DIAGRAMMATICALLY REDUCIBLE COMPLEXES AND HAKEN MANIFOLDS

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Abstract

We show that diagrammatically reducible two-complexes are characterized by the property: every finite subcomplex of the universal cover collapses to a one-complex. We use this to show that a compact orientable three-manifold with nonempty boundary is Haken if and only if it has a diagrammatically reducible spine. We also formulate an analogue of diagrammatic reducibility for higher dimensional complexes. Like Haken three-manifolds, we observe that if $n \geq 4$ and $M$ is a compact connected $n$-dimensional manifold with a triangulation, or a spine, satisfying this property, then the interior of the universal cover of $M$ is homeomorphic to Euclidean $n$-space.

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1. Introduction

In this paper we establish a connection between Haken 3-manifolds and diagrammatically reducible 2-complexes. More precisely, we show that a compact orientable 3-manifold $M$ with nonempty boundary is Haken if and only if it has a diagrammatically reducible spine $K$ (Theorem 4.4).

To carry out this construction, we first give a characterization of diagrammatically reducible 2-complexes, which is a result of independent significance (Theorem 2.4): A 2-complex $X$ is diagrammatically reducible if and only if every finite subcomplex of the universal cover of $X$ collapses to a 1-complex. In the case of finite 2-complexes, this was conjectured by Brick [Bk].
Diagrammatically reducible 2-complexes were introduced by Sieradski [Si] and were subsequently studied by Gersten [Ge1, Ge2] and others. They are an interesting class of aspherical 2-complexes, with applications in equations over groups. Haken manifolds are an important, well-behaved, class of compact 3-manifolds; see, for example, [He] as a general reference. Knot complements are examples of orientable Haken 3-manifolds. Gersten has previously shown [Ge2] that orientable Haken 3-manifolds have the homotopy type of a diagrammatically reducible 2-complex. And earlier Chiswell, Collins, and Huebschmann [CCH] had shown that bounded Haken 3-manifolds have the homotopy type of a Diagrammatically Aspherical 2-complex (a weaker property).

Our equivalent formulation of diagrammatic reducibility makes sense, with a minor modification, for higher dimensional complexes. With this in mind, we say that a simplicial complex $K$ satisfies the property $P_1$ if: every finite subcomplex of the universal cover of $K$ is contained in a finite subcomplex that collapses to a 1-complex. In dimension two it is not necessary to go to a larger subcomplex since subcomplexes of a finite 2-complex that collapses to a 1-complex also collapse to 1-complexes. In fact it is easy to see that a finite 2-complex collapses to a 1-complex if and only if every 2-dimensional subcomplex contains a 2-cell with a free face; see Section 2 for this terminology. Thus, for 2-complexes the $P_1$ condition is equivalent to diagrammatic reducibility, by Theorem 2.4.

Using this notion we extend a well-known result about Haken manifolds in dimension three. Namely, if $M^n$ is a compact, connected, $n$-dimensional manifold with a triangulation, or a spine, with the property $P_1$ ($n \geq 4$), then the interior of $M^n$ is covered by $\mathbb{R}^n$ (Theorem 3.2). By a spine of a PL manifold $M$ we mean a simplicial complex $K$ such that some triangulation of $M$ simplicially collapses to a subcomplex isomorphic to $K$. For a general reference on piecewise linear topology, we refer the reader to [RS].

It should be noted that there is also an interesting characterization, due to Gersten [Ge2], of diagrammatic reducibility in terms of branched coverings. It may be worth investigating what this condition means in higher dimensions, and possibly comparing with the $P_1$ condition above.

## 2. Diagrammatically reducible complexes

In this section we work in the category of combinatorial 2-complexes. Thus, for our purposes every 2-cell of a 2-complex is attached along a (finite) edge-circuit, and by a map of 2-complexes we mean a combinatorial map (that is, a map in which each open cell in the domain is mapped homeomorphically onto an open cell in the target).

Let $X$ be a 2-complex. We say that an open $(n - 1)$-cell $t$ is a free face of an
open \( n \)-cell \( e \) if it occurs exactly once in the boundary of \( e \) and it does not occur in the boundary of any other \( n \)-cell. Recall that under these circumstances, the passage from \( X \) to the subcomplex \( X \setminus (e \cup i) \) is called an elementary collapse. We say that \( X \) collapses to a subcomplex \( A \) if there is a finite sequence of elementary collapses passing from \( X \) to \( A \). (In this case, of course, \( X \) and \( A \) have the same homotopy type.)

