BULL. AUSTRAL. MATH. SOC. 17A65
VOL. 23 (1981), 423-428.

SOME DEGENERACY AND PATHOLOGY IN NON-ASSOCIATIVE RADICAL THEORY II

B.J. GARDNER

It is shown that in the universal classes of

(i) all commutative algebras,

(ii) all anti-commutative algebras and

(iii) all algebras satisfying $x^2 = 0$ (over any commutative, associative, unital ring)

the only radical classes with hereditary semi-simple classes are those for which membership is determined by additive structure. Some examples of non-hereditary semi-simple classes in the class of all power-associative algebras are also presented.

In recent years there have been a number of investigations into the question: When are semi-simple classes of rings or algebras hereditary? The most recent examples are the papers of Markovichev [3] and Nikitin [6]; for earlier references see [2] or the survey paper [1]. It is known that for the class of all algebras over any (commutative, associative, unital) ring, semi-simple classes are virtually never hereditary [2], while for alternative rings or algebras they are always hereditary. This prompts speculation concerning the radical behaviour of universal classes between alternative and arbitrary rings and, more generally, in universal classes of algebras satisfying some (non-associative) polynomial identity. In this

Received 24 December 1980.

https://doi.org/10.1017/S0004972700007292 Published online by Cambridge University Press

context the class of power-associative algebras looms large; one can ask whether, radically speaking, the class of power-associative algebras is more like the class of all algebras or more like the class of alternative algebras.

In fact, it can be deduced from an example of Mikheev [4] that for power-associative rings, the nil radical class has a non-hereditary semi-simple class. (Mikheev actually takes his example in the class of $(-1, 1)$-rings; the power-associative result follows because the ring considered retains all relevant properties in the larger universal class.) His example is of characteristic 2, however, and one suspects that in many respects characteristic 2 is "different"; for instance Nikitin [5] has shown that for $(-1, 1)$-algebras over a ring containing $1/2$ and $1/3$, supernilpotent radical classes always have hereditary semi-simple classes.

In [2] we showed that in the class of all algebras the only radical classes with hereditary semi-simple classes are those defined by additive structure. The proof of this result made use of two rings, $\Gamma(A)$ and $\Lambda(A)$, constructed from an arbitrary ring A. While $\Gamma(A)$ and $\Lambda(A)$ can certainly satisfy identities, they will rarely inherit interesting identities from A. Despite this, in [2] we were able to obtain some information about hereditary semi-simple classes in certain product varieties $V \circ V$, where $\Gamma(A)$ and $\Lambda(A)$ are in $V \circ V$ for every $A \in V$.

In this paper we introduce four ring constructions related to $\Gamma(\ )$ and $\Lambda(\ )$ and use them to show that hereditary semi-simple classes must be "additively determined" in the universal classes of

(i) commutative and

(ii) anticommutative algebras and

(iii) algebras satisfying $x^2 = 0$ (all algebras being not necessarily associative).

The algebras in (iii) (coinciding with those in (ii) when $1/2$ is a scalar) are power-associative. The results just described thus enable us to present examples of non-hereditary semi-simple classes of power-associative algebras without constraint on the ring of scalars.

https://doi.org/10.1017/S0004972700007292 Published online by Cambridge University Press

The results

We shall work in varieties (designated "universal") of algebras over a commutative, associative, unital ring; the zero algebra on the additive module of an algebra A will be called A^0 ; $\lhd$ will mean "is an ideal of".

The first result is crucial to our argument, but since it is proved by exactly the same arguments as were used to establish a special case in Propositions 2.2 and 2.3 and Theorem 2.4 of [2], we omit the details.

