ASTEROID PROPER ELEMENTS : THE BIG PICTURE

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Abstract. Four perturbation theories presently used to compute asteroid proper elements are reviewed, and their results are briefly discussed (Milani and Knežević, 1990, 1992, 1994, for low to moderate eccentricity/inclination main belt objects; Lemaitre and Morbidelli, 1994, for high e, I objects; Milani, 1993, for Trojans; Schubart, 1982, 1991 for Hildas). The most important recent improvements are described, in particular those pertaining to the upgrades of the previous analytic and semianalytic solutions. The dynamical structure of the asteroid main belt, as defined by the low order mean motion resonances and by linear and nonlinear secular resonances, is considered from the point of view of the effects of these resonances on the accuracy and/or reliability of the computation of proper elements and on the reliability of the identification of asteroid families.

1. What are the proper elements and what they are used for?

Asteroid proper elements are quasi-integrals of motion, stable over very long intervals of time; they represent a sort of "average" characteristics of motion in the sense that they result from a procedure of elimination of short– and long–periodic perturbations. The proper elements are used for two main purposes : (i) they serve as parameters for classification of asteroids into families; (ii) they are employed for studies of the dynamical structure of the asteroid belt.

Asteroid families are groupings of asteroids in the phase space of orbital elements. They were formed by catastrophic breakup of parent bodies, at unknown times in the remote past, and under circumstances (such as geometry of collision, physical properties of the parents, etc.) that are not yet fully understood. The parent body at the breakup might have had osculating elements significantly different from the corresponding proper values; since we do not know the epoch of the collision, the proper elements (due to their constancy in time) become the only tool to reliably recognize families, in particular the small ones consisting of a few members. Until recently, the very identification of the families and their memberships were not very reliable : the proper elements used for this identification were not accurate enough, and/or the classification methods suffered from subjectivity, biases, uncertainties due to insufficient data, etc. The situation is now greatly improved in both respects; the review of the current state of the art regarding proper elements and family classification is the subject of this and some other papers in this book (see e.g. Zappalà and Cellino, Milani, Burbine and Binzel).

The differences of proper orbital elements between family members are related to their relative velocities, averaged over the unknown quantity, the age of the family.

A. Milani et al. (eds.), Asteroids, Comets, Meteors 1993, 143–158.

In order to reliably recognize the families, the changes in the proper elements – over time spans of the order of the family ages– must not exceed an amount given by the typical relative velocities of the family members. The relative velocities of family members are known to be of the order of $100\,m/s$; for the so called standard metric (Zappalà et al., 1990) this corresponds to $6-7\times 10^{-3}$ in terms of the proper elements difference. Hence, proper elements stable –over very long time spans– to about 0.001 in proper eccentricity and sine of inclination and to 0.0005 in proper semimajor axis can be considered as good enough, those stable to 0.003 in $e, \sin I$ should be used with some downweighing, while those with instabilities ≥ 0.01 allow to identify asteroid families, but only in the less densely populated regions of the asteroid belt.

The dynamical structure of the asteroid belt is determined by the mean motion and secular resonances. The resonances split the portion of the phase space corresponding to the asteroid belt in several dynamically distinct zones; they affect the motion of nearby asteroids, give rise to gaps in the distributions of elements, etc. The location in the phase space of the resonances can be determined by computing simple combinations of the so–called fundamental frequencies (the rates of the proper angles), thus the knowledge of the dynamical structure of the belt is essentially equivalent to the ability of listing all the relevant resonances that bound or cut across the asteroid belt region. To our present knowledge, the resonances relevant for the dynamical structure of the main asteroid belt at low to moderate inclinations are the following : (1) the low degree mean motion resonances, like 2/1, 3/1, 4/1, 5/2, 7/3, and others giving rise to Kirkwood gaps; (2) the linear secular resonances $g-g_5$, $g-g_6$, and $s-s_6$ (g, s, g_j, s_j are the frequencies of the proper longitudes of perihelion and node of the asteroid and of the perturbing planets; some authors use another notation, e.g. ν_5, ν_6, ν_{16} for the three linear resonances); (3) nine nonlinear secular resonances that cut across the core asteroid belt ($2.2-3.2\,AU$ at low–to–moderate inclinations) and affect the motion of a significant number of asteroids : $g+s-g_5-s_6$, $z_1=g+s-g_6-s_6$, $g+s-g_5-s_7$, $g-2g_6+g_5$, $g-2g_6+g_7$, $g-3g_6+2g_5$, $s-s_6-g_5+g_6$, $z_2=2g+s-2g_6-s_6$, $z_3=3g+s-3g_6-s_6$. The latter are called nonlinear because the corresponding small divisors represent combinations of more frequencies and appear with terms of higher degree in eccentricity and/or inclination in the equations of motion.