For convenience, we say that a 2-complex is \textit{closed} if it is finite and none of its cells has a free face. Notice that every finite 2-complex collapses to a closed subcomplex.

Given a closed surface \( F \), we say that a map \( f : F \rightarrow X \) is a \textit{near immersion} if \( F \) supports a combinatorial cell structure for which \( f \) is a combinatorial map and \( f |_{F \setminus F^0} \) is an immersion. Here \( F^0 \) denotes the 0-skeleton of the cell structure of \( F \), and by an immersion we mean a local embedding. Then we have:

**Definition.** A 2-complex \( X \) is \textit{diagrammatically reducible} (abbreviated DR) if there is no near immersion of \( S^2 \) into \( X \).

The next lemma is used in the proof of the main result in this section. For use in the proof we make a definition: \textit{A complete set of cutting curves} on a closed orientable surface \( F \) is a collection of disjoint simple closed curves such that cutting the surface along these curves yields a genus zero surface.*

**Lemma 2.1.** Suppose \( f : F \rightarrow X \) is a near immersion, where \( F \) is a closed surface and \( X \) is a 1-connected 2-complex. Then there exists a near immersion \( S^2 \rightarrow X \) (that is, \( X \) is not DR).

**Proof.** By first subdividing \( X \), and \( F \) correspondingly, we may assume that \( X \) is a simplicial complex. We may also assume, by taking an orientable double cover, that \( F \) is orientable.

Choose a complete set of cutting curves \( \gamma_1, \ldots, \gamma_k \) for \( F \) such that each curve avoids the finitely many points at which \( f \) is not a local embedding. Then each \( f(\gamma_i) \) is an immersed curve in \( X \). By appropriately subdividing \( X \) (and pulling back the subdivision to \( F \)), we can arrange that the \( \gamma_i \) lie in the 1-skeleton and thus are embedded edge-circuits. Since \( X \) is simply connected, each \( f(\gamma_i) \) is null-homotopic and hence bounds a van Kampen diagram \((D_i, \phi_i)\) in \( X \). Recall that a van Kampen diagram \((D, \phi)\) in \( X \) is a finite 1-connected planar 2-complex \( D \) and a combinatorial map \( \phi : D \rightarrow X \); see, for example, [LS] for more details.

Form a 2-complex \( L \) by \textit{attaching} the diagram \( D_i \) to \( F \) along \( \gamma_i \), for each \( i = 1, \ldots, k \). It should be noted that under this \textit{attaching} some identifications of \( F \) along \( \gamma_i \) may be performed. Define a combinatorial map \( \phi : L \rightarrow X \) by \( \phi|_F = f \) and \( \phi|_{D_i} = \phi_i \) (\( 1 \leq i \leq k \)). Note that \( L \) is a closed, 1-connected 2-complex and that \( L \) embeds in \( S^3 \) (as shown in Figure 1) such that \( S^3 \setminus L \) is a disjoint union of open 3-cells.
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(a) (b)

FIGURE 1. Embedding of \( L \) in the 3-sphere in the case where \( F \) is the torus. In (a) the attached diagrams are disks however in general, as in (b), they are 'pinched' disks.

(two of them in this case). Furthermore, the map \( f \) factors through \( L \); \( f = \phi \circ \iota \), where \( \iota : F \to L \) is the natural map into the adjunction space \( L \).

Hence, there exists a 2-complex \( K \) with the following four properties:

1. \( K \) is a closed 1-connected 2-complex embedded in \( S^3 \).
2. \( S^3 \setminus K \) is a disjoint union of open 3-cells, each of which is attached to \( K \) by an immersion \( S^2 \to K \).
3. The map \( f \) factors through \( K \); that is, there exist combinatorial maps \( g : F \to K \) and \( h : K \to X \) such that \( f = h \circ g \).
4. Amongst all 2-complexes satisfying 1–3, \( K \) has a minimal number of 2-cells.

Now let \( j : S^2 \to K \) be the attaching map (immersion) of one of the 3-cells in \( S^3 \setminus K \). We claim that the map \( h \circ j : S^2 \to X \) is a near immersion.

