

# On non-dimensional forms of basal sliding laws and flow laws for ice-sheet and glacier modelling

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**ABSTRACT.** Ice sheets and glaciers flow through basal sliding and internal deformation, each governed by physical laws commonly expressed as power-law relations. These formulations include coefficients – the sliding coefficient and rate factor – whose values and units depend on the respective exponents. This dependency complicates the systematic exploration of parameter space, especially in ensemble simulations. To address this, we propose dimensionless formulations of both sliding and flow laws, in which the coefficients are of order unity and decoupled from the exponents. This separation simplifies sensitivity studies and parameter variations. The dimensionless laws are straightforward to implement in existing models; we demonstrate this with the SICOPOLIS ice-sheet model using three test simulations in an idealized set-up. These simulations illustrate that independent variation of exponents and coefficients is feasible and practical, supporting the use of dimensionless laws in efforts to better constrain ice dynamics in past and future climate scenarios.

## 1 INTRODUCTION

Ice sheets and glaciers flow due to two different processes, namely basal sliding and internal deformation. Basal sliding describes the sliding of glacier ice on the underlying substrate, which can be either hard bedrock or a deformable sediment layer between ice and bedrock. Internal deformation is governed by the

non-linear viscous properties of “hot” polycrystalline ice (that is, with a homologous temperature  $T/T_m$  near unity, where  $T$  is the absolute temperature and  $T_m$  the pressure melting point).

In a dynamic/thermodynamic ice sheet or glacier model, both processes must be included. Basal sliding, which in reality is a complex process that depends on a multitude of factors such as the basal temperature, roughness of the bedrock, softness of the subglacial sediment layer (if existing) and hydrological conditions, is usually parameterized by a sliding law that relates the sliding velocity to the basal stresses. Internal deformation can be modelled by a non-linear viscous flow law that describes the relation between the macroscopic deformation (strain rate) and internal stresses (e.g., Hooke, 2005; Greve and Blatter, 2009; Cuffey and Paterson, 2010).

Popular forms for such relations are the Weertman–Budd sliding law and the Nye–Glen flow law (see below for references). They have in common that they are expressed as power laws with some exponents, of which the optimal values are debated, and contain a factor to close the respective equation. This factor, the “sliding coefficient” in case of the sliding law and the “rate factor” in case of the flow law, may contain remaining dependencies, such as on the temperature. In a dimensional formulation, the units and numerical values of these factors depend strongly on the choice of the exponents, which makes it cumbersome to vary the exponents over their potential range of values, for instance, within an ensemble of simulations for a given scenario. To overcome this obstacle, we propose fully or partly dimensionless versions of the sliding and flow laws, which have in common that the respective factor is dimensionless and generally of order unity. These formulations decouple the value of the exponents from the value of the factor, so that the factors and exponents can be varied independently. We demonstrate this useful feature by some simple simulations with the ice-sheet model SICOPOLIS (SIMulation CODE for POLythermal Ice Sheets; SICOPOLIS Authors, 2025).

## 2 BASAL SLIDING LAWS

Basal sliding laws (aka basal friction laws) relate the shear stress (drag) at the base of an ice sheet or glacier,  $\tau_b$ , to the basal normal stress,  $N_b$  and the basal sliding velocity,  $v_b$ . In a general, implicit form, a basal sliding law can be expressed as

$$f(v_b, \tau_b, N_b) = 0, \quad (1)$$

where  $f$  is a function unspecified at this stage. The list of variables is not necessarily exhaustive as further dependencies, for instance on basal temperature or the presence of basal water, may be included. Note

that, in the presence of subglacial water, the basal normal stress is often understood as the difference between the ice overburden stress,  $N_{b,i}$ , and the basal water pressure,  $p_{b,w}$ ,

$$N_b = N_{b,i} - p_{b,w}, \quad (2)$$

and then called the reduced normal stress or, alternatively, the effective pressure.

