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DERIVATIONS OF HIGHER ORDER IN PRIME RINGS

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ABSTRACT. Let R be a prime ring of characteristic not 2 and d a derivation of R. It is shown that if d^{2n} is a derivation of R, where n is a positive integer, then $d^{2n-1} = 0$.

Let R be a prime ring of characteristics not 2. In [3], Poster shows that if the product of two derivations of R is a derivation, then one of these two derivations must be zero. In particular, if d is a derivation of R and $d^2 = 0$ then $d = 0$. Recently Chung and Luh [1] showed that if d is a derivation of R such that $d^{2n} = 0$, where n is a positive integer, then $d^{2n-1} = 0$, *i.e.*, the index of a nilpotent derivation of a prime ring is necessarily odd. A natural question arises: If d is a derivation of a prime ring R and if d^{2n} is a derivation, is $d^{2n-1} = 0$? This question has been settled by Martindale and Miers [2] if the characteristic of R is greater than n and if d and d^{2n} are both inner derivations of R.

In this paper we will be given an affirmative answer to this question for any derivation d of a prime ring of characteristic not 2, and thereby extend the results of Martindale and Miers as well as that of Chung and Luh.

THEOREM. *Let R be a prime ring of characteristic not 2 and d be a derivation of R. If n is a positive integer such that d^{2n} is a derivation of R, then $d^{2n-1} = 0$.*

Throughout this paper, R is a ring, d a derivation of R, $\mathbb{Z}$ the ring of integers and $d^k(R) = \{d^k(x) \mid x \in R\}$, for any positive integer k.

Let us begin with

LEMMA 1. *If d^{2n} is a derivation of R, then for any $x, y \in R$,*

$$\sum_{j=1}^{2n-1} \binom{2n}{j} d^{2n-j}(x)\, d^j(y) = 0. \tag{1}$$

PROOF. Since d and d^{2n} are derivations of R, by Leibniz' rule, for any $x, y \in R$,

$$d^{2n}(xy) = \sum_{j=0}^{2n} \binom{2n}{j} d^{2n-j}(x)\, d^j(y), \tag{2}$$

and

$$d^{2n}(xy) = d^{2n}(x)y + xd^{2n}(y). \tag{3}$$

Subtracting (3) from (2) side by side yields (1).

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LEMMA 2. *Let $k < n$ be two positive integers. Then there are integers $m_{1,k}, m_{2,k}, \ldots, m_{k-1,k}$, such that*

$$d^{n+k}(x)d^{n-k}(y) + d^{n-k}(x)d^{n+k}(y) = d^{2k}\big(d^{n-k}(x)d^{n-k}(y)\big) + \sum_{i=1}^{k-1} m_{i,k}d^{2k-2i}\big(d^{n-k+i}(x)d^{n-k+i}(y)\big) + (-1)^k 2d^n(x)d^n(y), \tag{4}$$

for all $x, y \in R$.

PROOF. Clearly, (4) is true by Leibniz' rule if $k = 1$. Now assume (4) holds for $k = 1, 2, \ldots, t-1$ where $t < n$. By Leibniz' rule, for any $x, y \in R$,

$$\begin{aligned} d^{2t}\big(d^{n-t}(x)d^{n-t}(y)\big) &= d^{n+t}(x)d^{n-t}(y) + d^{n-t}(x)d^{n+t}(y) \\ &\quad + \sum_{j=1}^{2t-1}\binom{2t}{j} d^{n+t-j}(x)d^{n-t+j}(y) \\ &= d^{n+t}(x)d^{n-t}(y) + d^{n-t}(x)d^{n+t}(y) \\ &\quad + \sum_{j=1}^{t-1}\binom{2t}{j}\big(d^{n+t-j}(x)d^{n-t+j}(y) \\ &\quad + d^{n-t+j}(x)d^{n+t-j}(y)\big) + \binom{2t}{t} d^n(x)d^n(y). \end{aligned}$$

Note that, by our assumption, for $j = 1, 2, \ldots, t-1$, since $1 \le t-j < t$,

$$\begin{aligned} d^{n+t-j}(x)d^{n-t+j}(y) + d^{n-t+j}(x)d^{n+t-j}(y) &= d^{2(t-j)}\big(d^{n-t+j}(x)d^{n-t+j}(y)\big) \\ &\quad + \sum_{i=1}^{t-j-1} m_{i,t-j}d^{2t-2j-2i}\big(d^{n-t+j+i}(x)d^{n-t+j+i}(y)\big) + (-1)^{t-j}2d^n(x)d^n(y). \end{aligned}$$