To see this, suppose \( h \circ j \) is not a near immersion. Then there exists a pair of distinct closed 2-cells \( \sigma \) and \( \tau \), in the pull-back cell structure of \( S^2 \), such that \( \sigma \cap \tau \) contains a 1-cell \( e \) and \( h(j(\sigma)) = h(j(\tau)) \). Thus, for each point \( x \in \sigma \), there is a unique point \( x' \in \tau \) such that \( h(j(x)) = h(j(x')) \), and \( x = x' \) if \( x \in e \). Let \( K' \) denote the 2-complex obtained by identifying \( j(x) \) and \( j(x') \), for each \( x \in \sigma \).

There are four ways in which the 2-cells \( \sigma \) and \( \tau \) can meet: in one edge, the union of two edges, one edge and a disjoint vertex, or the three edges making up the entire (common) boundary of the 2-cells. In any case, observe that the embedding of \( K \) in \( S^3 \) can be continuously deformed to an embedding of \( K' \), folding \( j(\sigma) \cup j(\tau) \) at \( e \) in the direction of the 3-cell bounded by the immersion \( j \). In the first two cases, the number of 3-cells in \( S^3 \setminus K' \) is the same as in \( S^3 \setminus K \). In the third case, the number of
3-cells is increased by one. And in the last case, the number of 3-cells is decreased by one.

The complement of this embedding of $K'$ is again a disjoint union of open 3-cells, and after collapsing any 2-cells of $K'$ with a free edge that may have been introduced, we see that (1) and (2) hold. Also (3) holds for $K'$, since $h$ factors through $K'$. But $K'$ has one less 2-cell than $K$, contradicting (4). Our claim therefore follows, and hence $X$ is not DR. \hfill $\square$

REMARK. As an alternative to viewing $K$ as embedded in $S^3$ in the above proof, the conditions (1) and (2) can be replaced by simply requiring the existence of a collection of immersions $S^2 \to K$ such that each open 2-cell of $K$ is hit exactly twice.

We note two easy consequences before turning to the theorem.

**COROLLARY 2.2.** Suppose $f : F \to X$ is a near immersion, where $F$ is a closed surface and $X$ is DR. Then the image of $f_* : \pi_1(F) \to \pi_1(X)$ is nontrivial.

**PROOF.** If $f_*$ is the trivial homomorphism, then $f$ lifts to the universal cover $\widetilde{X}$. But $\widetilde{X}$ is DR, contradicting Lemma 2.1. \hfill $\square$

**COROLLARY 2.3.** Let $X$ be a closed 2-complex. If $X$ is DR, then $\pi_1(X, x_0)$ is infinite (and torsion-free).

**PROOF.** By [CT1, Theorem 2.1] there is a near immersion $f : F \to X$, where $F$ is some closed surface. Thus, $\pi_1(X, x_0) \neq 1$ by Corollary 2.2. The result now follows since $X$ is an aspherical 2-complex; see [Ge1]. \hfill $\square$

**THEOREM 2.4.** A 2-complex $X$ is DR if and only if every finite subcomplex of the universal cover $\widetilde{X}$ collapses to a 1-complex.

**PROOF.** First assume that $X$ is DR and let $L$ be a finite subcomplex of $\widetilde{X}$. Then $L$ collapses to a closed subcomplex $L_0$, which we claim is a 1-complex. For if $L_0$ were 2-dimensional, then by [CT1, Theorem 2.1] there would be a near immersion $f : F \to L_0$, for some closed surface $F$. But that would imply, by Lemma 2.1, that $\widetilde{X}$ is not DR, a contradiction. (Clearly a 2-complex is DR if and only if its universal cover is DR.)

Conversely, suppose $f : S^2 \to X$ is a near immersion. Then $f$ lifts to a near immersion $f' : S^2 \to \widetilde{X}$ in the universal cover. But the image of a near immersion of a closed surface is a closed 2-dimensional subcomplex. Thus the image of $f'$ is a finite subcomplex of $\widetilde{X}$ that does not collapse to a 1-complex. \hfill $\square$
In the case of a finite 2-complex $X$ (or any 2-complex whose universal cover has only a countable number of cells), note that Theorem 2.4 can be stated as was conjectured by Brick [Bk]: $X$ is DR if and only if the universal cover of $X$ is the union of an ascending sequence of finite subcomplexes, each of which collapses to a 1-complex.