A popular form of a basal sliding law is the Weertman–Budd sliding law, which results when assuming an explicit form of Eq. (1) solved for  $v_b$ , with power-law dependencies on  $\tau_b$  and  $N_b$ :

$$v_b = C_b \frac{\tau_b^p}{N_b^q}, \quad (3)$$

where  $C_b$  is the sliding coefficient and  $(p, q)$  are the non-negative sliding exponents (Weertman, 1957; Budd and others, 1979, 1984; Budd and Jenssen, 1987). Alternatively, Eq. (3) can be solved for the shear stress,

$$\tau_b = C_b^* v_b^{1/p} N_b^{q/p}, \quad \text{with } C_b^* = C_b^{-1/p}, \quad (4)$$

where  $C_b^*$  is the basal friction coefficient. For the case  $q = 0$ , that is, ignoring the dependence on the normal stress  $N_b$ , the above forms are often referred to as the Weertman sliding law.

In principle,  $v_b$  and  $\tau_b$  are vector quantities. For simplicity, we formulate the sliding laws only with the respective magnitudes. However, to interpret the results correctly, it must be kept in mind that  $v_b$  and  $\tau_b$  are anti-parallel to each other due to the nature of friction.

Let us now non-dimensionalize the sliding law (3) by introducing scales (typical values) for the relevant quantities (e.g., Hutter and Jöhnk, 2004). We consider a situation near the edge of an ice sheet where basal sliding is most relevant:

$$[H] = 1 \text{ km} \quad (\text{typical thickness}), \quad (5a)$$

$$\varepsilon = 10^{-2} \quad (\text{typical surface slope}), \quad (5b)$$

$$[N_b] = \rho g [H] = 10^7 \text{ Pa} \quad (\text{typical normal stress}), \quad (5c)$$

$$[\tau_b] = \varepsilon [N_b] = 10^5 \text{ Pa} \quad (\text{typical shear stress}), \quad (5d)$$

$$[v_b] = 100 \text{ m a}^{-1} \quad (\text{typical sliding velocity}), \quad (5e)$$

where we have used approximate values  $\rho \approx 10^3 \text{ kg m}^{-3}$  for the ice density and  $g \approx 10 \text{ m s}^{-2}$  for the acceleration due to gravity, which is sufficiently accurate for the sake of a scaling analysis. This scaling is consistent with the linear sliding law  $[C_b = 10^{-3} \text{ m a}^{-1} \text{ Pa}^{-1}, (p, q) = (1, 0)]$  used for the EISMINT Phase 2 Simplified Geometry Experiments (Payne and others, 2000):

$$\underbrace{100 \text{ m a}^{-1}}_{v_b} = \underbrace{10^{-3} \text{ m a}^{-1} \text{ Pa}^{-1}}_{C_b} \times \underbrace{10^5 \text{ Pa}}_{\tau_b}. \quad (6)$$

An appropriate choice for the scale of the sliding coefficient results from Eq. (3) as

$$[C_b] = \frac{[v_b][N_b]^q}{[\tau_b]^p}. \quad (7)$$

We now use the above scales to introduce dimensionless quantities as follows:

$$v_b = [v_b] \tilde{v}_b, \quad (8a)$$

$$\tau_b = [\tau_b] \tilde{\tau}_b, \quad (8b)$$

$$N_b = [N_b] \tilde{N}_b, \quad (8c)$$

$$C_b = [C_b] \tilde{C}_b, \quad (8d)$$

where the quantities marked by the tilde symbol are the non-dimensional basal sliding velocity, shear stress, normal stress and sliding coefficient, respectively. Inserting Eqs. (8) in the sliding law (3) yields its fully non-dimensional form,

$$\tilde{v}_b = \tilde{C}_b \frac{\tilde{\tau}_b^p}{\tilde{N}_b^q}, \quad (9)$$

in which all quantities are supposed to be of order unity.

A dimensional form of Eq. (3) can be kept by only making use of the scaling (8d) of the sliding coefficient,

$$v_b = [C_b] \tilde{C}_b \frac{\tau_b^p}{N_b^q}, \quad (10)$$

which has the advantage that its implementation in an existing model based on dimensional quantities requires only minimal adaptations.

