dV(w). \qquad (4.13)
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\tilde{T}^n(h_s)(z)\\
&= \int_{\mathbb{D}^n} \frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^n\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)} \sum_{\tau\in\mathcal{S}_n}\frac{1}{\prod_{j=1}^{n-1}(1-\tau(w_j)s)^n}dV(w)\\
&= \int_{\mathbb{D}^n} \frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^n\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)} \sum_{\tau\in\mathcal{S}_n}\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\tau(w_j)})\\
&\quad\times\left(\frac{1}{\pi(1-\tau(w_j)s)^2}\right)dV(w)\\
&= \int_{\mathbb{D}^n}\sum_{\tau\in\mathcal{S}_n}\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\tau(\bar{w}_j)})\left(\frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^n\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)}\right)\\
&\quad\times\left(\frac{1}{\pi^{n-1}\prod_{j=1}^{n-1}(1-\tau(\bar{w}_j)s)^2}\right)dV(w)\\
&= \int_{\mathbb{D}^n}\sum_{\tau\in\mathcal{S}_n}\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\tau(\bar{w}_j)})\left(\frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^n\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)}\right)\\
&\quad\times \pi K_{\mathbb{D}^n}(w;\tau(s,\dots,s,0))dV(w)\\
&= \sum_{\tau\in\mathcal{S}_n}\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\tau(\bar{w}_j)})\left(\frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^{n-1}\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)}\right)\Bigg|_{\bar{w}=\tau(s,\dots,s,0)}.
\end{aligned}
$$

We claim that there is a constant c_n such that

$$
\begin{aligned}
&\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\tau(\bar{w}_j)})\left(\frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^{n-1}\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)}\right)\Bigg|_{\bar{w}=\tau(s,\dots,s,0)}\\
&\qquad= \frac{c_n s^{n(n-1)}}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)\prod_{l=1}^n(1-z_l s)^{n-1}}.
\end{aligned}
\tag{4.14}
$$

By symmetry, it suffices to show (4.14) for the case when τ is the identity map, i.e.,

$$
\begin{aligned}
&\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\bar{w}_j})\left(\frac{\prod_{1\le j<k\le n}(\bar{w}_j-\bar{w}_k)^2}{\pi^{n-1}\prod_{m=1}^n(1-z_m\bar{w}_m)\prod_{j,k=1}^n(1-z_j\bar{w}_k)}\right)\Bigg|_{\bar{w}=(s,\dots,s,0)}\\
&\qquad= \frac{c_n s^{n(n-1)}}{\prod_{m=1}^{n-1}(1-z_m s)\prod_{l=1}^n(1-z_l s)^{n-1}}.
\end{aligned}
\tag{4.15}
$$

Set $\bar{\partial}_j = \frac{\partial}{\partial \bar{w}_j}$. For a multi-index $\mathbf{l}=(l_1,\dots,l_n)$, set $\bar{\partial}^{\mathbf{l}} = \bar{\partial}_1^{l_1}\dots\bar{\partial}_n^{l_n}$. Then by the product rule, $\mathcal{D}_{\bar{w}_j}^k = \sum_{l=0}^k c_{k,l}\bar{w}_j^l\bar{\partial}_j^l$. Therefore

$$
\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\bar{w}_j}) = \prod_{j=1}^{n-1}\left(\sum_{l_j=0}^{n-2} d_{l_j}\bar{w}_j^{l_j}\bar{\partial}_j^{l_j}\right) = \sum_{\mathbf{l}\in\{0,1,\dots,n-2\}^{n-1}} d_{l_1}\dots d_{l_{n-1}}\bar{w}^{\mathbf{l}}\bar{\partial}^{\mathbf{l}},
$$

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for some constants d_{l_j}. Note that for $\mathbf{l} = (l_1, \ldots, l_{n-1}) \in \{0, 1, \ldots, n-2\}^{n-1}$,

$$\bar{\partial}^{\mathbf{l}} \left(\frac{\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2}{\pi^{n-1} \prod_{m=1}^{n} (1 - z_m \bar{w}_m) \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)} \right)$$

can be expressed as a linear combination of terms of the form

$$\bar{\partial}^{\mathbf{m}} \left(\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2 \right) \bar{\partial}^{\mathbf{l}-\mathbf{m}} \left(\frac{1}{\pi^{n-1} \prod_{m=1}^{n} (1 - z_m \bar{w}_m) \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)} \right),$$

where $\mathbf{m} = (m_1, \ldots, m_{n-1})$ with $m_j \le l_j$ for all j and $\mathbf{l} - \mathbf{m} = (l_1 - m_1, \ldots, l_{n-1} - m_{n-1})$.