3. Generalization of diagrammatic reducibility

Henceforth, we consider only simplicial complexes. Thus by a $k$-complex we now mean a simplicial complex of dimension $\leq k$. To indicate that a simplicial complex $L$ (simplicially) collapses to a subcomplex $K$, we write $L \searrow K$. See the book by Rourke and Sanderson [RS] for a general reference on piecewise linear topology.

**DEFINITION.** For each nonnegative integer $k$, we say that a simplicial complex $K$ satisfies the property $P_k$ provided: every finite subcomplex of $K$ is contained in a finite subcomplex that collapses to a $k$-complex.

We are only interested here in the cases $k = 0$ and $k = 1$. As we noted in the introduction, a 2-complex $X$ satisfies $P_1$ if and only if it is DR. Thus the condition $P_1$ can be viewed as a generalization of diagrammatic reducibility, for simplicial complexes of arbitrary dimension.

The next lemma is true for any nonnegative integer $k$.

**LEMMA 3.1.** Suppose $K$ is a subcomplex of a finite simplicial complex $L$ and that $L \searrow K$. If $K$ satisfies property $P_k$, then $L$ also satisfies $P_k$.

**PROOF.** We may assume that $L = K \cup \{s^n, s^{n-1}\}$, where $s^n$ and $s^{n-1}$ are open simplices that are not contained in $K$ and $s^{n-1}$ is a face of $s^n$. Let $X$ be a finite subcomplex of $\tilde{L}$. Observe that $\tilde{L}$ is obtained from $\tilde{K}$ by attaching lifts of $s^n$, each of which has a free face projecting to $s^{n-1}$. So $X \searrow A$ where $A$ is the subcomplex of $X$ obtained by deleting all the lifts of $s^n$ and $s^{n-1}$. Since $A \subseteq \tilde{K}$, there is a finite subcomplex $B$ of $\tilde{K}$, containing $A$, such that $B$ collapses to a $k$-complex. Put $Y = B \cup X$, a finite subcomplex of $\tilde{L}$ containing $X$. Then $Y \searrow B$ (by collapsing away each lift of $s^n$) which then collapses to a $k$-complex. \[\square\]

In the next section we show that every Haken 3-manifold has a triangulation satisfying property $P_1$, and it is well known that the interior of every Haken 3-manifold is covered by $\mathbb{R}^3$. We observe next that the same is true in higher dimensions.

**THEOREM 3.2.** Let $M^n$ be a compact, connected, $n$-dimensional manifold ($n \geq 4$) that has a triangulation or spine with the property $P_1$. Then the universal cover of $\text{Int} M^n$ is (topologically) homeomorphic to $\mathbb{R}^n$. 
PROOF. If $M^n$ has a spine satisfying $P_1$, then by Lemma 3.1 it also has a triangulation with this property. So let $M^n$ be triangulated in this fashion.

Let $C$ be a compact subset of $\tilde{M}$. We show that $C$ is contained in a PL $n$-cell. By property $P_1$, there is a finite connected subcomplex $X$ of $\tilde{M}$, that collapses to a 1-complex, such that $C \subseteq X$. Let $V$ be a regular neighbourhood of $X$ in $\tilde{M}$. Then $V$ is an $n$-dimensional handlebody (a 0-handle with 1-handles attached).

Since $\tilde{M}$ is simply connected, there exists a finite connected subcomplex $Y$ of $\tilde{M}$, containing $X$, such that $\pi_1(X) \to \pi_1(Y)$ is the trivial homomorphism. Let $W$ be a regular neighbourhood of $Y$, so that $W$ is an $n$-dimensional handlebody and $V \subseteq W$ induces a trivial homomorphism of fundamental groups.

Now, since $n \geq 4$, it follows by a general position argument that $V$ is ambient isotopic in $W$ to a subset of the 0-handle of $W$. This is a special case of the Zeeman Engulfing Theorem; see for example [Ru, Theorem 4.6.1]. Therefore, $V$ is contained in an $n$-cell, and hence this $n$-cell contains $C$.

Thus, every compact subset of $\tilde{M}$ is contained in an $n$-cell. It follows that $\text{Int} \tilde{M}$ is the union of an ascending sequence of open $n$-cells. The proof is completed by appealing to Brown’s Theorem [Bn].