In order to obtain the fully or partly dimensionless counterparts of Eq. (4), we note the scaling and

non-dimensionalization of the friction coefficient  $C_b^*$ :

$$C_b^* = [C_b^*] \tilde{C}_b^*, \quad \text{with } [C_b^*] = [C_b]^{-1/p} = \frac{[\tau_b]}{[v_b]^{1/p} [N_b]^{q/p}}. \quad (11)$$

The fully non-dimensional form of Eq. (4) results then as

$$\tilde{\tau}_b = \tilde{C}_b^* \tilde{v}_b^{1/p} \tilde{N}_b^{q/p}, \quad (12)$$

and the dimensional form in which only the scaling (11) of the friction coefficient is used reads

$$\tau_b = [C_b^*] \tilde{C}_b^* v_b^{1/p} N_b^{q/p}. \quad (13)$$

Why do we promote using Eqs. (10) or (13) instead of Eqs. (3) or (4) in an ice sheet or glacier model? In Table 1 we have compiled some parameter combinations that were used along with Weertman or Weertman–Budd sliding laws in the literature. The impossibility of comparing the various dimensional sliding coefficients  $C_b$  for different exponents  $(p, q)$  becomes immediately evident. They do not even have a common unit, and the respective numerical value tells nothing about the actual strength of basal sliding. In the second case,  $(p, q) = (1, 2)$ , the numerical value of  $C_b$  is greater than  $10^9$ ; however, the small dimensionless value means that it produces only very little basal sliding ( $\tilde{C}_b \approx 0.04$ ). By contrast, in the third case,  $(p, q) = (3, 0)$ , the numerical value of  $C_b$  is merely  $10^{-12}$ ; however, it corresponds to pronounced basal sliding ( $\tilde{C}_b = 10$ ). The dimensionless sliding coefficients  $\tilde{C}_b$  give a much better idea about what the respective value means physically, and allow comparing values across different sliding laws.

To further strengthen our point, suppose that we wish to test a sliding law with a new set of exponents, for instance  $(p, q) = (3, 1.5)$ . Working with the dimensional sliding coefficient  $C_b$ , we would not have any idea which order of magnitude may be suited for its numerical value, and which range of values mean strong or weak sliding. However, if the dimensionless sliding coefficient  $\tilde{C}_b$  is used, we can immediately start with an initial guess  $\tilde{C}_b = 1$  and, from there on, refine the sliding law by, e.g., tuning to observed flow speeds. According to the scaling (7), (8d), the dimensional equivalent of  $\tilde{C}_b = 1$  would be  $C_b = [C_b] = 10^{-2.5} \text{ m a}^{-1} \text{ Pa}^{-1.5} = 3.162 \times 10^{-3} \text{ m a}^{-1} \text{ Pa}^{-1.5}$ .

We have only discussed cases with a constant sliding parameter; however, the non-dimensionalization method is of course not limited to this. It can also be applied to a spatially variable sliding coefficient, which may arise from an inversion procedure (e.g., Morlighem and others, 2013). Alternative sliding laws, such as

| $(p, q)$ | $C_b$                                                   | $[C_b]$                                     | $\tilde{C}_b$ | Reference                            |
|----------|---------------------------------------------------------|---------------------------------------------|---------------|--------------------------------------|
| (1, 0)   | $10^{-3} \text{ m a}^{-1} \text{ Pa}^{-1}$              | $10^{-3} \text{ m a}^{-1} \text{ Pa}^{-1}$  | 1             | Payne and others (2000)              |
| (1, 2)   | $3.985 \times 10^9 \text{ m a}^{-1} \text{ Pa}$         | $10^{11} \text{ m a}^{-1} \text{ Pa}$       | 0.03985       | Budd and others (1984) <sup>†</sup>  |
| (3, 0)   | $10^{-12} \text{ m a}^{-1} \text{ Pa}^{-3}$             | $10^{-13} \text{ m a}^{-1} \text{ Pa}^{-3}$ | 10            | Cornford and others (2020)           |
| (3, 1)   | $1.607 \times 10^{-6} \text{ m a}^{-1} \text{ Pa}^{-2}$ | $10^{-6} \text{ m a}^{-1} \text{ Pa}^{-2}$  | 1.607         | Saito and others (2016) <sup>†</sup> |
| (3, 2)   | $6.72 \text{ m a}^{-1} \text{ Pa}^{-1}$                 | $10 \text{ m a}^{-1} \text{ Pa}^{-1}$       | 0.672         | Rückamp and others (2019)            |