Since $l_j \le n - 2$ for each j, the sum

$$|\mathbf{m}| = \sum m_j \le \sum l_j \le (n-1)(n-2).$$

Thus, the polynomial $\bar{\partial}^{\mathbf{m}} \left(\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2 \right)$ is of total degree $n(n-1) - |\mathbf{m}|$ which is at least $n(n-1) - (n-1)(n-2) = 2(n-1)$. Note also that for $\bar{w} = (s, \ldots, s, 0)$, the factor $(\bar{w}_j - \bar{w}_k) \ne 0$ if and only if either j or k equals n. It is not hard to see that the polynomial $\prod_{k=1}^{n-1} (\bar{w}_k - \bar{w}_n)^2$ is the only divisor of $\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2$ that has degree at least $2(n-1)$ and does not vanish at $(s, \ldots, s, 0)$. Hence,

$$\bar{\partial}^{\mathbf{m}} \left(\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2 \right) \Bigg|_{(s, \ldots, s, 0)} \ne 0$$

if and only if $|\mathbf{m}| = (n-2)(n-1)$, i.e., $\mathbf{m} = (n-2, \ldots, n-2)$. In this case, we have

$$\prod_{j=1}^{n-1} \bar{\partial}_j^{n-2} \left(\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2 \right) \Bigg|_{(s, \ldots, s, 0)} = c_n \prod_{k=1}^{n-1} (\bar{w}_k - \bar{w}_n)^2 |_{(s, \ldots, s, 0)} = c_n s^{2n-2}$$

for some constant c_n. Therefore,

$$\begin{aligned}
&\prod_{j=1}^{n-1} q_{n-2}(\mathcal{D}_{\bar{w}_j}) \left(\frac{\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2}{\pi^{n-1} \prod_{m=1}^{n} (1 - z_m \bar{w}_m) \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)} \right) \Bigg|_{\bar{w} = (s, \ldots, s, 0)} \\
&= \sum_{\mathbf{l} \in \{0, 1, \ldots, n-2\}^{n-1}} d_{l_1} \ldots d_{l_{n-1}} \bar{w}^{\mathbf{l}} \bar{\partial}^{\mathbf{l}} \left(\frac{\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2}{\pi^{n-1} \prod_{m=1}^{n} (1 - z_m \bar{w}_m) \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)} \right) \Bigg|_{\bar{w} = (s, \ldots, s, 0)} \\
&= \left(\frac{d_{n-2}^{n-1} \prod_{j=1}^{n-1} \left(\bar{w}_j^{n-2} \bar{\partial}_j^{n-2} \right) \left(\prod_{1 \le j < k \le n} (\bar{w}_j - \bar{w}_k)^2 \right)}{\pi^{n-1} \prod_{m=1}^{n} (1 - z_m \bar{w}_m) \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)} \right) \Bigg|_{\bar{w} = (s, \ldots, s, 0)} \\
&= \frac{d_{n-2}^{n-1} c_n s^{n(n-1)}}{\prod_{m=1}^{n-1} (1 - z_m s) \prod_{l=1}^{n} (1 - z_l s)^{n-1}},
\end{aligned} \tag{4.16}$$

which proves the claim (4.15) and gives

$$\tilde{T}^n(h_s)(z) = \sum_{\tau \in \mathcal{S}_n} \frac{d_{n-2}^{n-1} c_n s^{n(n-1)}}{\prod_{m=1}^{n-1} (1 - \tau(z_m) s) \prod_{l=1}^{n} (1 - z_l s)^{n-1}}. \tag{4.17}$$