Therefore,

$$\begin{aligned} d^{2t}\big(d^{n-t}(x)d^{n-t}(y)\big) &= d^{n+t}(x)d^{n-t}(y) + d^{n-t}(x)d^{n+t}(y) \\ &\quad + \sum_{j=1}^{t-1}\binom{2t}{j}\Big[d^{2(t-j)}\big(d^{n-t+j}(x)d^{n-t+j}(y)\big) \\ &\quad + \sum_{i=1}^{t-j-1} m_{i,t-j}d^{2t-2j-2i}\big(d^{n-t+j+i}(x)d^{n-t+j+i}(y)\big) \\ &\quad + (-1)^{t-j}2d^n(x)d^n(y)\Big] + \binom{2t}{t} d^n(x)d^n(y). \end{aligned}$$

By noting that $m_{k,1} = 0$ for all k, the summation

$$\begin{aligned} &\sum_{j=1}^{t-1}\binom{2t}{j}\sum_{i=1}^{t-j-1} m_{i,t-j}d^{2t-2j-2i}\big(d^{n-t+j+i}(x)d^{n-t+j+i}(y)\big) \\ &\quad = \sum_{j=1}^{t-1}\sum_{i=j+1}^{t-1}\binom{2t}{j} m_{i-j,t-j}d^{2(t-i)}\big(d^{n-t+i}(x)d^{n-t+i}(y)\big) \\ &\quad = \sum_{i=2}^{t-1}\Bigg(\sum_{j=1}^{i-1}\binom{2t}{j} m_{i-j,t-j}\Bigg) d^{2(t-i)}\big(d^{n-t+i}(x)d^{n-t+i}(y)\big). \end{aligned}$$

Thus,

$$
\begin{aligned}
d^{2t}&\big(d^{m-t}(x)d^{m-t}(y)\big) \\
&= d^{n+t}(x)d^{n-t}(y) + d^{n-t}(x)d^{n+t}(y) \\
&\quad + \binom{2t}{1} d^{2t-2}\big(d^{n-t+1}(x)d^{n-t+1}(y)\big) \\
&\quad + \sum_{i=2}^{t-1}\left(\binom{2t}{i} + \sum_{j=1}^{i-1}\binom{2t}{j} m_{i-j,t-j}\right) d^{2t-2i}\big(d^{n-t+i}(x)d^{n-t+1}(y)\big) \\
&\quad + \left(\sum_{i=1}^{t-1}\left((-1)^{t-i}\binom{2t}{i}2\right) + \binom{2t}{t}\right) d^n(x)d^n(y) \\
&= d^{n+t}(x)d^{n-t}(y) + d^{n-t}(x)d^{n+t}(y) \\
&\quad - \sum_{i=1}^{t-1} m_{i,t} d^{2t-2i}\big(d^{n-t+i}(x)d^{n-t+i}(y)\big) \\
&\quad - m_0 d^n(x)d^n(y), \text{ where } m_{1,t} = -\binom{2t}{1},
\end{aligned}
$$

$$
m_{i,t} = -\left(\binom{2t}{i} + \sum_{j=1}^{i-1}\binom{2t}{j} m_{i-j,t-j}\right) \text{ and}
$$

$$
\begin{aligned}
m_0 &= -\left(\sum_{i=1}^{t-1}\left((-1)^{t-i}\binom{2t}{i}2\right) + \binom{2t}{t}\right) \\
&= (-1)^{t+1}\left(\sum_{i=1}^{t-1}(-1)^i\binom{2t}{i}2 + (-1)^t\binom{2t}{t}\right) = (-1)^t 2.
\end{aligned}
$$

That is, (4) holds for $k = t$. This completes the proof.

