A COMPACTNESS THEOREM FOR AFFINE EQUIVALENCE-CLASSES OF CONVEX REGIONS

A. M. MACBEATH

In some parts of the Geometry of Numbers it is convenient to know that certain affine invariants associated with convex regions attain their upper and lower bounds. A classical example is the quotient of the critical determinant by the content (if the region is symmetrical) for which Minkowski determined the exact lower bound 2^{-n}. The object of this paper is to prove that for continuous functions of bounded regions the bounds are attained. The result is, of course, deduced from the selection theorem of Blaschke, and itself is a compactness theorem about the space of affine equivalence-classes.

1. Introduction. We consider affine transformations σ of Euclidean n-space (E^n) given by

$$(\sigma x)_i = y_i = \sum_{j=1}^{n} \sigma_{ij} x_j + \sigma_{i0} \qquad (i = 1, \ldots, n).$$

With each σ is associated the matrix, or homogeneous affine transformation

$$y_i = \sum_{j=1}^{n} \sigma_{ij} x_j \qquad (i = 1, \ldots, n).$$

This matrix will be denoted by $\mu(\sigma)$. The determinant of the matrix $\mu(\sigma)$ is called the determinant of σ, written $\det \sigma$. The mappings $\sigma \to \mu(\sigma) \to \det \sigma$ are homomorphic and $|\det \sigma|$ represents the factor by which the transformation alters content.

The affine transformations for which $\det \sigma \neq 0$ form a group, which will be denoted by G.

We define a *convex body* to be a bounded closed convex set *with inner points* in E^n. If $\sigma \in G$, σK is defined by the relation

$$x \in K \leftrightarrow \sigma x \in \sigma K.$$

Let $f(K)$ be a function defined on the space $\mathfrak{C}$ of all convex bodies in E^n: f is called an *affine invariant* if $f(K) = f(\sigma K)$ for all $\sigma \in G$, $K \in \mathfrak{C}$.

$\mathfrak{C}$ can be regarded as a metric space, for if $\delta(K, K')$ is the greatest distance of a point of K from K', i.e.,

$$\delta(K, K') = \sup_{x \in K} \left(\inf_{y \in K'} |x - y| \right),$$

then

$$\eta(K, K') = \max\left(\delta(K, K'), \delta(K', K)\right)$$

is a metric [1, p. 34].

Received May 16, 1949.

https://doi.org/10.4153/CJM-1951-008-7 Published online by Cambridge University Press

Define an *affine class* to be a set consisting of all convex bodies that can be derived from a fixed one by means of transformations of G. Let $\mathfrak{C}^*$ be the set of all affine classes. The topology of $\mathfrak{C}$ induces a topology on $\mathfrak{C}^*$, if we define a set of classes to be open in $\mathfrak{C}^*$ if and only if the union of these classes is open in $\mathfrak{C}$. Our main result is

THEOREM 1. *$\mathfrak{C}^*$ is a compact metric space.*

A weaker form of the result, often useful in applications, is

COROLLARY. *If $f(K)$ is an affine-invariant continuous real-valued function on $\mathfrak{C}$, then the upper and lower bounds of f are attained.*

2. Properties of G. A transformation $\sigma \in G$ is said to be *orthogonal* if the matrix $\mu(\sigma)$ is an orthogonal matrix and if also the terms σ_{i0} all vanish. The orthogonal transformations form a group O.

Now it is well known that every non-singular matrix can be expressed as the product of a positive definite symmetric matrix and an orthogonal one; but then the symmetric matrix can, by the usual reduction of quadratic forms, be written UDU^{-1}, where U is orthogonal and D is a diagonal matrix. It follows that every matrix can be written in the form

$$U_1 D U_2,$$

where the U are both orthogonal matrices and D is a diagonal one.