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We next compute the norm of $\tilde{T}^n(h_s)$

$$
\begin{aligned}
&\|\tilde{T}^n(h_s)(z)\|^p_{L^p_{\mathrm{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}\\
&= \int_{\mathbb{D}^n} \left| \sum_{\tau\in\mathcal{S}_n} \frac{d^{n-1}_{n-2} c_n s^{n(n-1)}}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)\prod_{l=1}^{n}(1-z_l s)^{n-1}} \right|^p \prod_{1\le j<k\le n} |z_j - z_k|^2 dV(z)\\
&= \int_{\mathbb{D}^n} \frac{d^{p(n-1)}_{n-2} c^p_n s^{pn(n-1)}}{\prod_{l=1}^{n}|1-z_l s|^{p(n-1)}} \left| \sum_{\tau\in\mathcal{S}_n} \frac{1}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)} \right|^p \prod_{1\le j<k\le n} |z_j - z_k|^2 dV(z).
\end{aligned}
\tag{4.18}
$$

Set

$$
U_n(s) = \left\{ w\in\mathbb{D} : \mathrm{Arg}(1-ws) \in \left(-\frac{\pi}{6(n-1)}, \frac{\pi}{6(n-1)} \right) \right\}.
$$

Then for any $z=(z_1,\dots,z_n)\in(U_n(s))^n$ and $\tau\in\mathcal{S}_n$,

$$
\mathrm{Arg}\left\{ \frac{1}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)} \right\} \in \left(-\frac{\pi}{6}, \frac{\pi}{6} \right),
$$

which yields that

$$
\left| \sum_{\tau\in\mathcal{S}_n} \frac{1}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)} \right| \gtrsim \frac{1}{\prod_{m=1}^{n-1}|1-z_m s|}.
$$

Using this inequality, we have

$$
\begin{aligned}
&\|\tilde{T}^n(h_s)(z)\|^p_{L^p_{\mathrm{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}\\
&= \int_{\mathbb{D}^n} \frac{d^{p(n-1)}_{n-2} c^p_n s^{pn(n-1)}}{\prod_{l=1}^{n}|1-z_l s|^{p(n-1)}} \left| \sum_{\tau\in\mathcal{S}_n} \frac{1}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)} \right|^p \prod_{1\le j<k\le n} |z_j - z_k|^2 dV(z)\\
&\gtrsim \int_{(U_n(s))^n} \frac{1}{\prod_{l=1}^{n}|1-z_l s|^{p(n-1)}} \left| \sum_{\tau\in\mathcal{S}_n} \frac{1}{\prod_{m=1}^{n-1}(1-\tau(z_m)s)} \right|^p \prod_{1\le j<k\le n} |z_j - z_k|^2 dV(z)\\
&\gtrsim \int_{(U_n(s))^n} \frac{\prod_{1\le j<k\le n}|z_j-z_k|^2}{\prod_{m=1}^{n-1}|1-z_m s|^p \prod_{l=1}^{n}|1-z_l s|^{p(n-1)}} dV(z)\\
&= \int_{(U_n(s))^n} \frac{\prod_{1\le j<k\le n}|z_j-z_k|^2}{\prod_{l=1}^{n}|1-z_l s|^{2(n-1)}} \cdot \frac{1}{\prod_{m=1}^{n-1}|1-z_m s|^p \prod_{l=1}^{n}|1-z_l s|^{p(n-1)-2(n-1)}} dV(z).
\end{aligned}
\tag{4.19}
$$

By a similar $(n-1)$-step partial fraction procedure, we obtain the following analogue of (4.10)

$$
\frac{\prod_{1\le j<k\le n}(z_j-z_k)}{\prod_{l=1}^{n}(1-z_l s)^{n-1}} = \sum_{(l_1,\dots,l_n)\in\mathcal{S}_n} \frac{\mathrm{sgn}((l_1,\dots,l_n)) s^{-\frac{1}{2}n(n-1)}}{\prod_{t=1}^{n}(1-z_{l_t}s)^{n-t}}.
$$