2. Theories and methods for computation of proper elements

Hirayama (1918) discovered asteroid families by looking into the distributions of osculating orbital elements, but he used from the very beginning the concept of proper elements; in his later works (Hirayama, 1923; 1928) he used explicit computations of proper elements by means of a linear theory. Brouwer (1951) used an enhanced Lagrange-type analytical solution of the linear secular perturbation equations and the major planet theory by Brouwer and Van Woerkom (1950) to produce proper elements for all asteroids numbered at that time; his procedure and results were rather advanced with respect to the previous ones and therefore they were used for quite some time (c.f. Anders, 1965; Arnold, 1969; Van Houten et al., 1970; Lindblad and Southworth, 1971). The real breakthrough occurred, however, when Williams (1969) developed a new semianalytic theory, which used no expansion of the perturbing function in eccentricity and inclination of the asteroid, and thus provided results of about the same accuracy regardless of e, I. Proper elements computed by means of this theory have been published by Williams (1979) and by Williams and Hierath (1987), and are sometimes still in use (they can also be found in Williams, 1989). The positions of the surfaces of the linear secular resonances, which are the most important because they bound the main asteroid belt and open large gaps in the distribution of asteroids, were determined using this theory by Williams and Faulkner (1981). The theory of Williams represents a cornerstone in the development of the theories of asteroid motion, and the beginning of the contemporary era in this respect. Afterwards new theories appeared, and there are at present four theories and/or methods in use for computation of asteroid proper elements. These are the purely analytical theory by Milani and Knežević (1990, 1992, 1994), which provides the most accurate results for low to moderate eccentricity/inclination main belt objects, the semianalytic theory by Lemaitre and Morbidelli (1994) which should be used for high e, I objects, a numerical method by Milani (1993) for the Trojans, and a similar one by Schubart (1982, 1991) for the Hildas. All these four catalogues of proper elements are in the public domain, and can be obtained by `anonymous ftp`, as described in Milani et al., this volume. A method to compute proper elements in another case, for asteroids close to some secular resonances, has been developed recently by Morbidelli (1993); the underlying theoretical concept and the results are, however, described in the paper by Froeschlé and Morbidelli in this volume.

2. 1. THEORY BY MILANI AND KNEŽEVIĆ

The theory by Milani and Knežević was developed from the analytical theory of secular perturbations of asteroids by Yuasa (1973), which takes into account terms in the development of the perturbing Hamiltonian up to degree four in eccentricity and inclination, in the first order with respect to perturbing mass, and those of the second degree, in the second order. This theory used the canonical formalism (Hori, 1966) to remove the short– and the long–periodic terms from the Hamiltonian, by means of two successive canonical transformations. Many improvements have been introduced by Milani and Knežević with respect to the original theory. The derivation of the second order terms has been revised, the missing indirect part of the Hamiltonian has been added, the long periodic Hamiltonian has been rearranged to include the second order effects in the linear theory, an iterative procedure has been used to solve for proper elements and frequencies, and last but not least, some errors have been removed from the original equations.

Since it is based on a truncated development of the perturbing Hamiltonian, this theory is particularly suitable for low–to–moderate inclination/eccentricity objects (almost 90% of the main belt asteroids belong to this "core belt"), for which in most cases it provides proper elements with high accuracy. On the other hand, in the high inclination/eccentricity regions and near the main resonances it fails to provide reliable results; the extension of the developments to even higher orders and/or degrees is very difficult and can be achieved by means of the same technique only to some extent (see later, items (b) and (c)).

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As claimed by Milani and Knežević (1994), the results achieved by means of the analytical theory are more than good enough for family identification purposes (in the sense described above) for more than 90% of the asteroids belonging to the core belt. This is supported by an extensive set of stability tests, including 35 objects representative of the various regions of the belt and of the families found by Zappalá et al. (1994); the tests consisted of a numerical integration of all the selected objects for 5 Myr (within a dynamical model including the four outer major planets), deriving the time series of proper elements and monitoring their variations over that time span. The results obtained by these tests are summarized in Figure 1, where the square root of the sum of the squares of the RMS of the proper $e, \sin I$ changes over a time span of $5\,Myr$ is plotted versus the proper semimajor axis.