**Table 1.** Sliding exponents  $(p, q)$ , dimensional sliding coefficients  $C_b$ , scales  $[C_b]$  and dimensionless sliding coefficients  $\tilde{C}_b$  for several Weertman ( $q = 0$ ) or Weertman–Budd ( $q > 0$ ) sliding laws used in the literature.

<sup>†</sup>: Rather than using the normal stress  $N_b$ , these sliding laws were formulated with the pressure head  $Z = N_b/(\rho g)$ . We converted the sliding coefficients given in these studies accordingly, using  $\rho = 910 \text{ kg m}^{-3}$  and  $g = 9.81 \text{ m s}^{-2}$ .

the Coulomb-limited rules discussed by Cornford and others (2020), allow similar non-dimensionalization, although we refrain from working out the details here.

### 3 FLOW LAWS

A similar problem of units and hugely varying numerical values arises for the flow law of polycrystalline ice. It is a viscous flow law that relates the strain-rate (stretching) tensor  $d_{ij}$  to the stress deviator  $t_{ij}^D$ . The strain-rate tensor is defined as

$$d_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (i, j = 1, 2, 3), \quad (14)$$

where  $x_i$  denotes the Cartesian coordinates ( $x_1 = x$ ,  $x_2 = y$ ,  $x_3 = z$ ), and  $v_i$  is the velocity vector. The stress deviator is the traceless part of the full stress tensor  $t_{ij}$ ,

$$t_{ij} = -p \delta_{ij} + t_{ij}^D, \quad (15)$$

where  $p = -t_{ii}/3$  is the pressure (we assume the Einstein summation convention: summation over the twice-appearing index  $i$  implied, thus  $t_{ii}$  is the trace of the stress tensor), and  $\delta_{ij}$  is the Kronecker delta symbol, in other words, the unit tensor in index notation.

For the flow law, usually collinearity between the symmetric tensors  $d_{ij}$  and  $t_{ij}^D$  is assumed. We note

the form given by Greve and Blatter (2009),

$$d_{ij} = Af(\tau_e)t_{ij}^D, \quad (16)$$

where  $A$  is the rate factor,  $\tau_e = [(t_{ij}^D t_{ij}^D)/2]^{1/2}$  the effective stress (summation over both  $i$  and  $j$  implied), and  $f(\tau_e)$  is the creep function. The rate factor depends on the temperature relative to the pressure melting point,  $T'$ , via an Arrhenius law (e.g., Cuffey and Paterson, 2010), but it is sometimes chosen as a constant parameter for simplicity. In the Nye–Glen flow law (Glen, 1955; Nye, 1957), the creep function is expressed as a power law,

$$f(\tau_e) = \tau_e^{n-1}, \quad (17)$$

so that

$$d_{ij} = A\tau_e^{n-1}t_{ij}^D, \quad (18)$$

where  $n$  is the stress exponent. A value of  $n = 1$  would correspond to a Newtonian fluid; however, the deformability of ice differs markedly from that behaviour, and the value is frequently chosen as  $n = 3$ , or within the range from 1.5 to 4.2 (Cuffey and Paterson, 2010) (while recent evidence from laboratory experiments actually supports  $n = 1$  for temperate ice; Schohn and others, 2025).

The Nye–Glen flow law (18) can also be inverted for the stress deviator,

$$t_{ij}^D = A^*d_e^{-(1-1/n)}d_{ij}, \quad \text{with } A^* = A^{-1/n}, \quad (19)$$

where  $A^*$  is the associated rate factor and  $d_e = [(d_{ij}d_{ij})/2]^{1/2}$  the effective strain rate (e.g., Greve and Blatter, 2009).

