10

Monopole Pair Production

In this chapter we will study the analysis by Affleck and Manton [4] of the decay of constant, external magnetic fields due to the production of magnetic monopole–anti-monopole pairs. The calculation is analogous to a calculation of the decay of external electric fields by Schwinger [109] due to the production of electron–positron pairs. In both cases the effect is due to non-perturbative tunnelling transitions.

10.1 't Hooft–Polyakov Magnetic Monopoles

In Chapter 9, we saw the solutions that correspond to magnetic monopoles, in the Georgi–Glashow model [54]; however, as we were in 2+1 dimensions these solutions were instantons in Euclidean three dimensions. Clearly the same solutions in 3+1 dimensions correspond to static soliton solutions and correspond to particle states of the 3+1-dimensional theory. There is a perturbative spectrum of particles corresponding to quantization of the small oscillations about the trivial vacuum. These particles correspond to a massless photon, a charged massive vector boson, and a neutral scalar from the Higgs field. We will consider the limit that the Higgs field mass and the vector gauge boson masses are very heavy while the photon remains massless. In this limit the monopoles are heavy, essentially point particles. We will see that in the presence of a constant external magnetic field, the Euclidean equations of motion admit instanton solutions that describe the production of monopole–anti-monopole pairs. The form of the instanton is surprisingly simple.

10.2 The Euclidean Equations of Motion

The solutions to the Euclidean equations of motion for a 't Hooft–Polyakov magnetic monopole in a constant external magnetic field must exist in general, as the initial value problem for the corresponding set of non-linear differential

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equations is well-defined. The solutions must be well-approximated by the solutions to the equations for point-like monopoles, certainly in the limit that the masses of the Higgs field and the massive vector bosons are taken to be very large. Then, apart from the self-action of each monopole being very large, the additional contribution to the action from the Euclidean trajectories of the monopoles will not diverge. The state of the system in the presence of a constant magnetic field should correspond to a meta-stable state, similar in principle to a false vacuum. There will be a finite probability for the creation of a monopole–anti-monopole pair. Creation of the pair of course costs energy; however, separating the monopoles in an external magnetic field gives back energy. After a separation to a critical radius, it is energetically favourable for the monopoles to separate to infinity. Thus the analogy to the decay of a meta-stable state is quite apt. The result is an exact analogy to the Schwinger calculation [109] of the decay of a constant electric field due to the creation of charged boson–anti-boson pairs. Schwinger found the amplitude

$$\Gamma = \frac{e^2 E^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-n\pi m^2/eE}}{n^2} (1 + o(e^2)), \tag{10.1}$$

where E is the amplitude of the external electric field and m is the boson mass.

Manton and Affleck [4] found the result

$$\Gamma = \frac{g^2 B^2}{8\pi^3} e^{-\left(\pi M^2/gB + g^2/4\right)} \left(1 + o\left(\frac{g^3 B}{M^2}\right) + o(e^2)\right) \tag{10.2}$$

with g the magnetic charge, B the amplitude of the magnetic field, and M the mass of the monopole, which corresponds to the first term in the expansion found by Schwinger, interchanging electric charge and field with magnetic charge and field.

To find this amplitude, we will look for a solution to the classical Euclidean equations of motion that interpolate between the false vacuum in the presence of the constant background magnetic field, and the configuration containing a monopole–anti-monopole pair which are separating to infinity in the background magnetic field. The Euclidean solution will actually be a bounce-type instanton, thus we expect the pair will move apart up to a critical separation and then bounce back and return to each other and annihilate. The bounce point will correspond to the point at which the tunnelling occurs in Minkowski spacetime, and after the appearance of the physical monopoles in Minkowski spacetime, the magnetic field will pull them apart to infinite separation. The bounce should have one negative mode and all the rest positive. The negative mode will give rise to the imaginary part of the functional integral, with the appropriate analytical continuation. Effectively, the imaginary part of the functional integral is given by

$$\mathfrak{Im}\left(TVKe^{-S_E}\right), \tag{10.3}$$

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where

$$K = \frac{det^{-1/2}\left(\left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{\text{inst.}}\right)}{det^{-1/2}\left(\left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_0\right)}. \tag{10.4}$$

There are also some zero modes that give the usual complications, which we will deal with using the Faddeev–Popov method. Our conventions will be the following for an $SU(2)$ gauge field $A_\mu = A_\mu^a T^a$ and a scalar field in the triplet representation, $\phi = \phi^a T^a$, where $T^a_{bc} = \epsilon^{abc}$ are the anti-symmetric 3×3 matrix representation of $SU(2)$,

$$\mathcal{L} = \frac{1}{e^2}\left(\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D_\mu\phi)^a(D_\mu\phi)^a + \frac{\lambda}{4e^2}\left(|\phi|^2 - M_W^2\right)^2\right), \tag{10.5}$$

where $[T^a, T^b] = \epsilon^{abc}T^c$, $|\phi|^2 = \phi^a\phi^a$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ and $D_\mu\phi = \partial_\mu\phi - [A_\mu, \phi]$, and M_W provides the mass scale. The equations of motion are

$$\begin{aligned} D_\mu F_{\mu\nu} &= [D_\nu\phi, \phi] \\ D_\mu D_\mu\phi &= \frac{\lambda}{e^2}\left(|\phi|^2 - M_W^2\right)\phi. \end{aligned} \tag{10.6}$$