As a consequence we have the following (the case $n = 3$ is handled in the next section): Let $K$ be a finite, connected, diagrammatically reducible 2-complex. If $M$ is any $n$-dimensional thickening of $K$, that is, triangulated $n$-manifold that collapses to $K$, then $\text{Int} M$ is covered by $\mathbb{R}^n$. Of course, not every finite 2-complex has a 3-dimensional thickening, but they all have $n$-dimensional thickenings, for every $n \geq 4$.

4. Haken three-manifolds

Turning to 3-dimensional manifolds we next show that an orientable Haken 3-manifold with nonempty boundary has a spine which is DR, in a strong sense.

THEOREM 4.1. Let $M$ be an orientable Haken 3-manifold with nonempty boundary. Then $M$ has a 2-dimensional spine $K$ satisfying the property $P_0$ (in particular, $K$ is DR).

We first establish two preliminary results. Here, and elsewhere, we say that an embedding $j : A \to X$, or its image $j(A)$, is incompressible if $j_* : \pi_1(A) \to \pi_1(X)$ is injective for any choice of base point in $j(A)$.

LEMMA 4.2. Suppose $K$ and $\Sigma$ are finite simplicial complexes and $g : \Sigma \times \{-1, 1\} \to K$ is a simplicial map such that $g|_{\Sigma \times \{-1\}}$ and $g|_{\Sigma \times \{1\}}$ are incompressible embeddings. If $K$ and $\Sigma$ both have the property $P_0$, then $L = K \cup_g (\Sigma \times [-1, 1])$ also satisfies $P_0$. 

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PROOF. We may assume that $L$ and $\Sigma$ are connected, and that $K$ has one or two components. Then $\pi_1(L)$ is either an HNN extension of $\pi_1(K)$ or an amalgamated free product of the fundamental groups of the distinct components of $K$; in each case the splitting is over a subgroup isomorphic to $\pi_1(\Sigma)$. The universal cover of $L$ therefore consists of copies of the universal covers of the components of $K$ connected by copies of the universal cover of $\Sigma \times [-1, 1]$ in a ‘tree-like’ fashion.

Denote by $p : \widetilde{L} \to L$ the universal covering map, and let $X$ be a finite connected subcomplex of $\widetilde{L}$. Then $X$ meets only finitely many closures of components of $p^{-1}(\Sigma \times (-1, 1))$, each of which is a copy of $\widetilde{\Sigma} \times [-1, 1]$. Denote these components $((\widetilde{\Sigma} \times [-1, 1]), \ldots, ((\widetilde{\Sigma} \times [-1, 1])_m$. By hypothesis, we can choose a subcomplex of the form $A_1 = T_i \times [-1, 1]$ of $((\widetilde{\Sigma} \times [-1, 1])_i$ where $T_i$ is a finite collapsible subcomplex of $\widetilde{\Sigma}$, large enough that $X \cap ((\widetilde{\Sigma} \times [-1, 1])_i \subset A_i$ (for $i = 1, \ldots, m$). Put $Y = X \cup A_1 \cup A_2 \cdots \cup A_m$, a finite subcomplex of $L$.

Then $Y$ meets only finitely many components of $p^{-1}(K)$, say $\widetilde{K}_1, \ldots, \widetilde{K}_n$, each of which is a copy of the universal cover of a component of $K$. Choose, as we may by the hypothesis on $K$, a collapsible subcomplex $B_j$ of $\widetilde{K}_j$ such that $Y \cap \widetilde{K}_j \subset B_j$ for each $j = 1, \ldots, n$. Set $Z = B_1 \cup \cdots \cup B_n \cup A_1 \cup \cdots \cup A_m$, a finite subcomplex of $L$ containing $X$. Note that $Z$ consists of the complexes $B_i$ joined by ‘generalized 1-handles’ $A_j$ in a ‘tree-like’ manner.

We complete the proof by observing that $Z$ is collapsible. Initially collapse each $A_i = T_i \times [-1, 1]$ onto the subcomplex $(T_i \times (-1, 1)) \cup (\ast_i \times [-1, 1])$ where $\ast_i$ is some vertex of $T_i$. In this way we collapse $Z$ onto a subcomplex consisting of the parts $B_j$ joined together by arcs (in a ‘tree-like’ fashion). Then we can collapse each part $B_j$ onto a spanning tree in its 1-skeleton, thus collapsing $Z$ onto a tree in its 1-skeleton. Finally we collapse this tree to a vertex, as required. \(\square\)

It is obvious that 1-dimensional simplicial complexes have property $P_0$. We next observe that the same is true for triangulations of compact aspherical surfaces.