LEMMA 1. *Let A be an algebra in a given universal variety W which also contains A^0. Suppose there exist algebras B, C in W such that*

(i) $A \oplus A^0 \lhd B$, $B/A \oplus A^0 \cong A^0$,

(ii) $A \oplus A^0 \lhd C$, $C/A \oplus A^0 \cong A$,

(iii) $A \oplus \left(A^2\right)^0$ *is the ideal of both B and C generated by A and*

(iv) $A^2 \oplus A^0$ *is the ideal of both B and C generated by A^0.*

Then if R is a radical class in W whose semi-simple class S is hereditary, we have

$$A \in R \Leftrightarrow A^0 \in R \quad \textit{and} \quad A \in S \Leftrightarrow A^0 \in S \,.$$

We next introduce our constructions. Let A be any algebra. We define four algebras $\Gamma^+(A)$, $\Gamma^-(A)$, $\Lambda^+(A)$ and $\Lambda^-(A)$ on the direct sum of three copies of the additive module of A by the following multiplications:

$$\left(x_1, y_1, z_1\right)\left(x_2, y_2, z_2\right) = \begin{cases} \left(x_1x_2+z_1y_2+z_2y_1,\ z_2x_1+z_1x_2,\ 0\right) & \text{in } \Gamma^+(A) \,, \\ \left(x_1x_2+z_1y_2-z_2y_1,\ z_2x_1-z_1x_2,\ 0\right) & \text{in } \Gamma^-(A) \,, \\ \left(x_1x_2+z_1y_2+z_2y_1,\ z_2x_1+z_1x_2,\ z_1z_2\right) & \text{in } \Lambda^+(A) \,, \\ \left(x_1x_2+z_1y_2-z_2y_1,\ z_2x_1-z_1x_2,\ z_1z_2\right) & \text{in } \Lambda^-(A) \,. \end{cases}$$

The relevant properties of the constructed algebras are summarized in the next result.

PROPOSITION 2. *(i)* $A \oplus A^0 \lhd \Gamma^+(A)$, $\Gamma^-(A)$, $\Lambda^+(A)$ *and* $\Lambda^-(A)$; $\Gamma^+(A)/A \oplus A^0 \cong A^0 \cong \Gamma^-(A)/A \oplus A^0$; $\Lambda^+(A)/A \oplus A^0 \cong A \cong \Lambda^-(A)/A \oplus A^0$.

(ii) If A *is commutative, then so are* $\Gamma^+(A)$ *and* $\Lambda^+(A)$.

(iii) If A *is anticommutative, then so are* $\Gamma^-(A)$ *and* $\Lambda^-(A)$.

(iv) If A *satisfies the identity* $x^2 = 0$ *, then so do* $\Gamma^-(A)$ *and* $\Lambda^-(A)$.

(v) In each constructed algebra the ideal generated by A *(respectively* A^0 *) is* $A \oplus \left(A^2\right)^0$ *(respectively* $A^2 \oplus A^0$ *).*

Proof. *(i)* The embedding of $A \oplus A^0$ in each algebra is given by

$$(a, b) \mapsto (a, b, 0) .$$

(ii)-(iv) and the rest of *(i)* are straightforward.

(v) Consider $\Gamma^+(A)$. We have

$$(x, y, z)\left(a, \sum b_ic_i, 0\right) = \left(xa + \sum z\left(b_ic_i\right), za, 0\right)$$

for any $x, y, z, a, b_i, c_i \in A$, so by commutativity, $A \oplus \left(A^2\right)^0 \lhd \Gamma^+(A)$. On the other hand, since for any $u, v_i, w_i \in A$,

$$\left(u, \sum v_iw_i, 0\right) \doteq (u, 0, 0) + \sum \left(w_i, 0, 0\right)\left(0, 0, v_i\right)$$

is in the ideal generated by A , the latter ideal must be $A \oplus \left(A^2\right)^0$. The other parts of *(v)* are proved by similar arguments. //

THEOREM 3. *Let* $\mathcal{W}$ *be either*

(i) the class of all commutative algebras,

(ii) the class of all anticommutative algebras or

(iii) the class of algebras satisfying the identity $x^2 = 0$.