LEMMA 3. *If d^{2n} is a derivation of R, where n is an integer ≥ 2, then, for any x, $y \in R$, $2d^n(x)d^n(y) \in \sum_{j=1}^{n-1} \mathbb{Z} d^{2(n-j)}\big(d^j(x)d^j(y)\big)$.*

PROOF. From (1), we have

$$
\sum_{j=1}^{n-1}\binom{2n}{j}\big(d^{2n-j}(x)d^j(y) + d^j(x)d^{2n-j}(y)\big) + \binom{2n}{n} d^n(x)d^n(y) = 0. \tag{5}
$$

Note that, for $j = 1, \ldots, n-1$,

$$
d^{2n-j}(x)d^j(y) + d^j(x)d^{2n-j}(y) = d^{n+k}(x)d^{n-k}(y) + d^{n-k}(x)d^{n+k}(y),
$$

where $k = n - j$, and hence, by Lemma 2,

$$d^{2n-j}(x)d^j(y) + d^j(x)d^{2n-j}(y) = d^{2n-2j}\big(d^j(x)d^j(y)\big) + \sum_{i=1}^{n-j-1} m_{i,n-j}d^{2n-2j-2i}\big(d^{j+i}(x)d^{j+i}(y)\big) + (-1)^{n-j}2d^n(x)d^n(y).$$

Thus,

$$\sum_{j=1}^{n-1}\binom{2n}{j}\big(d^{2n-j}(x)d^j(y) + d^j(x)d^{2n-j}(y)\big) = \sum_{j=1}^{n-1}\binom{2n}{j}d^{2n-2j}\big(d^j(x)d^j(y)\big) + \sum_{j=1}^{n-1}\binom{2n}{j}\sum_{i=1}^{n-j-1} m_{i,n-j}d^{2n-2j-2i}\big(d^{j+1}(x)d^{j+i}(y)\big) + \sum_{j=1}^{n-1}\binom{2n}{j}(-1)^{n-j}2d^n(x)d^n(y).$$

It follows by (5) that

$$\sum_{j=1}^{n-1}\binom{2n}{j}d^{2n-2j}\big(d^j(x)d^j(y)\big) + \sum_{j=1}^{n-1}\binom{2n}{j}\sum_{i=1}^{n-j-1} m_{i,n-j}d^{2n-2j-2i}\big(d^{j+i}(x)d^{j+i}(y)\big) + \left(\binom{2n}{n} + \sum_{j=1}^{n-1}\binom{2n}{j}(-1)^{n-j}2\right)d^n(x)d^n(y) = 0. \tag{6}$$

Since $\binom{2n}{n} + \sum_{j=1}^{n-1}\binom{2n}{j}(-1)^{n-j}2 = (-1)^{n-1}2$ and all but the last term in the left-hand side of (6) belong to $\sum_{j=1}^{n-1}\mathbb{Z}d^{2n-2j}\big(d^j(x)d^j(y)\big)$, we have $2d^n(x)d^n(y) \in \sum_{j=1}^{n-1}\mathbb{Z}d^{2n-2j}\big(d^j(x)d^j(y)\big)$ as we desired.

LEMMA 4. *If d^{2n} is a derivation of R, where n is an integer ≥ 2, then, for any integer $k \geq 0$,*

$$2^{k+1}\Big(\big(d^{n+k}(R)\big)^2\Big) \subset d^{2n+2k-2}\Big(\big(d(R)\big)^2\Big) + d^{2n+2k-4}\Big(\big(d^2(R)\big)^2\Big) + \cdots + d^{2k+2}\Big(\big(d^{m-1}(R)\big)^2\Big). \tag{7}$$

PROOF. We proceed by induction on k. By Lemma 3, we have $2\big(d^n(R)\big)^2 \subset \sum_{j=1}^{n-1} d^{2n-2j}\Big(\big(d^j(R)\big)^2\Big)$. So (7) holds for $k = 0$. Now we assume that (7) holds for $k = t$. That is,

$$2^{t+1}\big(d^{n+t}(R)\big)^2 \subset d^{2n+2t-2}\Big(\big(d(R)\big)^2\Big) + d^{2n+2t-4}\big((d^2R)^2\big) + \cdots + d^{2t+2}\Big(\big(d^{n-1}(R)\big)^2\Big).$$