Applying the homomorphism $\sigma \rightarrow \mu(\sigma)$, we find that, if $\sigma \in G$ then σ can be put in the form

$$\sigma = \alpha_1 \delta \alpha_2, \tag{1}$$

where $\alpha_1, \alpha_2 \in O$ and δ is such that $\mu(\delta)$ is a diagonal matrix, i.e.,

$$\delta_{ij} = 0 \qquad (j \neq 0, i).$$

With each element σ of G associate the point of $E^{n(n+1)}$ whose coordinates are the $n(n+1)$ numbers

$$\sigma_{ij} \qquad (i = 1, \ldots, n; \; j = 0, \ldots, n).$$

The elements of G are thus in one-one correspondence with the complement of the variety $\det \sigma = 0$ in $E^{n(n+1)}$. The topology so induced characterizes G as a topological group, for the maps $G \times G \rightarrow G$ and $G \rightarrow G$ given by the operations of multiplication and taking inverses are continuous.

We shall use the absolute value symbol $|\sigma|$ to denote the Euclidean distance from the origin of the point of $E^{n(n+1)}$ corresponding to σ. Then $|\sigma|$ is unaltered by left or right multiplication by an element of O.

A closed set $\Sigma \subset G$ is compact if

$$\inf_{\sigma \in \Sigma} |\det \sigma| > 0, \tag{i}$$

$$\sup_{\sigma \in \Sigma} |\sigma| < \infty, \tag{ii}$$

for then the corresponding set in $E^{n(n+1)}$ is bounded and closed.

LEMMA 1. *If K is a fixed convex body, the mapping of G into $\mathfrak{C}$ given by*

$$\sigma \to \sigma K$$

is continuous.

Proof. Let $R > 1$ be an upper bound for the distance of points of τK from the origin, where τ is fixed. Then, if $x \in \tau K$ and e denotes the identity element of G,

$$|(\sigma x)_i - x_i| = |((\sigma - e)x)_i| \leqslant |\sigma - e|R(n+1).$$

Hence

$$\eta(\sigma\tau K, \tau K) \leqslant R|\sigma - e|n(n+1) \to 0$$

as $\sigma \to e$. This proves the lemma.

In the following two lemmas S denotes the solid unit sphere with its centre at the origin, and the A are positive constants.

LEMMA 2. *The set $\sum$ of $\sigma \in G$ satisfying the conditions*

$$S \supset \sigma S; \quad \det \sigma \geqslant A$$

is compact.

Proof. It follows from Lemma 1 that the set of σ satisfying the first of these two conditions is closed. Since the set $\det \sigma \geqslant A$ is obviously closed, $\sum$ is closed. Now the first of our compactness conditions is satisfied by hypothesis, so it is enough to show that $\sum$ corresponds to a bounded set in $E^{n(n+1)}$.

By (1) we can write $\sigma = \alpha_1 \delta \alpha_2$. Then, since $\alpha S = S$ for all $\alpha \in O$,

$$S = \alpha_1^{-1} S \supset \alpha_1^{-1}\sigma S = \delta S.$$

Apply this to the points $(0, \ldots, 0)$ and $(n^{-\frac{1}{2}}, \ldots, n^{-\frac{1}{2}})$. We find

$$\sum_{i=1}^{n} \delta_{i0}^2 \leqslant 1, \qquad \sum_{i=1}^{n} (n^{-\frac{1}{2}}\delta_{ii} + \delta_{i0})^2 \leqslant 1.$$

From the triangle inequality, $\sum \delta_{ii}^2 \leqslant 4n$. Hence $|\delta|^2 \leqslant 1 + 4n$; so, since transformations of O do not alter lengths in $E^{n(n+1)}$, $|\sigma|^2 \leqslant 1 + 4n$. This proves the lemma.

LEMMA 3. *Let K, K' be two convex bodies. Then the set $\sum_1$ of $\sigma \in G$ satisfying the conditions*

$$\sigma K' \supset K, \quad \det \sigma \leqslant A_2$$

is compact.

Proof. From Lemma 1, $\sum_1$ is closed. We shall show also that it is a subset of a compact set.