Hence (4.19) becomes

$$\|\tilde{T}^n(h_s)(z)\|^p_{L^p_{\mathrm{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}$$
$$\gtrsim \int_{(U_n(s))^n} \frac{\prod_{1\le j<k\le n}|z_j-z_k|^2}{\prod_{l=1}^n|1-z_ls|^{2(n-1)}}\cdot\frac{dV(z)}{\prod_{m=1}^{n-1}|1-z_ms|^p\prod_{l=1}^n|1-z_ls|^{p(n-1)-2(n-1)}}$$
$$\gtrsim \int_{(U_n(s))^n}\left|\sum_{(l_1,\dots,l_n)\in\mathcal{S}_n}\frac{\mathrm{sgn}((l_1,\dots,l_n))}{\prod_{t=1}^n(1-z_{l_t}s)^{n-t}}\right|^2\frac{dV(z)}{\prod_{m=1}^{n-1}|1-z_ms|^p\prod_{l=1}^n|1-z_ls|^{p(n-1)-2(n-1)}}. \tag{4.20}$$

We further restrict our region of integration to obtain more precise estimates. For $j\in\{1,\dots,n\}$ and $s\in(1-(5n!)^{-2n},1)$, we set

$$U_n(s,j)=U_n(s)\bigcap\left\{z:(5n!)^{2j}(1-s)<\left|z-\frac{1}{s}\right|<1\right\},$$

and set $\mathbf{U}(s)=U_n(s,1)\times U_n(s,2)\times\cdots\times U_n(s,n)$. It is worth noting that we implement a positive lower bound $1-(5n!)^{-2n}$ for s here so that $U_n(s,j)$ is nonempty and $\mathbf{U}(s)$ is asymmetric in its components. As the reader will see, we need this extra restriction for s but not when $n=2$ since in higher dimensions, the desired integral estimates cannot be achieved solely by Forelli–Rudin estimate. The asymmetry of $\mathbf{U}(s)$ is also used.

By a polar coordinate computation,

$$\begin{aligned}\int_{U_n(s,j)}\frac{dV(z)}{|1-zs|^k}&=s^{-k}\int_{U_n(s,j)}\frac{dV(z)}{|z-s^{-1}|^k}\\&=s^{-k}\int_{-\frac{\pi}{6(n-1)}}^{\frac{\pi}{6(n-1)}}\int_{(5n!)^{2j}(1-s)}^{1}r^{1-k}\,dr\,d\theta\\&=\begin{cases}\frac{\pi}{3s^k(k-2)(n-1)}\left((5n!)^{2j(2-k)}(1-s)^{2-k}-1\right) & k>2\\ -\frac{\pi}{3s^2(n-1)}\left(2j\log 5n!+\log(1-s)\right) & k=2\end{cases}.\end{aligned}\tag{4.21}$$

For functions $f(s)$ and $g(s)$, we write $f(s)\sim g(s)$ if

$$\lim_{s\to1^-}\frac{f(s)}{g(s)}=1.$$

Then, (4.21) yields

$$\int_{U_n(s,j)}\frac{dV(z)}{|1-zs|^k}\sim\begin{cases}\frac{\pi(5n!)^{2j(2-k)}(1-s)^{2-k}}{3s^k(k-2)(n-1)} & k>2\\ -\frac{\pi\log(1-s)}{3s^2(n-1)} & k=2\end{cases}.\tag{4.22}$$

Recall that for $\tau\in\mathcal{S}_n$, we let $\tau(j)$ be the index satisfying $z_{\tau(j)}=\tau(z_j)$. For $p=\frac{2n}{n-1}$, the triangle inequality and Cauchy–Schwarz inequality implies