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Replacing R by $d(R)$, multiplying by 2 on both sides and using Lemma 3 yields

$$\begin{aligned}2^{t+2}\left(d^{n+t+1}(R)\right)^2 &\subset 2d^{2n+2t-2}\left(\left(d^2(R)\right)^2\right)\\ &\quad + 2d^{2n+2t-4}\left(\left(d^3(R)\right)^2\right) + \cdots + 2d^{2t+2}\left(\left(d^n(R)\right)^2\right)\\ &\subset d^{2n+2t-2}\left(\left(d^2(R)\right)^2\right)\\ &\quad + d^{2n+2t-4}\left(\left(d^3(R)\right)^2\right) + \cdots + d^{2t+4}\left(\left(d^{n-1}(R)\right)^2\right)\\ &\quad + d^{2t+2}\left(\sum_{j=1}^{n-1} d^{2n-2j}\left(\left(d^j(R)\right)^2\right)\right)\\ &\subset d^{2n+2t}\left(\left(d(R)\right)^2\right)\\ &\quad + d^{2n+2t-2}\left(\left(d^2(R)\right)^2\right) + \cdots + d^{2n+4}\left(\left(d^{n-1}(R)\right)^2\right).\end{aligned}$$

That is, (7) holds for $k = t + 1$. Hence (7) holds for any integer $k \geq 0$.

LEMMA 5. *If d^{2n} is a derivation of R, where n is an integer ≥ 2, then $2^{n+1}d^{2n}(R)$ is a subring of R.*

PROOF. Clearly, $2^{n+1}d^{2n}(R)$ is an additive abelian group, and hence we need only to show that it is closed under multiplication. By Lemma 4, for $k = n$, $2^{n+1}\left(d^{2n}(R)\right)^2 \subset d^{4n-2}\left(\left(d(R)\right)^2\right) + d^{4n-4}\left(\left(d^2(R)\right)^2\right) + \cdots + d^{2n+2}\left(\left(d^{n+1}(R)\right)^2\right)$ which is clearly contained in $d^{2n+2}(R)$. Thus,

$$\left(2^{n+1}d^{2n}(R)\right)^2 = 2^{n+1}\left(2^{n+1}\left(d^{2n}(R)\right)^2\right) \subset 2^{n+1}d^{2n+2}(R) \subset 2^{n+1}d^{2n}(R).$$

LEMMA 6. *If R is a prime ring, d^{2n} is a derivation of R and $2n$ is not divisible by the characteristic of R, then either $d^{2n-1} = 0$ or $\ker d = \ker d^2$.*

PROOF. Suppose $\ker d \neq \ker d^2$. Then there exists an $a \in R$ such that $d^2(a) = 0$ but $d(a) \neq 0$. Replacing y by a in (1) yields $2nd^{2n-1}(x)d(a) = 0$ and hence $d^{2n-1}(x)d(a) = 0$, for all $x \in R$. It follows that $d^{2n}(x)d(a) = 0$, for all $x \in R$. Since d^{2n} is a derivation, $0 = d^{2n}(xy)d(a) = d^{2n}(x)yd(a) + xd^{2n}(y)d(a) = d^{2n}(x)yd(a)$ for all $x, y \in R$. By the primeness of R, $d^{2n}(x) = 0$ for all $x \in R$, or $d^{2n} = 0$. By a result in [1], $d^{2n-1} = 0$.

LEMMA 7. *Suppose R is a prime ring, d^{2n} is a derivation of R and $2n$ is not divisible by the characteristic of R. If $d^m = 0$, where m is an integer $\geq 2n$, then $d^{2n-1} = 0$.*

PROOF. Suppose, to the contrary, that $d^{2n-1} \neq 0$. Then by Lemma 6, $\ker d = \ker d^2$. Let k be the smallest integer such that $2n - 1 < k \leq m$ and $d^k = 0$. Since $d^{k-2}(R) \subset \ker d^2$, we have $d^{k-2}(R) \subset \ker d$, or $d^{k-1} = 0$, a contradiction.

LEMMA 8. *Suppose R is a prime ring, d^{2n} is a derivation of R and $2n$ is not divisible by the characteristic of R. If $d^{2n-1} \neq 0$, then $2^{n+1}d^{2n}(R)$ is a prime subring of R.*

PROOF. Let $S = 2^{n+1}d^{2n}(R)$ and $D = d^{2n}$. Then D is a non-zero derivation of R and $S = 2^{n+1}D(R)$ is a subring of R by Lemma 5. We want to show that S is a prime ring. Suppose not, let a, $b \in R$ be such that $D(a) \neq 0$, $D(b) \neq 0$ and $\big(2^{n+1}D(a)\big)\big(2^{n+1}D(x)\big)\big(2^{n+1}D(b)\big) = 0$ for all $x \in R$. That is,

$$D(a)D(x)D(b) = 0, \text{ for all } x \in R \tag{8}$$

Replacing x by $\big(D(x)\big)y$ in (8) yields

$$D(a)\big(D^2(x)y + D(x)D(y)\big)D(b) = 0, \text{ for all } x, y \in R. \tag{9}$$