Similar to the procedure in Sect. 2, we now introduce scales (typical values) for the relevant quantities, considered suitable for areas where rather large deformations take place:

$$[\tau] = 10^5 \text{ Pa} \quad (\text{typical deviatoric stress}), \quad (20a)$$

$$[d] = 2.5 \times 10^{-2} \text{ a}^{-1} \quad (\text{typical strain rate}), \quad (20b)$$

where the scale  $[\tau]$  is deemed appropriate for both  $t_{ij}^D$  and  $\tau_e$ . Using Eq. (18) entails the choice for the

scale of the rate factor:

$$[A] = \frac{[d]}{[\tau]^n}. \quad (21)$$

We introduce the dimensionless quantities

$$t_{ij}^D = [\tau] \tilde{t}_{ij}^D, \quad (22a)$$

$$\tau_e = [\tau] \tilde{\tau}_e, \quad (22b)$$

$$d_{ij} = [d] \tilde{d}_{ij}, \quad (22c)$$

$$A = [A] \tilde{A}, \quad (22d)$$

where the quantities marked by the tilde symbol are the non-dimensional components of the stress deviator, effective stress, components of the strain-rate tensor and rate factor, respectively.

For  $n = 3$ , this yields  $[A] = 2.5 \times 10^{-17} \text{ a}^{-1} \text{ Pa}^{-3} = 7.922 \times 10^{-25} \text{ s}^{-1} \text{ Pa}^{-3}$ . This value is close to the recommendation by Cuffey and Paterson (2010) for  $T' = -6^\circ\text{C}$ , which demonstrates the validity of our scaling.

We obtain the fully non-dimensional form of the flow law (18) as

$$\tilde{d}_{ij} = \tilde{A} \tilde{\tau}_e^{n-1} \tilde{t}_{ij}^D, \quad (23)$$

(all quantities supposed to be of order unity). A form with only  $A$  scaled results if we only apply the scaling (22d):

$$d_{ij} = [A] \tilde{A} \tau_e^{n-1} t_{ij}^D. \quad (24)$$

Analogous to the sliding law (10), this form can be implemented in a model based on dimensional quantities with only minimal changes.

To obtain the fully or partly dimensionless versions of Eq. (19), we note the scaling and non-dimensionalization of the associated rate factor  $A^*$ ,

$$A^* = [A^*] \tilde{A}^*, \quad \text{with } [A^*] = [A]^{-1/n} = \frac{[\tau]}{[d]^{1/n}}, \quad (25)$$



**Fig. 1.** Dimensionless rate factor  $\tilde{A}$  as a function of the temperature relative to pressure melting  $T'$ , following the recommendation by Cuffey and Paterson (2010): Arrhenius law with activation energies  $Q = 60 \text{ kJ mol}^{-1}$  for  $T' \leq -10^\circ\text{C}$ ,  $Q = 115 \text{ kJ mol}^{-1}$  for  $T' \geq -10^\circ\text{C}$ ,  $A = 3.5 \times 10^{-25} \text{ s}^{-1} \text{ Pa}^{-3}$  for  $T' = -10^\circ\text{C}$  and  $n = 3$ .

and for the effective strain rate, the scale (20b) is used,

$$d_e = [d] \tilde{d}_e. \quad (26)$$

The fully non-dimensional form of Eq. (19) is then

$$\tilde{t}_{ij}^D = \tilde{A}^* \tilde{d}_e^{-(1-1/n)} \tilde{d}_{ij}, \quad (27)$$

and the form with only  $A^*$  scaled reads

$$t_{ij}^D = [A^*] \tilde{A}^* \tilde{d}_e^{-(1-1/n)} d_{ij}, \quad (28)$$

Figure 1 shows the temperature-dependent rate factor following the recommendation by Cuffey and Paterson (2010), non-dimensionalized with the scaling (21), (22d). However, while the original recommendation in dimensional form is valid only for  $n = 3$ , this dimensionless form can be used for any value of the stress exponent  $n$ . It is therefore much more flexible.