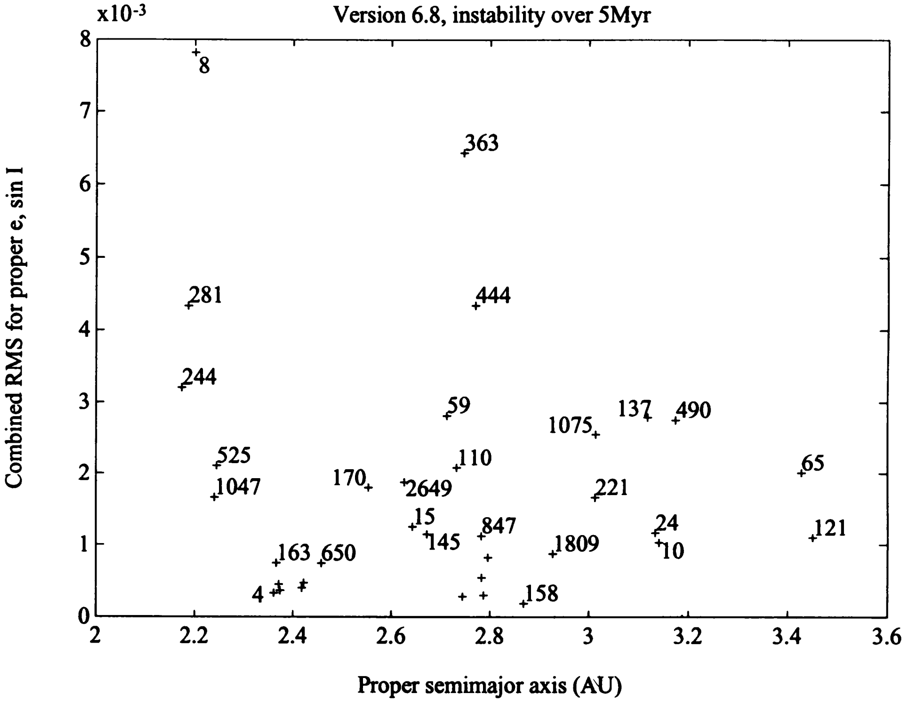

Fig. 1. Summary of the proper elements accuracy tests (from Milani and Knežević, 1994).

There are really but a few cases of seriously degraded accuracy, and they are always due to some resonance (either the 2/1 mean motion resonance or some secular one); it is not even a matter of computing accurate proper elements in these cases, the analytical procedure being divergent, but it is possible to provide a warning and to switch to an adapted theory less sensitive to the resonant effect.

As an illustration of the dynamical structure of the belt, all the most important resonances mentioned in Section 1, affecting the asteroid motion and the determination of proper elements, are plotted (for the average value 0.14 of the proper eccentricity) in a proper inclination vs. proper semimajor axis plane in Figure 2. Contour lines are given for values of the associated divisors of $\pm 0.5\, arcsec/yr$, except for the major $g - g_6$ resonance for which the lines correspond to values of $-2, 0, +2\, arcsec/yr$.

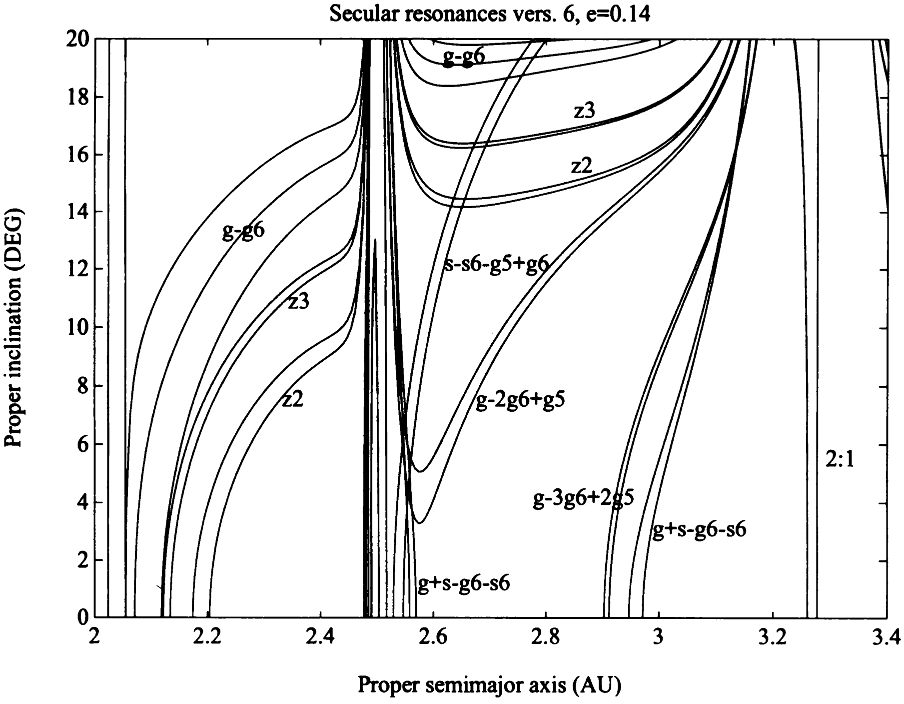

Fig. 2. Positions of secular resonances in the core belt as determined by means of the theory of Milani and Knežević, vers. 6.8.

Among of the nonlinear secular resonances, $g - 2g_6 + g_5$, z_1, and z_2 are the most important both for the effects they cause and for the number of asteroids they affect (since they cut through the most densely populated regions). Many asteroid families are also crossed by the resonances, which in case of a large family (e.g. Eos, see Milani and Knežević, 1992) does not affect the overall recognition of the family, but can affect the list of members; in the case of a small family (e.g. the spurious Lydia family, see Zappalà et al., 1994) their influence can prevent even a reliable identification.

With respect to the version 5.7 of the theory reported at ACM91, the present version 6.8 contains several improvements; since they are described in detail elsewhere (Milani and Knežević, 1994), here we shall only briefly summarize the most important ones.