If $\mathcal{R}$ *is a radical class in* $\mathcal{W}$ *whose semi-simple class* S *is hereditary, then for any* $A \in \mathcal{W}$ *,*

$$A \in R \Leftrightarrow A^0 \in R \quad \textit{and} \quad A \in S \Leftrightarrow A^0 \in S \; .$$

Proof. By Proposition 2, we can obtain the results from Lemma 1, using $\Gamma^+(\ )$ and $\Lambda^+(\ )$ for *(i)*, $\Gamma^-(\ )$ and $\Lambda^-(\ )$ for *(ii)* and *(iii)*. //

A radical class is an *A-radical* class if it contains, along with any algebra R , all other algebras with additive modules isomorphic to that of R . A *strict radical class* is one whose semi-simple class is closed under subalgebras.

COROLLARY 4. *In the universal varieties of commutative algebras, anticommutative algebras and algebras satisfying* $x^2 = 0$, *the following conditions are equivalent for a radical class* R :

(i) R *has a hereditary semi-simple class;*

(ii) R *is strict;*

(iii) R *is an A-radical class.*

Since algebras satisfying $x^2 = 0$ are clearly power-associative, by using Lemma 1 and Proposition 2 for individual algebras rather than the whole universal class, we get

THEOREM 5. *In the universal variety of power-associative algebras, if* R *is a radical class whose semi-simple class* S *is hereditary and* A *is an algebra satisfying* $x^2 = 0$, *then*

$$A \in R \Leftrightarrow A^0 \in R \quad \textit{and} \quad A \in S \Leftrightarrow A^0 \in S \; .$$

Let $L(X)$, $U(X)$ denote, respectively, the lower and upper radical classes generated by a homomorphically closed hereditary class X . (This is sufficiently general for our purposes.) Let Z denote the class of all zero algebras.

COROLLARY 6. *In the universal variety of power-associative algebras, there are non-hereditary semi-simple classes. Specifically, if* K *is a non-empty class of simple non-nilpotent algebras satisfying* $x^2 = 0$ *and* R *is a radical class such that either* $L(Z) \subseteq R \subseteq U(K)$ *or* $L(K) \subseteq R \subseteq U(Z)$, *then* R *does not have a hereditary semi-simple class.*

Proof. Note that there *are* simple algebras as described, for example, Lie algebras. If $S \in K$ and $L(Z) \subseteq R \subseteq U(K)$, then $S^0 \in R$ and $R(S) = 0$, while if $L(K) \subseteq R \subseteq U(Z)$, then $S \in R$ and $R(S^0) = 0$; in either case, Theorem 5 says that R has a non-hereditary semi-simple class. //

COROLLARY 7. *In the universal variety of power-associative algebras, neither the Baer lower radical class* $L(Z)$ *nor the idempotent radical class* $U(Z)$ *has a hereditary semi-simple class.*

References

[1] B.J. Gardner, "Some current issues in radical theory", *Math. Chronicle* 8 (1979), 1-23.

[2] B.J. Gardner, "Some degeneracy and pathology in non-associative radical theory", *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 22-23 (1979-80), 65-74.

[3] A.C. Марковичев [A.S. Markovichev], "О наследственности радикалов колец типа (γ, δ)" [On the hereditariness of radicals of (γ, δ)-rings], *Algebra i Logika* 17 (1978), 33-55.

[4] I.M. Mikheev, "Prime right-alternative rings", *Algebra and Logic* 14 (1976), 34-36.

[5] A.A. Никитин [A.A. Nikitin], "О наднильпотентных радикалах (-1, 1)-колец" [On supernilpotent radicals of (-1, 1)-rings], *Algebra i Logika* 12 (1973), 305-311.

[6] A.A. Никитин [A.A. Nikitin], "О наслелственности радикалов колец" [On the hereditariness of radicals of rings], *Algebra i Logika* 17 (1978), 303-315.

Department of Mathematics,
University of Tasmania,
GPO Box 252C,
Hobart,
Tasmania 7001,
Australia.

https://doi.org/10.1017/S0004972700007292 Published online by Cambridge University Press