Consider the case  $n = 4$ , which has recently been discussed by, e.g., Bons and others (2018); Millstein and others (2022); Getraer and Morlighem (2025). The unit of the rate factor  $A$  must then be  $\text{a}^{-1} \text{ Pa}^{-4}$ , but what about suitable numerical values? When using the dimensionless formulation promoted here, there is no problem;  $\mathcal{O}(1)$  values will be suitable for the high-temperature regime (i.e., temperatures close to pressure melting) just like for any other value of  $n$ , and we can use the function shown in Figure 1 as a starting point. According to Eq. (21), the scale for  $A$  changes to  $[A] = 2.5 \times 10^{-22} \text{ a}^{-1} \text{ Pa}^{-4}$  (compared to  $[A] = 2.5 \times 10^{-17} \text{ a}^{-1} \text{ Pa}^{-3}$  for  $n = 3$ ), so that  $\tilde{A} = 1$  corresponds to  $A = 2.5 \times 10^{-22} \text{ a}^{-1} \text{ Pa}^{-4}$ .

The non-dimensionalization method is not limited to the Nye-Glen flow law discussed above. A straightforward extension is for its regularized version with the creep function  $f(\tau_e) = \tau_e^{n-1} + \tau_0^{n-1}$ , where  $\tau_0$  is the residual stress, a small constant introduced to avoid the infinite-viscosity limit for vanishing stresses or strain rates (Greve and Blatter, 2009). For this flow law, Eqs. (20)–(22) remain applicable. However,

it cannot be analytically inverted for the stress deviator; this is only possible by numerically solving an implicit equation. Further flow laws shall not be considered here.

## 4 TESTS WITH THE ICE-SHEET MODEL SICOPOLIS

Replacing the previously implemented, dimensional versions, the Weertman–Budd basal sliding law formulated with the dimensionless sliding coefficient, Eqs. (10) and (13), and the Nye–Glen flow law formulated with the dimensionless rate factor, Eqs. (24) and (28), have both been introduced in the ice-sheet model SICOPOLIS v25 (SICOPOLIS Authors, 2025), revision 3e7e6c939 of 17 June 2025. Details of the implementation can be found in Sect. 6 (“Modelling choices”) of the ReadTheDocs manual at <https://sicopolis.readthedocs.io/> (last access: 26 June 2025).

We briefly demonstrate the benefit of the new formulation by considering experiment H of the EISMINT Phase 2 Simplified Geometry Experiments (Payne and others, 2000). The original set-up of this experiment uses the Weertman sliding law with  $(p, q) = (1, 0)$  and  $C_b = 10^{-3} \text{ m a}^{-1} \text{ Pa}^{-1}$ , only applied for a temperate base, while no-slip conditions are assumed for a cold base. The value of the sliding coefficient corresponds to  $\tilde{C}_b = 1$  (Table 1). The applied flow law is Nye–Glen with  $n = 3$ .

We define three versions of the experiment. For all cases, to avoid the singularity associated with the binary switch between fully developed sliding and no-slip conditions, we allow for some exponentially decaying sub-melt sliding (e.g., Hindmarsh and Le Meur, 2001; Greve, 2005; Dunse and others, 2011) by setting  $\tilde{C}_b \rightarrow \tilde{C}_b \exp(T'_b/\gamma_{\text{sms}})$ , where  $T'_b$  is the basal temperature relative to the pressure melting point (in °C, hence  $T'_b$  is non-positive), and  $\gamma_{\text{sms}} = 3^\circ\text{C}$  is the sub-melt-sliding parameter. Further, for all cases, we apply the dimensionless rate factor shown in Figure 1, which differs slightly from the original set-up. Experiment H1 employs linear Weertman sliding and the Nye–Glen flow law with  $n = 3$  as in the original set-up. For experiment H2, sliding has been changed to a Weertman–Budd law with  $(p, q) = (3, 2)$  [no basal water pressure considered,  $p_{b,w} = 0$ , cf. Eq. (2)], and for experiment H3, the stress exponent has been changed to  $n = 4$  (Table 2).