Since K has inner points $K \supset \tau_1 S$ for some $\tau_1 \in G$. Further K' is bounded, so $K' \subset \tau_2 S$. Hence the conditions defining $\sum_1$ imply

$$\sigma\tau_2 S \supset \tau_1 S, \qquad \det \sigma \leqslant A_2$$

or

$$S \supset \tau_2^{-1}\sigma^{-1}\tau_1 S, \qquad \det \sigma \leqslant A_2; \tag{2}$$

https://doi.org/10.4153/CJM-1951-008-7 Published online by Cambridge University Press

i.e., $\sum_1$ is contained in the set $\sum_2$ defined by the conditions (2). Apply to this set the homeomorphism $\sigma \to \tau_2^{-1}\sigma^{-1}\tau_1$. It follows from Lemma 2 that $\sum_2$ is compact. Hence $\sum_1$, being a closed subset of it, is also compact.

3. The invariant ρ. The proof of Theorem 1 depends on the introduction of a function ρ, defined as follows. Let K, K' be two convex bodies in E^n.

$$\rho(K,K') = \inf \frac{V(\sigma K')}{V(K)} \; [\sigma \in G, \sigma K' \supset K],$$

where V denotes the content of the body.

By definition of ρ there is a sequence $\{\sigma_i\}$ of elements of G such that $\sigma_i K' \supset K$, $\dfrac{V(\sigma_i K')}{V(K)} \to \rho(K,K')$. Since $V(\sigma_i K') = |\det \sigma_i| V(K')$, $\det \sigma_i$ tends to a limit and so is bounded. Hence, by the compactness proved in Lemma 3, there is a subsequence such that $\lim \sigma_{i_\nu} = \tau \in G$; so by Lemma 1,

$$\tau K' \supset K, \quad \rho(K,K') = \frac{V(\tau K')}{V(K)} .$$

Thus $\rho(K,K')$, which by definition is not less than unity, is equal to unity only if K, K' belong to the same affine class.

Since affine transformations preserve ratios of content,

$$\rho(K, K') = \rho(\sigma K, \tau K') \qquad (\sigma, \tau \in G), \tag{3}$$

so ρ is really a function on $\mathfrak{C}^* \times \mathfrak{C}^*$. Next

$$\rho(K_1,K_3) \leqslant \rho(K_1,K_2)\, \rho(K_2,K_3). \tag{4}$$

For let
$$\rho(K_1,K_2) = \frac{V(\sigma K_2)}{V(K_1)} ; \quad \rho(K_2,K_3) = \frac{V(\tau K_3)}{V(K_2)} .$$

Then
$$\rho(K_1,K_3) \leqslant \frac{V(\sigma\tau K_3)}{V(K_1)} = \frac{V(\tau K_3)}{V(K_2)} \frac{V(\sigma K_2)}{V(K_1)} .$$

LEMMA 4. *The function $\rho(K,K')$ is continuous.*

Proof. We prove first that, for each fixed K, $\rho(K,L) \to 1$ as $L \to K$.

Since K has inner points it contains a sphere, say centre O and radius r. Then if $\eta(K,L) < \frac{1}{2}r$, L contains a sphere centre O, radius $\frac{1}{2}r$. Let $(1+t)L$ be the body obtained by expanding L homothetically about O in the ratio $(1+t){:}1$. Then $(1+t)L$ is an affine image of L and contains all the points at distance $\frac{1}{2}tr$ or less from L; so, if $\epsilon < 1$,

$$\eta(K,L) < \tfrac{1}{2}\,\epsilon r \to (1+\epsilon)L \supset K$$
$$\to \rho(K,L) \leqslant V((1+\epsilon)L){:}V(K) = (1+\epsilon)^n \frac{V(L)}{V(K)}$$

https://doi.org/10.4153/CJM-1951-008-7 Published online by Cambridge University Press

and the result follows from continuity of the content [2, p. 61, 62; 1, p. 38]. A similar argument shows that $\rho(L,K) \to 1$ as $L \to K$.

Now from (4) we have the relations

$$\rho(K'_1, K'_2) \leqslant \rho(K'_1, K_1)\, \rho(K_1, K_2)\, \rho(K_2, K'_2)$$

$$\rho(K_1, K_2) \leqslant \rho(K_1, K'_1)\, \rho(K'_1, K'_2)\, \rho(K'_2, K_2).$$

On letting $K'_1 \to K_1$, $K'_2 \to K_2$, we find that $\lim \rho(K'_1, K'_2) = \rho(K_1, K_2)$ and continuity is established.