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$$\int_{(U_n(s))^n}\left|\sum_{\tau\in\mathfrak{S}_n}\frac{\operatorname{sgn}((l_1,\dots,l_n))}{\prod_{t=1}^n(1-z_{\tau(t)}s)^{n-t}}\right|^2\frac{1}{\prod_{m=1}^{n-1}|1-z_ms|^p\prod_{l=1}^n|1-z_ls|^{p(n-1)-2(n-1)}}dV(z)$$
$$\gtrsim\int_{\mathbf{U}(s)}\left|\sum_{\tau\in\mathfrak{S}_n}\frac{\operatorname{sgn}((l_1,\dots,l_n))}{\prod_{t=1}^n(1-z_{\tau(t)}s)^{n-t}}\right|^2\frac{1}{\prod_{m=1}^{n-1}|1-z_ms|^p\prod_{l=1}^n|1-z_ls|^{p(n-1)-2(n-1)}}dV(z)$$
$$\gtrsim\int_{\mathbf{U}(s)}\left(\frac{1}{\prod_{t=1}^n|1-z_ts|^{2n-2t}}-\left|\sum_{\substack{\tau\in\mathfrak{S}_n\\ \tau\neq I}}\frac{\operatorname{sgn}((l_1,\dots,l_n))}{\prod_{t=1}^n(1-z_{\tau(t)}s)^{n-t}}\right|^2\right)$$
$$\times\frac{dV(z)}{\prod_{m=1}^{n-1}|1-z_ms|^{\frac{2n}{n-1}}\prod_{l=1}^n|1-z_ls|^2}$$
$$\gtrsim\int_{\mathbf{U}(s)}\left(\frac{1}{\prod_{t=1}^n|1-z_ts|^{2n-2t}}-\sum_{\substack{\tau\in\mathfrak{S}_n\\ \tau\neq I}}\frac{n!}{\prod_{t=1}^n|1-z_{\tau(t)}s|^{2n-2t}}\right)$$
$$\times\frac{dV(z)}{\prod_{m=1}^{n-1}|1-z_ms|^{\frac{2n}{n-1}}\prod_{l=1}^n|1-z_ls|^2}$$
$$=\int_{\mathbf{U}(s)}\left(\frac{1}{\prod_{t=1}^n|1-z_ts|^{2n-2t}}-\sum_{\substack{\tau\in\mathfrak{S}_n\\ \tau\neq I}}\frac{n!}{\prod_{t=1}^n|1-z_ts|^{2n-2\tau^{-1}(t)}}\right)$$
$$\times\frac{dV(z)}{\prod_{m=1}^{n-1}|1-z_ms|^{\frac{2n}{n-1}}\prod_{l=1}^n|1-z_ls|^2}.$$

We claim that in the integral above, the first term will dominate the rest terms, and thus determines the size of the entire integral. We start by showing that the first term dominates the sum of those terms with $\tau^{-1}(n)\neq n$. Note that

$$\int_{\mathbf{U}(s)}\frac{dV(z)}{\prod_{t=1}^n|1-z_ts|^{2n-2\tau^{-1}(t)}\prod_{m=1}^{n-1}|1-z_ms|^{\frac{2n}{n-1}}\prod_{l=1}^n|1-z_ls|^2}$$
$$=\int_{\mathbf{U}(s)}\frac{dV(z)}{|1-z_ns|^{2n+2-2\tau^{-1}(n)}\prod_{m=1}^{n-1}|1-z_ms|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}}$$
$$=\int_{U_n(s,n)}\frac{dV(z_n)}{|1-z_ns|^{2n+2-2\tau^{-1}(n)}}\prod_{m=1}^{n-1}\int_{U_n(s,m)}\frac{dV(z_m)}{|1-z_ms|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}}. \tag{4.23}$$

Since $1\le m\le n-1$, the denominator factor $|1-z_ms|$ in (4.23) has power strictly greater than 2. The factor $|1-z_ns|$ has power 2 only if $\tau^{-1}(n)=n$, or equivalently $\tau(z_n)=z_n$. By the Forelli–Rudin estimates (2.8) and the fact that $\{\tau^{-1}(1),\dots,\tau^{-1}(n)\}=\{1,\dots,n\}$,

$$\int_{U_n(s,n)}\frac{dV(z_n)}{|1-z_ns|^{2n+2-2\tau^{-1}(n)}}\prod_{m=1}^{n-1}\int_{U_n(s,m)}\frac{dV(z_m)}{|1-z_ms|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}}$$
$$\approx\begin{cases}(1-s)^{-n^2-n} & \tau(n)\neq n\\ -\frac{\log(1-s)}{(1-s)^{n^2+n}} & \tau(n)=n\end{cases}. \tag{4.24}$$