**LEMMA 4.3.** If $\Sigma$ is a 2-dimensional simplicial complex homeomorphic to a compact aspherical surface, then $\Sigma$ satisfies property $P_0$.

**PROOF.** If $\Sigma$ has nonempty boundary, then $\Sigma$ has a 1-dimensional spine and the result follows from Lemma 3.1. So assume that $\Sigma$ is a closed surface. Then each component of $\widetilde{\Sigma}$ is a triangulation of the plane, and it is easy to see that every finite subcomplex of a triangulation of the plane is contained in a collapsible one. \(\square\)

**PROOF OF THEOREM 4.1.** We assume, without loss of generality, that $M$ is connected.

It is well known (see [He, Theorem 13.3]) that $M$ admits a hierarchy of the following
form:

\[ M = M_0 \supset M_1 \supset \cdots \supset M_n = B^3, \]

where \( M_i \) is obtained from \( M_{i-1} \) by cutting along a properly embedded surface \( F_i \subset M_{i-1} \) which satisfies:

1. \( F_i \) is incompressible in \( M_{i-1} \);
2. \( F_i \) is compact, connected, and orientable;
3. \( \partial F_i \neq \emptyset \);
4. (implicit from \( M_n = B^3 \)) \( F_i \) does not separate \( M_{i-1} \).

We now associate to such a hierarchy of \( M \) a 2-dimensional spine—which satisfies property \( P_0 \).

To begin, let \( K_i \) denote a 1-dimensional spine for each \( F_i, i = 1, \ldots, n \). For simplicity, we assume that \( K_i \) is collapsed as much as possible. In particular, \( K_i \) is a point if \( F_i = D^2 \). Recall that \( M_i \) is obtained from \( M_{i-1} \) by cutting along \( F_i \) means that we view \( F_i \times [-1, 1] \subset M_{i-1} \) such that

\[ \partial (F_i \times [-1, 1]) \cap \partial M_{i-1} = \partial (F_i \times [-1, 1]) \cap \partial M_{i-1} = \partial F_i \times [-1, 1] \]

and \( M_i = M_{i-1} - \left[ F_i \times (-1, 1) \right] \). Evidently, there are two copies of \( F_i \) in \( \partial M_i : F_i^+ = F_i \times \{1\} \) and \( F_i^- = F_i \times \{-1\} \). Let \( K_i^+ \) and \( K_i^- \) denote the copies of \( K_i \) in \( F_i^+ \) and \( F_i^- \), respectively.

We next construct certain 1-complexes \( C_i \subset \partial M_i \) for \( i = 1, \ldots, n \). Initially, set \( C_1 = K_1^+ \cup K_1^- \). We assume (without loss) that \( F_2 \) meets \( C_1 \) transversely in a finite number of points, say \( \{p_1, \ldots, p_k\} \). For \( i = 1, \ldots, k \), let \( A_i \) denote an embedded arc in \( F_2 \) such that \( A_i \) joins \( p_i \) to \( K_2 \), \( \text{Int} A_i \) misses \( K_2 \cup \partial F_2 \), and \( A_i \cap A_j = \emptyset \) if \( i \neq j \).

Set \( S_2 = K_2 \cup (\bigcup_{i=1}^k A_i) \).

Since \( F_2 \) meets \( C_1 \) transversely, we may assume that \( C_1 \cap (F_2 \times [-1, 1]) = \{p_1, \ldots, p_k\} \times [-1, 1] \); that is, that \( C_1 \) meets \( F_2 \times [-1, 1] \) in \([-1, 1]\)-fibers. Now \( C_2 \) is defined by cutting \( C_1 \) along \( \{p_1, \ldots, p_k\} \) and gluing \( S_2^- \) and \( S_2^+ \) to this cut 1-complex, where \( S_2^\pm \) are the copies of \( S_2 \) in \( F_2^\pm \), respectively. In other words, \( C_2 = \left[ C_1 - ((p_1, \ldots, p_k) \times [-1, 1]) \right] \cup S_2^- \cup S_2^+ \).

The process of passing from \( C_i \) to \( C_2 \) is now repeated in obtaining \( C_{i+1} \) from \( C_i \) for \( i = 1, \ldots, n-1 \).