For the elimination of the short-periodic perturbations and the computation of the asteroid mean elements, the main improvements are as follows. (i) For the removal of the short-periodic effects of Jupiter and Saturn, the previous version of the M&K theory used a development of the perturbing function up to degree four in the first order with respect to the perturbing mass for the direct perturbations, and up to degree three for the indirect perturbations. For reasons of formal completeness of the theory, the terms of degree four were added to the indirect part as well; as shown by Knežević (1992b), this changes the results only by a small amount. (ii) For the perturbation by the inner planets, about 90% of the effects were taken into account by referring the osculating elements to the barycentre of the Sun and the terrestrial planets; this is important because the indirect perturbations of the inner planets can be amplified by small divisors and therefore change the fundamental secular frequencies by a significant amount. (iii) A quality code for mean elements was introduced, to notify the user of the troublesome cases for which the procedure fails to provide accurate mean elements. It allows to automatically detect (and remove from the catalogue) the asteroids locked in the main mean motion resonances. (iv) Technical improvements implemented in the software include the application of a "generic term" technique (Knežević, 1992a), the automatic handling of the reference system changes and of the variable epochs for the input data, the computation of the mean mean anomaly (intended, however, for other purposes, such as asteroid identification), etc.

The improvements implemented in the procedure of removal of the long-periodic perturbations include : the introduction of (a) nonlinear forced terms, (b) second order terms of degree four with squares of the small divisor of the 3/1 resonance, and (c) degree six/eight terms with the z_2 divisor; also (d) an automated switching to an adapted theory has been introduced.

(a) The secular perturbation theory of the major planets contains many terms giving rise to significant forced effects in the motion of asteroids. In the previous versions of the theory of Milani and Knežević (1990, 1992), only the forced terms taken from a linear secular perturbation theory for the planets had been accounted for. Therefore, comparatively large instabilities of the proper elements were found for many asteroids in the vicinity of the resonant surfaces associated with the omitted nonlinear terms; as an example, the results for 1272 *Gefion*, contained an oscillation with frequency $g - 2g_6 + g_5$ and amplitude 4.8×10^{-3} in proper e. In order to remove this source of instability the eight largest forced nonlinear terms from the synthetic theory of major planets by Nobili et al. (1989) have been included in the computation; the result of this simple operation was a significant improvement of the accuracy of proper elements for many test cases previously affected by this problem (the stability for 1272 *Gefion* improved by almost an order of magnitude). However, the theories of the major planets derived from numerical integration include some periodic terms which have not been recognized as integer combinations of fundamental frequencies ("chaotic terms"), and these have not been included, although they have only slightly smaller amplitudes, because their effects cannot be described by an analytical theory.

(b) Although the second order, degree two terms had been included in the previous versions of the theory, the results for some asteroids near the 3/1 mean motion resonance were degraded; this is due to the fact that the most important contribution to the second order effects might come from the degree four part. Among the terms of degree four, in turn, only the terms containing the square of the small divisor are important close to resonance, and these have been added in the present version of the theory. This addition proved to be effective in removing the instabilities from the time series of proper elements for many tested asteroids, the most significant improvement (by a factor of $\simeq 5$) being achieved in the case of 650 *Amalasuntha*. For the 2/1 resonance, on the other hand, the results of adding the same kind of terms were ambiguous, namely, some improvement in some test cases and a slight degradation in others. However, the proper elements for these cases show the features of the chaotic behaviour (intermittency and random walk, positive Lyapunov exponents), and this seems to imply that they are accurate already to the "chaotic limit"; from this we should conclude that no improvement is possible by means of a deterministic theory. Nevertheless the proper elements are, in most of these chaotic cases, stable within a very narrow range; this "stable chaos" phenomenon is far from being fully understood (Milani and Nobili, 1992; Milani, these proceedings; Milani, 1994).

(c) Another only partially successful improvement was the inclusion of the degree six and eight terms with the $z_2 = 2g + s - 2g_6 - s_6$ small divisor. This divisor is the most important one in a cascade of secular resonances that cluster inside the main $g - g_6$ one, and cut across the densely populated Flora region of the core belt. The result of adding this very complex term was rather modest; for the two test cases where the z_2 term was the main source of instability —1047 *Geisha* and 525 *Adelaide*— only about 50% of the corresponding oscillation was removed. Not surprisingly, the Flora region is the only one where family identifications and memberships still have a comparatively low reliability (see Zappalà and Cellino, these proceedings). Further study will be needed to solve this problem.

(d) Since the M&K theory is an iterative one, divergence sometimes occurs, indicating that a long periodic effect contains a divisor close to zero. In a sample of 12,573 asteroids, this occurred in 1,045 cases. The solution applied in such cases is to flag them by means of a special "resonance code", and to try to obtain a convergent result anyway — by switching to an adapted theory from which the nearly resonant term has been removed. Thus, the computation of proper elements becomes more reliable, but somewhat less accurate. In order to identify automatically the resonant term which is the cause of the divergence and therefore has to be eliminated from the theory, a very complicated procedure has been introduced, which now involves an improved iteration scheme (more accurate initial values and tighter convergence control) and checks the size of the terms in three successive iterations, comparing their differences and amplitudes with some predefined threshold values. Because of this complexity the procedure of automatic selection of an appropriate adapted theory cannot be absolutely reliable; sometimes the algorithm fails to identify the responsible resonance, sometimes the procedure diverges even after removal of the supposed "main culprit", etc.; in fact, Milani and Knežević

(1994) estimated that the selection of the suitable adapted theory has a 90% reliability, thus the percentage of cases which have not been properly flagged is less than 1% of the total catalogue of proper elements.