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Thus, for s sufficiently close to 1, the integral in (4.23) with $\tau(n) = n$ dominates the ones with $\tau(n) \neq n$. Therefore, we can further assume that

$$\int_{\mathbf{U}(s)} \left(\frac{1}{2} \frac{1}{\prod_{t=1}^{n} |1 - z_t s|^{2n-2t}} - \sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(n) \neq n}} \frac{n!}{\prod_{t=1}^{n} |1 - z_t s|^{2n-2\tau^{-1}(t)}} \right) \times \frac{dV(z)}{\prod_{m=1}^{n-1} |1 - z_m s|^{\frac{2n}{n-1}} \prod_{l=1}^{n} |1 - z_l s|^2} \geq 0,$$

which implies

$$\begin{aligned} &\int_{\mathbf{U}(s)} \left(\frac{1}{\prod_{t=1}^{n} |1 - z_t s|^{2n-2t}} - \sum_{\substack{\tau \in \mathcal{S}_n \\ \tau \neq I}} \frac{n!}{\prod_{t=1}^{n} |1 - z_t s|^{2n-2\tau^{-1}(t)}} \right) \frac{dV(z)}{\prod_{m=1}^{n-1} |1 - z_m s|^{\frac{2n}{n-1}} \prod_{l=1}^{n} |1 - z_l s|^2} \\ &\gtrsim \left(\frac{1}{2} \prod_{m=1}^{n-1} \int_{U_n(s,m)} \frac{dV(z_m)}{|1 - z_m s|^{\frac{2n^2}{n-1}+2-2m}} - \sum_{\substack{\tau \in \mathcal{S}_{n-1} \\ \tau \neq I}} \prod_{m=1}^{n-1} \int_{U_n(s,m)} \frac{n! dV(z_m)}{|1 - z_m s|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}} \right) \\ &\quad \times \int_{U_n(s,n)} \frac{dV(z_n)}{|1 - z_n s|^2}. \end{aligned} \tag{4.25}$$

Now we turn to show that the positive term in the last line of (4.25) also dominates the rest terms. Since all these terms share the same z_n part, the estimate (4.24) is no longer able to distinguish one from another. Thus here, we will make use of the asymmetry of $\mathbf{U}(s)$ in z_j variables to prove the claim.

By (4.22), we have

$$\begin{aligned} \prod_{m=1}^{n-1} \int_{U_n(s,m)} \frac{dV(z_m)}{|1 - z_m s|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}} &\sim \prod_{m=1}^{n-1} \frac{\pi (5n!)^{2m(2\tau^{-1}(m) - \frac{2n^2}{n-1})} (1-s)^{2\tau^{-1}(m) - \frac{2n^2}{n-1}}}{3 s^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)} \left(\frac{2n^2}{n-1} - 2\tau^{-1}(m)\right)(n-1)} \\ &= \frac{\pi^{n-1} (1-s)^{-n^2-n} (5n!)^{-2n^3}}{3^{n-1} s^{n^2+n+2} (n-1)^{n-1}} \prod_{m=1}^{n-1} \frac{(5n!)^{4m\tau^{-1}(m)}}{\left(\frac{2n^2}{n-1} - 2\tau^{-1}(m)\right)} \\ &= \frac{\pi^{n-1} (1-s)^{-n^2-n} (5n!)^{-2n^3}}{3^{n-1} s^{n^2+n+2} (n-1)^{n-1}} \frac{(5n!)^{4\sum_{m=1}^{n-1} m\tau^{-1}(m)}}{\prod_{m=1}^{n-1} \left(\frac{2n^2}{n-1} - 2m\right)}. \end{aligned} \tag{4.26}$$

Hence, for any permutation $\tau \in \mathcal{S}_{n-1}$ with $\tau \neq I$,

$$\frac{\prod_{m=1}^{n-1} \int_{U_n(s,m)} \frac{dV(z_m)}{|1 - z_m s|^{\frac{2n^2}{n-1}+2-2m}}}{\prod_{m=1}^{n-1} \int_{U_n(s,m)} \frac{dV(z_m)}{|1 - z_m s|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}}} \sim (5n!)^{4\sum_{m=1}^{n-1}(m^2 - m\tau^{-1}(m))} \geq (4n!)^4. \tag{4.27}$$

Here $\sum_{m=1}^{n-1}(m^2 - m\tau^{-1}(m)) \geq 1$ follows by Cauchy–Schwarz inequality and the fact that the sum $\sum_{m=1}^{n-1}(m^2 - m\tau^{-1}(m))$ is an integer. Substituting these estimates into (4.25), we finally obtain