We now describe the spine \( K \) for \( M \) by stating the intersection of \( K \) with the ‘generalized handles’ of \( M \) associated to its hierarchy: \( K \) is defined by the property that \( K \cap M_n \) is the cone on \( C_n \), and \( K \cap (F_i \times [-1, 1]) \) is \( S_i \times [-1, 1] \) if \( i > 1 \), and \( K \cap (F_i \times [-1, 1]) = K_1 \times [-1, 1] \).

It is relatively straightforward to see that \( K \) is a spine for \( M \). First of all, \( F_1 \times [-1, 1] \) collapses to \((K_1 \times [-1, 1]) \cup (F_1 \times [-1, 1]) \). Note that \( S_2 \) is a spine of \( F_2 \) and \( F_2 \times [-1, 1] \) collapses to \((S_2 \times [-1, 1]) \cup (F_2 \times [-1, 1]) \). Proceeding sequentially in
this manner we obtain \( M \setminus (K \cup M_n) \) and finally \( M \setminus K \) since \( K \cap M_n \) is the cone over \( C_n \).

We show that \( K \) satisfies property \( P_0 \) inductively. Note that the preceding paragraph actually shows more, namely that \( K \cap M_i \) is a spine of \( M_i \) for \( i = 1, \ldots, n \). The induction starts at \( K \cap M_n \), which is collapsible and hence satisfies \( P_0 \). Then observe that \( S^*_i, S^-_i \hookrightarrow K \cap M_i \) are incompressible embeddings and \( K \cap M_{i-1} = (K \cap M_i) \cup (S_i \times [-1, 1]) \). The inductive step, and hence the proof, is thus completed by Lemma 4.2.

We next observe that the converse of Theorem 4.1 holds, thus giving a characterization of orientable Haken 3-manifolds with boundary.

**Theorem 4.4.** A compact orientable 3-manifold \( M \) with nonempty boundary is Haken if and only if it has a diagrammatically reducible 2-dimensional spine \( K \).

**Proof.** Suppose \( M \) is a compact 3-manifold with a DR spine \( K \), and choose a triangulation of \( M \) that collapses to \( K \). Then, by Lemma 3.1, the triangulation of \( M \) satisfies the condition \( P_1 \) (as \( K \) satisfies \( P_1 \) by Theorem 2.4). Recall that \( K \) and hence \( M \), is aspherical. It is well known that an irreducible, compact, aspherical 3-manifold with boundary is Haken. Thus, the proof is completed by the claim (which also holds for closed manifolds): Every compact 3-manifold with a triangulation satisfying \( P_1 \) is irreducible.

To see this, let \( S \) be a PL 2-sphere in \( M \). Then \( S \) lifts to a 2-sphere \( \tilde{S} \) in \( \tilde{M} \) which, by property \( P_1 \), is contained in some finite connected subcomplex \( X \) of \( \tilde{M} \) that collapses to a 1-complex. Then a regular neighbourhood of \( X \) in \( M \) must be a 3-dimensional handlebody (consisting of a 0-handle and 1-handles). Since such handlebodies are irreducible, we conclude that \( \tilde{S} \) bounds a 3-cell which projects to a 3-cell in \( M \) bounded by \( S \), as required.

**Remark 4.5.** For closed 3-manifolds the situation is more complicated. On the one hand, a construction similar to that of the spine for Theorem 4.1, using induction on the length of a hierarchy, shows that every closed Haken 3-manifold has a triangulation satisfying \( P_0 \) (and thus \( P_1 \)). However, the converse is false for the following reason. There are closed 3-manifolds which are not Haken, but for which some finite sheeted cover is Haken (virtually Haken manifolds). Let \( M \) be such a 3-manifold and let \( M' \) be a finite cover of \( M \) which is Haken. Then \( M' \) supports a triangulation satisfying the property \( P_0 \). By a standard fact from PL topology, there is a subdivision of the triangulation of \( M' \) and a triangulation of \( M \) for which the covering projection is a simplicial map. This subdivided triangulation of \( M' \) also satisfies \( P_0 \), which follows from the fact that every subdivision of a 3-dimensional collapsible simplicial complex...
remains collapsible [Ch]. Since $M$ and $M'$ have the same universal cover, it follows that the triangulation of $M$ also satisfies $P_0$.

We do not know whether every closed 3-manifold with a triangulation satisfying $P_0$ is virtually Haken.

References


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