Finally, let us just mention that among the technical improvements to the long periodic software, the most important one is the implementation of the generic term technique (Knežević, 1994), by which a previous 1200 statements computer routine has been replaced by one with 50 statements plus a 1000 lines file storing the information on 300 terms of the literal development of the perturbing Hamiltonian. All the software is placed in the public domain and can be accessed through anonymous `ftp`, at the same server which is used to obtain the proper elements.

It is interesting to note that Milani and Knežević believe that at the present level of development of their analytical theory, they are already beginning to explore the so called "Arnold web" of narrow high degree resonances having small strengths, which sometimes can nevertheless affect the stability of proper elements to some extent. Fundamentally new ideas will be needed if the instability tolerances are to be tightened to values below the present 0.001 level; even more substantial new developments will be needed, if such stability requirements should be extended from the $\simeq 10^7\, yr$ time spans covered by the present stability tests to the lifetime of the solar system.

2. 2. THEORY BY LEMAITRE AND MORBIDELLI

The fundamental difference of the semianalytic theories, such as the one by Lemaitre and Morbidelli (1994), with respect to the analytic theories, such as the one by Milani and Knežević, is due to an observation made already by Kozai (1962). The dynamics at high inclinations and high eccentricities is dominated by the $e^2 i^2 \cos 2\omega$ perturbation, which in such cases must be included in the integrable part of the Hamiltonian in order to obtain meaningful results. Kozai accomplished this by simplifying the dynamical model : by assuming that the perturbing planets are on circular, planar orbits, the secular perturbation Hamiltonian (after averaging out the short periodic perturbations) contains only one angular variable, namely the argument of perihelion ω. Thus this first approximation secular Hamiltonian is integrable, and can be used as a starting point for a perturbative theory. Since this integrable approximation is geometrically very different from the linear theory, there is a big difference in the very definition of the proper elements, at least for the high eccentricities and/or inclinations for which this class of theories should be used.

The first perturbative treatment using the Kozai approximation as a starting point was developed by Williams (1969); in his theory, the portions of the long periodic perturbing function containing e', I' to the first power are accounted for (the primed letters indicate the mean elements of the perturbing planets). The perturbations are computed by numerical averaging of the perturbing function and its derivatives, with respect to the fast variables (the mean anomalies/longitudes of both the asteroid and the perturbing planets). The theory by Williams was very effective, and for many asteroids achieved a level of accuracy comparable to that of the more recent analytical theories; for this reason, Williams' proper elements have been used for a long time. However, there are two main limitations in Williams' theory. First, the theory was developed by means of the non–canonical formalism of classical perturbation theory. To first order in the perturbing masses, the results do not depend upon the formalism; however, it was already difficult to reproduce the results by Williams on the basis of his papers; to go beyond, to a more comprehensive theory, without a more advanced formalism, was impossible. Second, it is by no means true that the effects of higher order in the perturbing mass are always small. For asteroids near (but by no means inside) the main mean motion resonances (such as $2/1, 3/1$), a complicated nonlinear effect of the removal of the short periodic terms results in a secular perturbation containing the square of the perturbing masses, but also the inverse square of the small divisor of the near resonance. A theory accounting for this effect can differ from a purely first order theory by as much as 50% in the fundamental frequencies, as in the Themis region. Thus the kind of accuracy needed to identify asteroid families could not be reached in a significant portion of the main belt, unless the theory was extended to the second order in the perturbing masses. Because of the technical limitations of the classical formalism, this was not possible with Williams theory. As a result, the first theories to achieve an accurate treatment of the main second order effects were the analytical theories, first by Yuasa and then by Milani and Knežević, as discussed in Section 2.1. However, this leaved out the region of the main belt with high eccentricity and/or inclination, were the analytical theories cannot work in a satisfactory way.

The first limitation of the semianalytic theories was removed by Henrard (1990), who developed a new semianalytic perturbation scheme within the canonical formalism of a transformation theory (that is, the solution is directly expanded as a series, without going through the expansion of the differential equations in a series). The Henrard method essentially consists of the use of the Arnold–Jost theorem to define action–angle canonical variables for the integrable problem; these variables are theoretically defined by line integrals, and they can be practically computed by numerical quadrature. Then the entire perturbation scheme to be used to compute the solution of a non–integrable problem can be expressed by means of numerical quadratures of the same kind. When this procedure is applied to the Kozai Hamiltonian and to the perturbations of degree one in e', I', Henrard's canonical method reproduces Williams' proper elements and the theory of the location of the secular resonances by Williams and Faulkner (1981); this was shown by Morbidelli and Henrard (1991a, 1991b). Moreover, the formalism is powerful enough to allow the computation of the second order effects, again by numerical quadrature of a more complicated expression. In this way Lemaitre and Morbidelli (1994) have succeeded in computing proper elements with a theory including the second order effects and without truncation in e, I. Clearly, this theory is particularly suitable for objects with high inclination and/or eccentricity, where the analytical theories fail to provide meaningful results. The Lemaitre and Morbidelli method also uses a fixed point iteration to find the proper frequencies and the proper elements from the mean ones (to understand why an iterative algorithm is always needed, see Milani and Knežević, 1990; Milani, 1994). The iteration is divergent near the secular resonance surfaces; however, this method can handle the cases in which the main argument of the Kozai theory, ω, is in libration (divergence can occur near the separatrix), as it is the case for some very high inclination orbits.