LEMMA 5. *There is an absolute constant $c(n)$, depending only on the dimension of the space, such that*

$$\rho(K,K') \leqslant c(n)$$

for every two bodies K,K'.

Proof. Consider a simplex S of maximum content contained in K. Suppose its vertices are $A_0, \ldots, A_n$ and its faces $a_0, \ldots, a_n$. Let $b_0, \ldots, b_n$ be the hyperplanes through $A_0, \ldots, A_n$ respectively parallel to the opposite faces. These are hyperplanes of support of K, for if, say, b_0 were not, there would be points of K further from the hyperplane a_0 than A_0. If A' were such a point, then $A', A_1, \ldots, A_n$ would form a simplex of larger content, contradicting the original choice of S. Hence K is contained in the simplex $\sum$ defined by the hyperplanes $b_0, \ldots, b_n$. Now $V(\sum):V(S) = n^n$. Since all simplexes belong to the same affine class, we have

$$\rho(S,K)\, \rho(K,S) \leqslant n^n, \tag{5}$$

$$\rho(K,S) \leqslant n^n \text{ and } \rho(S,K') \leqslant n^n; \tag{6}$$

thus, from (4),

$$\rho(K,K') \leqslant n^{2n}. \tag{7}$$

It is now easy to prove the compactness part of Theorem 1. It follows from (6) that if S is a fixed simplex of unit content there is, for each K, a σK contained in S such that $V(\sigma K) \geqslant n^{-n}$. Then let $\mathfrak{B}$ be the set of all closed convex subsets of S which have content not less than n^{-n}. $\mathfrak{B}$ is compact, by Blaschke's theorem [1, p. 34; 2, p. 62-8] and by continuity of the content. Consider now the mapping of $\mathfrak{B}$ into $\mathfrak{C}^*$ where each body is mapped on the class of which it is a member. This map is continuous and the image is the whole of $\mathfrak{C}^*$; so, since $\mathfrak{B}$ is compact, its image $\mathfrak{C}^*$ is compact.[1]

[1]The condition $V(K) \geqslant n^{-n}$ is necessary only because we restrict ourselves to convex sets with inner points. This is the only interesting case, for if an affine invariant is continuous for all closed convex sets it is a constant. Thus if f is such a function and I denotes a closed interval, I can be derived from K by a singular affine map, and is the limit of a sequence of sets which are derived from K by non-singular affine maps. Hence $f(K) = f(I)$ and f is a constant.

From (4) it follows that the function

$$\Delta(K,K') = \log \rho(K,K') + \log \rho(K',K)$$

satisfies the triangle inequality, so it characterizes the affine classes as a metric space $\mathfrak{C}^{**}$. Let φ be the mapping of $\mathfrak{C}^*$ onto $\mathfrak{C}^{**}$ which maps each class, considered as a member of $\mathfrak{C}^*$ into the same class, considered as a member of $\mathfrak{C}^{**}$. Then, since Δ is a continuous function on $\mathfrak{C}^*$, the set of K such that

$$\Delta(K',K) < \epsilon \qquad (K' \text{ fixed})$$

is open in $\mathfrak{C}^*$. Since every open set in $\mathfrak{C}^{**}$ is a union of such sets, and so open in $\mathfrak{C}^*$, φ is a continuous map. Now $\mathfrak{C}^*$ is a compact space and $\mathfrak{C}^{**}$ is metric, so φ is topological; i.e. Δ gives a metrization for the space $\mathfrak{C}^*$. This completes the proof of Theorem 1.

Among the inequalities (5), (6), (7), only (5) is best possible. Equality holds in (5) if K is a sphere, but equality never holds in (6), since, from (5)

$$\rho(S,K) = n^n \rightarrow \rho(K,S) = 1 \rightarrow K = \sigma S \rightarrow \rho(S,K) = 1,$$

a contradiction. Since $\mathfrak{C}^*$ is compact, the upper bound is attained and so must be less than n^n.

4. A better bound for ρ. The above simple proof that ρ that is bounded was suggested to the author in a conversation with Dr. Mahler. This section gives the author's original proof, which implies a better bound for ρ. The proofs of the first two lemmas were suggested by a referee, and are shorter than the author's original proofs.