Note that since

$$2^{2n+2}D(x)D(y) \in S, \qquad 2^{2n+2}D(a)D(x)D(y)D(b) = 0,$$

or $D(a)D(x)D(y)D(b) = 0$, for all $x, y \in R$. Thus (9) becomes $D(a)D^2(x)yD(b) = 0$, for all $x, y \in R$. By the primeness of R again, we obtain that

$$D(a)D^2(x) = 0, \text{ for all } x \in R. \tag{10}$$

In (10), we replace x by $D(x)y$. We obtain

$$D(a)\big(D^3(x)y + 2D^2(x)D(y) + D(x)D^2(y)\big) = 0,$$

for all $x, y \in R$. By (10), we get

$$D(a)D(x)D^2(y) = 0, \text{ for all } x, y \in R. \tag{11}$$

Similarly, in (10), replacing x by $xD(y)$ yields

$$D(a)\big(D^2(x)D(y) + 2D(x)D^2(y) + xD^3(y)\big) = 0.$$

It follows, by (10) and (11), that $D(a)xD^3(y) = 0$, for all $x, y \in R$. Consequently, $D^3 = 0$ or $d^{6n} = 0$. By Lemma 7, $d^{2n-1} = 0$, a contradiction. Hence S is a prime ring.

LEMMA 9. *Let $S_0, S_1, S_2, \ldots, S_{m-1}$ be subrings of a ring R, $m > 1$, and d a derivation of R with $d(S_{m-1}) \subset S_{m-1} \subset d(S_{m-2}) \subset S_{m-2} \subset \cdots \subset d(S_0) \subset S_0$. Suppose*

$$d^m(x)a_m + d^{m-1}(x)a_{m-1} + \cdots + d(x)a_1 = 0,$$

for all $x \in S_0$, where $a_1, a_2, \ldots, a_m \in R$. Then $d(y_{m-1})d(x_{m-1})d(x_{m-2})d(x_{m-3})\cdots d(x_1)a_m = 0$ for all $x_i \in S_i$, $i = 1, 2, \ldots, m-1$, and $y_{m-1} \in S_{m-1}$.

PROOF. We proceed by induction on m, the length of the chain $S_{m-1} \subset S_{m-2} \subset \cdots \subset S_0$.

Suppose $m = 2$, $d(S_1) \subset S_1 \subset d(S_0) \subset S_0$, and

$$d^2(x)a_2 + d(x)a_1 = 0, \text{ for all } x \in S_0. \tag{12}$$

Then, for any $x_1, y_1 \in S_1$, since $y_1d(x_1) \in d(S_0)$, $y_1d(x_1) = d(x)$ for some $x \in S_0$. Replacing $d(x)$ by $y_1d(x_1)$ in (12) yields

$$d\big(y_1 d(x_1)\big)a_2 + y_1 d(x_1)a_1 = 0,$$

or

$$d(y_1)d(x_1)a_2 + y_1\big(d^2(x_1)a_2 + d(x_1)a_1\big) = 0.$$

Thus, $d(y_1)d(x_1)a_2 = 0$ for all $x_1, y_1 \in S_1$. Therefore, Lemma 9 is true for $m = 2$.

Now assume $m > 2$, and assume

$$d^m(x)a_m + d^{m-1}(x)a_{m-1} + \cdots + d(x)a_1 = 0, \text{ for all } x \in S_0. \tag{13}$$

Then, for any $x_1, y_1 \in S_1$, since $y_1 d(x_1) \in d(S_0)$, $y_1 d(x_1) = d(x)$ for some $x \in S_0$. Substituting $d(x)$ by $y_1 d(x_1)$ in (13). We get

$$d^{m-1}\big(y_1 d(x_1)\big)a_m + d^{m-2}\big(y_1 d(x_1)\big)a_{m-1} + \cdots + y_1 d(x_1)a_1 = 0. \tag{14}$$