Although this theory is still under development, there are already some interesting results that have been reported recently (Lemaitre and Morbidelli, 1994). In the first place, there are some encouraging results on the accuracy achieved with proper elements for a Pallas-like asteroid, which can be considered as a real challenge, being close to the separatrix between circulation and libration of ω. Even by using "version 1" of the theory (with Jupiter as the only perturber, and with only three secular frequencies g_5, g_6, s_6), the changes in the proper eccentricity and sine of inclination both stay within ± 0.015 (within $\pm 1.5\, arcsec/yr$ for the frequencies) over the time span of $2.5\, Myr$ of a numerical integration test. This stability level is good enough for the purpose of family identification in a region with such a sparse background. Indeed Lemaitre and Morbidelli have been able to identify a family around the asteroid *2 Pallas* (see Fig. 4 in Lemaitre and Morbidelli, 1994); such a family had already been proposed by Williams (1989), but with a lower number of members. The accuracy for the less dynamically complex cases can be much better. As an example, for the asteroid *185 Eunike*, located above the $g - g_6$ resonance, the rms instabilities of the proper eccentricity and proper sine of inclination, as inferred from a $4\, Myr$ integration, amount to 0.006 and 0.002, respectively.

TABLE I

Proper elements derived by means of the theory by Lemaitre and Morbidelli, and their differences with respect to those obtained from the theory by Milani and Knežević.

No.	a	e	Δ_e	$\sin I$	$\Delta \sin I$
4	2.36151	0.0882	-0.0105	0.1137	0.0024
15	2.64366	0.1480	-0.0005	0.2261	0.0002
44	2.42276	0.1743	0.0000	0.0555	0.0025
142	2.41867	0.1589	-0.0006	0.0559	0.0002
145	2.67273	0.1685	-0.0006	0.2036	0.0014
158	2.86881	0.0372	-0.0085	0.0372	-0.0003
163	2.36712	0.2078	-0.0024	0.0824	0.0005
170	2.55374	0.0963	-0.0049	0.2600	0.0004
808	2.74515	0.1267	-0.0066	0.0836	-0.0012
847	2.78276	0.0680	0.0012	0.0620	-0.0010
1272	2.78380	0.1298	0.0007	0.1559	-0.0006
1378	2.37482	0.1580	-0.0052	0.0513	-0.0013
1726	2.78741	0.0320	-0.0149	0.0742	-0.0021
1809	2.92742	0.0751	-0.0007	0.0381	0.0015
1932	2.37179	0.1966	-0.0035	0.0363	-0.0032

The theory by Lemaitre and Morbidelli is still evolving quickly. Recently the proper elements "version 2" have become available : they account for the perturbations by Jupiter and Saturn with the frequencies $g_5, g_6, g_7, s_6, 2g_6 - g_5$; some tests have already been performed with s_7, s_8 as well. Two comparisons were made with

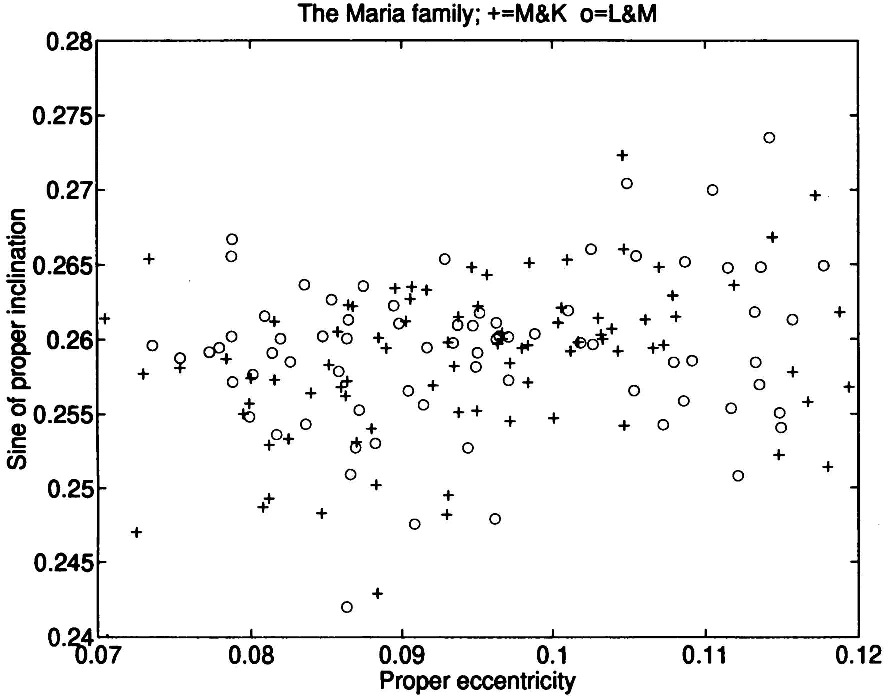

Fig. 3. The Maria family; crosses are data derived by means of the analytical theory by Milani and Knežević, and open circles are obtained from the semianalytical theory by Lemaitre and Morbidelli.