All experiments are carried out with SICOPOLIS v25. Ice dynamics is modelled by the depth-integrated viscosity approximation (DIVA) (Goldberg, 2011; Lipscomb and others, 2019; Grandadam, 2024), and for ice thermodynamics we employ the melting-CTS enthalpy method (CTS: cold-temperate transition surface) (Greve and Blatter, 2016). For the horizontal resolution, we use 10 km (rather than the 25 km from the original set-up), and we integrate over a model time of 100 ka starting from ice-free initial conditions, which

| Experiment | $(p, q)$ | $\tilde{C}_b$ | $n$ | $\tilde{A}$               |
|------------|----------|---------------|-----|---------------------------|
| H1         | (1, 0)   | 1             | 3   | $\tilde{A}_{\text{CP10}}$ |
| H2         | (3, 2)   | 1             | 3   | $\tilde{A}_{\text{CP10}}$ |
| H3         | (1, 0)   | 1             | 4   | $\tilde{A}_{\text{CP10}}$ |

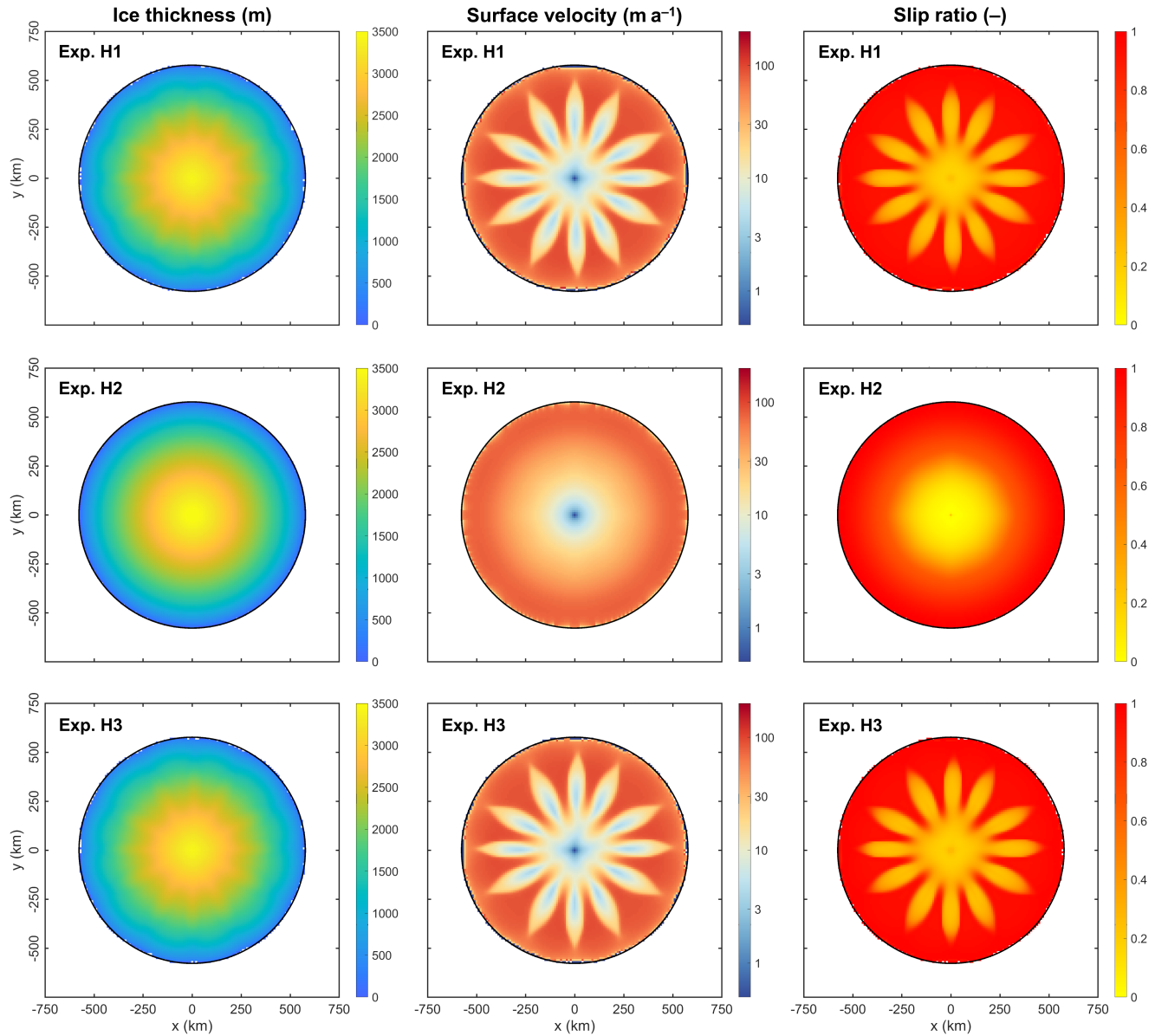
**Table 2.** Set-up of the experiments H1, H2 and H3: Sliding exponents  $(p, q)$ , dimensionless sliding coefficient  $\tilde{C}_b$ , stress exponent  $n$ , dimensionless rate factor  $\tilde{A}$  ( $\tilde{A}_{\text{CP10}}$  denotes the non-dimensionalized rate factor by Cuffey and Paterson (2010) as shown in Figure 1). Note that  $\tilde{C}_b$  and  $\tilde{A}$  are the same for all experiments.