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$$\begin{aligned}
&\int_{U_n(s,n)} \frac{dV(z_n)}{|1-z_ns|^2}\\
&\times\left(\frac{1}{2}\prod_{m=1}^{n-1}\int_{U_n(s,m)}\frac{dV(z_m)}{|1-z_ms|^{\frac{2n^2}{n-1}+2-2m}}-\sum_{\tau\in\mathcal{S}_{n-1}}\prod_{m=1}^{n-1}\int_{U_n(s,m)}\frac{n!dV(z_m)}{|1-z_ms|^{\frac{2n^2}{n-1}+2-2\tau^{-1}(m)}}\right)\\
&\gtrsim\int_{U_n(s,n)}\frac{dV(z_n)}{|1-z_ns|^2}\prod_{m=1}^{n-1}\int_{U_n(s,m)}\frac{dV(z_m)}{|1-z_ms|^{\frac{2n^2}{n-1}+2-2m}}\left(\frac{1}{2}-\sum_{\tau\in\mathcal{S}_{n-1}}\frac{n!}{(4n!)^4}\right)\\
&\geq\frac{1}{4}\int_{U_n(s,n)}\frac{dV(z_n)}{|1-z_ns|^2}\prod_{m=1}^{n-1}\int_{U_n(s,m)}\frac{dV(z_m)}{|1-z_ms|^{\frac{2n^2}{n-1}+2-2m}}\approx-(1-s)^{-n^2-n}\log(1-s),
\end{aligned}\tag{4.28}$$

which implies that $\|\tilde{T}^n(h_s)\|^p_{L^p_{\text{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}\gtrsim-(1-s)^{-n^2-n}\log(1-s)$. Thus

$$\frac{\|\tilde{T}^n(h_s)\|^p_{L^p_{\text{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}}{\|h_s\|^p_{L^p_{\text{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)}}\gtrsim-\log(1-s)\to\infty$$

as $s\to1$, proving that $\tilde{T}^n$ is unbounded on $L^p_{\text{sym}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^2)$ for $p=\frac{2n}{n-1}$. ∎

5 Some remarks

1. In [HW20a], we studied weak-type estimates of the Bergman projection on the Hartogs triangle and showed the projection is of weak-type $(4,4)$ but not of weak-type $(\frac{4}{3},\frac{4}{3})$. These results together with the Marcinkiewicz interpolation also recover the sharp L^p regular range $(\frac{4}{3},4)$ for the projection on the Hartogs triangle. Similarly, weak-type (p,p) estimates of $P_{\mathbb{G}^n}$ when $p=\frac{2n}{n\pm1}$ could lead to an alternative approach for Theorem 1.1.

2. In [CJY23], Chen, Jin, and Yuan obtained the Sobolev L^p boundedness for $P_{\mathbb{G}}$ from $W^{k,p}(\mathbb{G})$ to some weighted $W^{k,p}$ spaces for $p>2$. With L^p irregularity results obtained for $P_{\mathbb{G}^n}$, it would be interesting to investigate the $W^{k,p}$ (ir)regularity for $P_{\mathbb{G}^n}$. In addition to estimates for $P_{\mathbb{G}^n}$, one may further consider L^p boundedness and compactness of operators that are related to the Bergman projection, such as Toeplitz operators and Hankel operators.

3. The symmetrized polydisc $\mathbb{G}^n$ can be viewed as the quotient domain $\mathbb{D}^n/\mathcal{S}_n$, where $\mathcal{S}_n$ is the group of permutations of variables acting on $\mathbb{D}^n$. It is interesting to see whether our method can be generalized to obtain similar results on other quotient domains of $\mathbb{D}^n$. For instance, the L^p norm of $P_{\mathbb{G}^n}$ is equivalent to the L^p norm of $P_{\mathbb{D}^n}$ over $L^p_{\text{anti}}(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^{2-p})$, a subspace of $L^p(\mathbb{D}^n,|J_{\mathbb{C}}\Phi_n|^{2-p})$ that is related to $\mathcal{S}_n$. On this subspace, we are able to construct the operator T^n_2 which equals $P_{\mathbb{D}^n}$. It is interesting to see if such a proving strategy can be abstracted to work for general quotient domains.

Acknowledgments The authors would like to thank Gian Maria Dall'Ara, Yuan Yuan, Yuan Zhang, Włodzimierz Zwonek, and the anonymous referee for their valuable comments and suggestions.

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https://doi.org/10.4153/S0008414X24000634 Published online by Cambridge University Press