the results by Milani and Knežević, in order to have an idea on the overall reliability of the semianalytic data, and to find out the values of the inclination corresponding to the transition region where the users of the proper elements catalogues are supposed to switch from one theory to another. The data on the latter comparison will be reported elsewhere (Knežević et al., 1994); we shall only state here that the transition region seems to be somewhere between 15° and 17° of inclination, somewhat below the $g - g_6$ resonance. Another comparison is summarized in Table I. Proper elements derived by means of the current version of the theory by Lemaitre and Morbidelli are given together with the differences of these proper values with respect to those by Milani and Knežević, for a number of asteroids in the low-to-moderate inclination zone of the core belt. For most of the objects included in this comparison the agreement is good, but there are still some unexpectedly large discrepancies (4 *Vesta*). Note that in some of these cases the frequencies g, s, as computed by Lemaitre and Morbidelli, are closer to the numerical fits than the corresponding values by Milani and Knežević. Obviously the Lemaitre and Morbidelli proper elements should not be used for low eccentricity and inclination cases, because they are less stable and their computation is less efficient than it is the case for analytic proper elements, but it is important to be able to confirm that, in most cases, they give consistent results. Some of the discrepancies still have to be explained; in particular the stability of the numerical quadrature methods used by the semianalytic methods requires further study to assess the reliability of the computation of proper elements. However, for high inclination and/or eccentricity asteroids the proper elements by Lemaitre and Morbidelli are already at a level of stability and reliability which is useful for the identification of asteroid families, and without doubt the next run of asteroid family classification will use a "composite catalogue" (in the sense of Milani et al., this volume) of proper elements computed by different authors.

An independent test of the significance of the discrepancies in proper elements coming from the two theories is given in Figure 3, which shows the comparatively high inclination region which includes the Maria family, in the proper inclination vs. proper semimajor axis plane, with data from two theories superimposed on the same plot. This is the region of transition between the two theories, in the sense explained above, and proper elements from both theories are supposed to be of about the same accuracy; indeed the agreement of the proper elements and frequencies can be rated as reasonably good.

Two obvious conclusions can be drawn from this figure : the overall shape and identification of the family do not critically depend on the choice of the proper elements theory, while the membership of a particular asteroid (especially at the edge of the family) can be affected. The best idea for this transition region, at least for the moment, might be to use both sets of elements in the classification procedure and to compare the results.

2. 3. METHOD BY MILANI FOR TROJANS

The computation of proper elements for Trojans by means of a synthetic theory, that is, by post-processing the output of a numerical integration, has been first done by Bien and Schubart (1987); they computed the orbits of 41 Trojans for $\simeq 150,000\, yr$, and derived "orbital parameters" for 40 of them. The same technique was improved upon, and applied to a larger set of Trojan orbits, by Milani (1993, and also this volume). The orbits of 184 Trojans have been computed for $1\, Myr$ (for 20 of them the computation has been extended to $5\, Myr$), in a model including the full perturbations by the four giant planets, and with a correction to account for most of the indirect effect of the inner planets. Two files contain the proper elements and frequencies, and their stability as determined by a running box test.

The proper elements as given by Milani are the following : da, a measure of the amplitude of the oscillation of the semimajor axis during one libration period, in AU; as an alternative, also D, a measure of the amplitude of the libration in longitude (in degrees); a proper eccentricity e_p, obtained by filtering out the forced terms; the sine of a proper inclination i_p. The fundamental frequencies f, g, s associated with these three amplitudes are also given. About 87% of the cases have proper elements stable enough for most purposes, with the maximum excursion in the running box test less than $0.001\, AU$ for da, and less than 0.0015 for e_p and $\sin i_p$. Although the dynamical meaning of these "synthetic" proper elements is quite different from the case of an analytic/semianalytic theory, these proper elements are meant to be used exactly in the same way as the others, as an example to identify asteroid families. For any other detail see Milani (1993, and this volume).

2. 4. METHOD BY SCHUBART FOR HILDAS

A synthetic method, based upon the post-processing of numerical integrations, has been developed by Schubart (1982, 1991) to derive "proper parameters" for the Hilda-type asteroids, located in the 3/2 mean motion resonance with Jupiter. The basis of the method is again numerical integration of orbits in the framework of a simplified dynamical model (Sun, Jupiter, Saturn, and solar mass augmented for the masses of the inner planets), covering time spans from $36,500\,yr$ to $109,500\,yr$. Specially defined proper parameters are used in this case, since these asteroids are resonant and no classical secular perturbation is applicable in commensurabilities; the libration gives rise to an oscillation of semimajor axis, for example, and the mean value of this oscillation does not represent a useful quantity, being nearly the same for all Hildas.