is sufficient to reach a steady state for all three experiments.

Selected results (ice thickness, surface velocity, slip ratio = ratio of basal to surface velocity) are shown in Figure 2. Our main message is that all three experiments produce reasonable, consistent results for the same dimensionless sliding coefficient and rate factor, even though the exponents in the sliding law and the flow law have been varied. As explained above, in a dimensional formulation, this would require changes in the numerical values of the sliding coefficient and the rate factor by several orders of magnitude. The results of experiments H1 and H3 (flow-law exponent  $n$  changed) are very similar to each other and show a fingering instability that results from the thermomechanical coupling, which was already discussed in the original EISMINT publication by Payne and others (2000). By contrast, this instability does not occur in experiment H2 (sliding-law exponents  $p$  and  $q$  changed), where the solution maintains an almost perfect circular symmetry. While an interesting topic, we refrain from a deeper discussion of these instabilities here, yet point the interested reader to the EISMINT paper and further studies by, e.g., Fowler and Johnson (1996), Payne and Dongelmans (1997), Sayag and Tziperman (2008) and Hindmarsh (2009).

## 5 SUMMARY

We presented non-dimensionalized forms of basal sliding laws and flow laws for use in dimensional ice-sheet and glacier models. Compared to their dimensional counterparts, these forms have the advantage that the coefficients (sliding coefficient, rate factor) become independent of the respective exponents. All of these parameters are to some extent uncertain and therefore candidates for systematic variation in ensemble simulations of ice sheets and glaciers, which becomes much easier with the dimensionless formulations as the parameters can be varied independently. This is particularly relevant for efforts to reproduce observations of the recent history of ice sheets and glaciers as a starting point for predictions of their future changes. As claimed, implementation in an existing dimensional ice-sheet model (SICOPOLIS v25) could be done



**Fig. 2.** Simulated ice thickness, surface velocity and slip ratio (ratio of basal to surface velocity) for the three experiments (H1, H2, H3) described in the main text (Sect. 4) after 100 ka model time.

with minimal adaptations. We demonstrated the practicability of the method by three test simulations for a simple, idealized geometry, in which we varied the sliding-law exponents and the flow-law exponent while keeping the dimensionless sliding coefficient and rate factor constant.

## CODE AND DATA AVAILABILITY

SICOPOLIS (SICOPOLIS Authors, 2025) is free and open-source software, published on a persistent Git repository hosted by GitHub (<https://github.com/sicopolis/sicopolis/>). The run-specs headers and output data produced for this study are available at Zenodo, <https://doi.org/10.5281/zenodo.17373818>.

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