LEMMA 6. *Let P, Q be two points of a convex body K in E^n, where the distance $|PQ| = l$. Let K' be the orthogonal projection of K on π, a hyperplane perpendicular to PQ. Let V' be the $(n-1)$-dimensional content of K'. Then*

$$V(K) \geqslant lV'/n.$$

Proof. Symmetrize with respect to the hyperplane π [2, p. 44; 1, p. 69 sqq.] Let K_1 be the body obtained in this way. K_1 is convex and $V(K_1) = V(K)$. The intersection of K_1 and π is the projection K'. To the points P, Q correspond two points P_1, Q_1 of K_1 and $P_1Q_1 = l$.

By convexity, K contains the two cones with common base K' and vertices P, Q, whose total content is

$$lV'/n.$$

LEMMA 7. *There is a parallelotope Π containing K such that*[2]

$$V(\Pi) \leqslant n!V(K).$$

Proof by induction on n. Let P, Q be two points of K with maximum dis-

[2]A similar result for *symmetrical* convex bodies is proved in [3, p. 97-8].

tance $= l$. The two normal hyperplanes to the line PQ at P,Q are hyperplanes of support of K. Let K' be the projection of K on one of these hyperplanes. By the induction assumption there is an $(n-1)$-dimensional parallelotope Π' containing K' for which

$$V'(\Pi') \leqslant (n-1)!V'(K'),$$

where V' denotes $(n-1)$-dimensional content. Now K is contained in the parallelotope Π with base K' and altitude l. Hence

$$V(\Pi) = lV'(\Pi') \leqslant (n-1)!lV'(K'),$$

so by Lemma 6,
$$V(\Pi) \leqslant n!V(K).$$

Lemma 8. *There is a parallelotope* Π *contained in* K, *such that*

$$V(\Pi) \leqslant n^{-n}V(K).$$

Proof by induction on n. Assume, without loss of generality, that, among the hyperplanes $x_1 = \text{const.}$, $x_1 = 0$ is the one or one of those whose section with K has maximum $(n-1)$-dimensional content. If $A(\alpha)$ is the $(n-1)$-dimensional content of the section $x_1 = \alpha$, then

$$V(K) = \int_{-q}^{p} A(u)\,du \leqslant (p+q)\,A(0), \tag{8}$$

where p, $-q$ are the max and min of x_1-coordinates of points of K.

Let R be the intersection of K with the hyperplane $x_1 = 0$. By convexity, K contains the cones obtained by joining R to the two points P, Q that have max and min x_1-coordinate respectively. The hyperplanes $x_1 = p/n$, $-q/n$ intersect these cones in two congruent and similarly placed $(n-1)$-dimensional convex sets, each homothetic to R and each of $(n-1)$-content

$$A(0)\,(1-1/n)^{n-1}.$$

By the induction assumption, we can inscribe in these regions two congruent and similarly placed $(n-1)$-parallelotopes, each of content at least $A(0)/n^{n-1}$; and these, being distant $(p+q)/n$ apart, span an n-parallelotope of content at least

$$A(0)\,(p+q)/n^n \geqslant n^{-n}V(K)$$

by 8. This proves Lemma 8.

Lemmas 7, 8 combined with (4) give the sharper inequality

$$c(n) \leqslant n!\,n^n. \tag{9}$$

However, it is easy to convince oneself that neither Lemma 7 nor Lemma 8 is best possible, except Lemma 7 for $n = 2$. (Then equality holds if K is a triangle.) The problem of finding the exact value of $c(n)$ (which must be attained, by compactness, for some K, K') is left open. It seems natural to conjecture that $c(n)$ is attained when K is a simplex and K' is an ellipsoid.

The author would like to record his gratitude to Dr. Mahler and Mr. J. F. Nash for illuminating discussions and criticisms.

References

[1] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper* (Ergebnisse der Math.) (Berlin, 1943).

[2] W. Blaschke, *Kreis und Kugel* (Leipzig, 1916).

[3] K. Mahler, *Ein Übertragungsprinzip für konvexe Körper.* Casopis pro Pestovani Mat. a Fys., vol. 68 (1939), 93-102.

Princeton University