The parameters chosen by Schubart to represent the long term characteristics of motion for Hilda asteroids are : $\overline{\sigma}_A$ – a mean amplitude of libration of $\overline{\sigma}$, where the latter is the critical argument in which the argument of perihelion is replaced by its value transformed in such a way to account for the forced effect; $\overline{e}_p$ – a mean value of e_p, where e_p is also a value freed from the forced effect; and i_p – a proper parameter of inclination, a mean value derived from the time series of osculating elements. For the actual derivation of these three parameters, Schubart used a combination of digital filtering and graphical representation of the remaining variations, determining the final results from the respective plots.

Proper parameters were derived in this way for a total of 57 Hildas, two of which, with small eccentricity and a critical argument not in libration (1256 *Normannia* and 4196 1982*SA*13), are treated in a somewhat different way. Although the proper values obtained by Schubart seem to be fairly reliable (almost the same results have been obtained when parameters for nine objects were recomputed by means of an advanced model and different numerical definitions), certain improvements are possible, as an example extended integrations to filter out the very long periodic effects. The dynamical meaning of Schubart's parameters is again very different from that of either analytical or semianalytical proper elements, nevertheless they can be used to identify asteroid families by translating a difference in proper elements into a relative velocity by means of some metric. At least one family of Hildas has been identified by Schubart (1991).

3. Conclusion

Let us try to summarize this overview of the present state of the art regarding the asteroid proper elements :

- There are four different theories and methods currently in use for the computation of proper elements of different populations or groups of asteroids and contributing to the general catalogue (see Milani et al., this volume) : the analytic theory by Milani and Knežević, the semianalytic theory by Lemaitre

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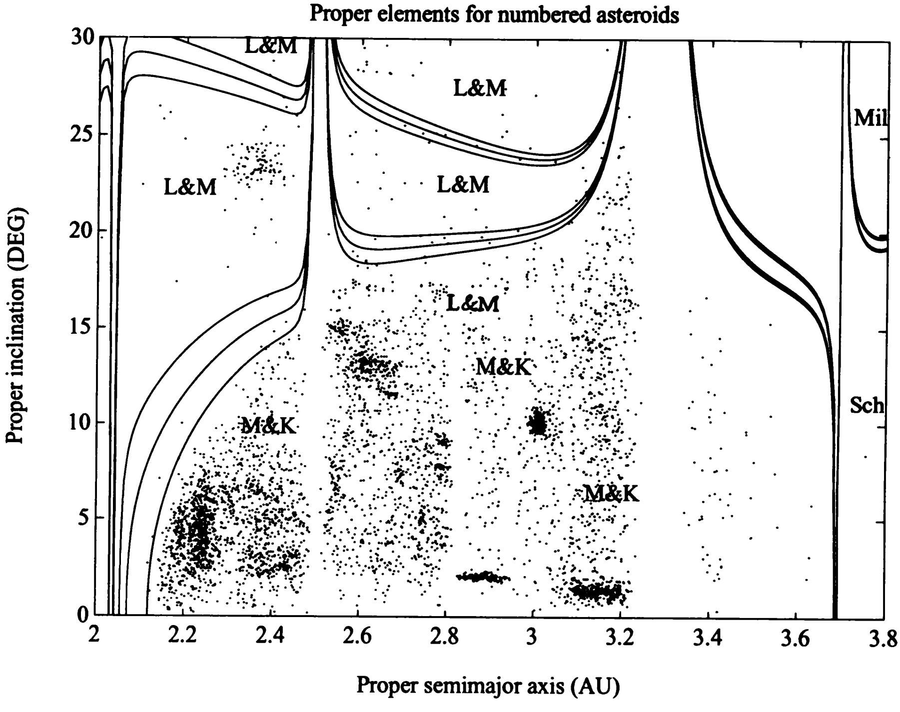

Fig. 4. Asteroid proper elements : the big picture. Labels on the plot pertain to theories recommended for use in particular regions of the belt.

and Morbidelli, the numerical method by Milani for Trojan asteroids, and the numerical-graphical method by Schubart for Hildas.

- Being computed by means of different procedures and even defined in different ways, proper elements are of different accuracy and long term reliability; the user has to bear these differences in mind and has to be particularly cautious when comparing or mixing data from various sources. Most of the data are of accuracy that permits fairly reliable identification of asteroid families against the background, given the number of asteroid orbits presently known.
- As a kind of recipe, one can recommend at the moment to use the theory by Milani and Knežević below 15° of inclination, and the theory by Lemaitre and Morbidelli above 17° (and for special asteroid groupings, like Hungarias and Phocaeas); the region in between can be considered as a transition one in which

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the two theories could be used equivalently or alternatively; for resonant Trojans and Hildas there are special methods by Milani and Schubart. In Figure 4. this recipe is summarized on the plot representing almost the entire main asteroid belt — only the proper elements for numbered asteroids are given here, and positions of only three linear secular resonances are indicated; the labels on the plot indicate the theory to be used in each region.

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