By Leibniz' rule for each term of (14) and by (13), we obtain

$$d^{m-1}(y_1)d(x_1)a_m + d^{m-2}(y_1)\big((m-1)d^2(x_1)a_m + d(x_1)a_{m-1}\big) + \cdots + y_1\big(d^m(x_1)a_m + \cdots + d(x_1)a_1\big) = 0,$$

which is of the form

$$d^{m-1}(y_1)b_{m-1} + d^{m-2}(y_1)b_{m-2} + \cdots + d(y_1)b_1 = 0,$$

for all $y_1 \in S_1$. Note that $S_{m-1} \subset S_{m-2} \subset \cdots \subset S_1$ is a chain of length $m-1$. By the induction hypothesis,

$$d(y_{m-1})d(x_{m-1})d(x_{m-2})\cdots d(x_2)b_{m-1} = 0,$$

for all $x_i \in S_i$, $i = 2, 3, \ldots, m-1$, and $y_{m-1} \in S_{m-1}$. That is,

$$d(y_{m-1})d(x_{m-1})d(x_{m-2})\cdots d(x_2)d(x_1)a_m = 0,$$

for all $x_i \in S_i$, $i = 2, 3, \ldots, m-1$, $y_{m-1} \in S_{m-1}$. This completes the proof.

LEMMA 10. *Suppose R is a prime ring and d^{2n} is a derivation of R. If $2n$ is not divisible by the characteristic of R, then $d^{2n-1} = 0$.*

PROOF. Suppose to the contrary that $d^{2n-1} \neq 0$. Let $S_0 = R$, and $S_i = 2^{n+1}d^{2n}(S_{i-1})$,

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$i = 1, 2, \ldots, 2n$. By Lemmas 7 and 8, $S_0, S_1, S_2, \ldots, S_{2n-1}$ are prime subrings of R, and $d(S_{2n-1}) \subset S_{2n-1} \subset d(S_{2n-2}) \subset S_{2n-2} \subset \cdots \subset S_0 = R$. From Lemma 1,

$$\binom{2n}{1} d^{2n-1}(x)d(y) + \binom{2n}{2} d^{2n-2}(x)d^2(y) + \cdots + \binom{2n}{2n-1} d(x)d^{2n-1}(y) = 0,$$

for all $x, y \in R$. By Lemma 9, since $2n$ is not divisible by the characteristic of R, we have

$$d(y_{2n-1})d(x_{2n-1})d(x_{2n-2})\cdots d(x_1)d(y) = 0, \tag{15}$$

for all $x_i \in S_i$, $i = 1, 2, \ldots, 2n-1$, $y_{2n-1} \in S_{2n-1}$ and $y \in R$. Replacing y by zy in (15) yields

$$d(y_{2n-1})d(x_{2n-1})d(x_{2n-2})\cdots d(x_1)zd(y) = 0,$$

for all $x_i \in S_i$, $i = 1, 2, \ldots, 2n-1$, $y_{2n-1} \in S_{2n-1}$, and $y, z \in R$. By the primeness of R, we have

$$d(y_{2n-1})d(x_{2n-1})d(x_{2n-2})\cdots d(x_1) = 0, \tag{16}$$

for all $x_i \in S_i$, i=1,2,…,$2n-1$, and $y_{2n-1} \in S_{2n-1}$. Now replace x_1 by y_1x_1 in (16). By the primeness of S_1, it follows that

$$d(y_{2n-1})d(x_{2n-1})d(x_{2n-2})\cdots d(x_2) = 0,$$

for all $x_i \in S_i$, $i = 2, 3, \ldots, 2n-1$, and $y_{2n-1} \in S_{2n-1}$. Continuing this process, finally, we get

$$d(y_{2n-1}) = 0 \text{ for all } y_{2n-1} \in S_{2n-1}.$$

Note that $S_{2n-1} = 2^{(2n-1)(n+1)}d^{2(2n-1)n}(R)$. Thus, we have $2^{(2n-1)(n+1)}d^{2(2n-1)n}(R) = 0$. Since 2 is not the characteristic of R, $d^{2(2n-1)n} = 0$. By Lemma 7, $d^{2n-1} = 0$, a contradiction.

We are now in a position to prove our main result.

PROOF OF THE THEOREM. In view of the last lemma, we need only consider the case that the characteristic p of R is a divisor of n. Let $n = p^k q$, where $p \nmid q$ and $k \geq 1$, and let $D = d^{p^k}$. Then, clearly, D is a derivation of R, $d^{2n} = D^{2q}$, $p \nmid 2q$, and D^{2q} is a derivation of R. By Lemma 10, $D^{2q-1} = 0$, or $d^{2n-p^k} = 0$. Therefore, $d^{2n-1} = 0$, as we desired.

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