If we take $A_4 = 0$ and all fields independent of x_4, the equations of motion reduce to the static, Euclidean three-dimensional equations that we have already studied in Chapter 9, and there is a finite energy, stable, static non-trivial solution of the equations corresponding to the magnetic monopole. The action of the monopole is, of course, not finite as the solution is independent of x_4. The mass is

$$M = \frac{4\pi M_W}{e^2}k\left(\lambda/e^2\right) \quad \text{where } k \approx 1 \text{ for } \lambda/e^2 \leq 1 \tag{10.7}$$

and the magnetic charge is $g = 4\pi/e$, the core radius is $r_{cl} = g^2/M$ and the "Abelian" field strength can be defined as $f_{\mu\nu} = F^a_{\mu\nu}\phi^a/eM_W$. The Abelian field strength satisfies the Maxwell equation if $|\phi|^2 = M_W^2$ and $D_\mu\phi = 0$. In the limit of $\lambda \to \infty$, $e^2 \to \infty$ but λ/e^2 remaining finite, the monopole core size goes to zero and it looks very much like a point monopole.

10.3 The Point Monopole Approximation

Then in an external, constant magnetic field, the monopole solution cannot remain static. In Euclidean time, it must respect the Euclideanized magnetic "Lorentz" force law

$$M\frac{d^2 z_\mu}{ds^2} = -g\tilde{f}_{\mu\nu}\frac{dz_\nu}{ds}, \tag{10.8}$$

where z_μ is the position of the magnetic charge, s is a world line parameter normalized so that $\frac{dz_\nu}{ds}\frac{dz_\nu}{ds} = 1$ and $\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}f_{\sigma\tau}$. This equation is simply the dual of the usual Euclidean "Lorentz" force for a charged particle in electric and magnetic fields

$$M\frac{d^2 z_\mu}{ds^2} = -e f_{\mu\nu}\frac{dz_\nu}{ds}. \tag{10.9}$$

For the magnetic field with constant magnitude B in the three-direction, $f_{12} = B$ which means $\tilde{f}_{34} = B$. Then a solution of the equation of motion (10.8) is simply $z_1 = z_2 = 0$ and

$$z_3 = \frac{M}{gB}\cos\left(\frac{gB}{M}s\right) \quad z_4 = \frac{M}{gB}\sin\left(\frac{gB}{M}s\right). \tag{10.10}$$

The solution is obviously a circle. This is the analytic continuation of the corresponding Minkowski space solution, which would be a hyperbola.

10.4 The Euclidean Action

The point monopole equations of motion are, of course, approximative, but we can derive them in the limit of a weak external magnetic field [4]. This circular Euclidean solution is exactly the bounce solution that we are looking for. We can equally well think of the solution in the (x_3, x_4) plane as the creation of a monopole–anti-monopole pair, the two separating to a finite critical distance and then bouncing back together and annihilating. The diameter of the circle is the critical separation and corresponds to the point to which the pair separates in the Euclidean solution, but also the separation at which the pair appears in the tunnelling process, in Minkowski space. The circular solution neglects the Coulomb attraction between the monopole–anti-monopole pair. We will see that the Coulomb interaction does not greatly affect the instanton. To analyse the corrections, we consider the following decomposition of the action

$$S_E = \int d^4x\left(\mathcal{L} - \frac{1}{4}\tilde{f}_{\mu\nu}\tilde{f}_{\mu\nu}\right) + \int d^4x\frac{1}{4}\left(\tilde{f}_{\mu\nu}\tilde{f}_{\mu\nu} - f^{ext.}_{\mu\nu}f^{ext.}_{\mu\nu}\right), \tag{10.11}$$

where we have separated the Lagrangian into the first term that governs the dynamics above the Abelian gauge field and subtracted the action of the external gauge field. We define the dual Abelian gauge field into the core of the monopole as

$$\begin{aligned}\partial_\mu \tilde{f}_{\mu\nu} &= \tilde{j}_\nu \\ \partial_\mu f_{\mu\nu} &= 0\end{aligned} \tag{10.12}$$

where $\tilde{j}_\nu$ is an appropriate, conserved, Abelian definition of the dual current into the core. Outside the core, $j_\nu = 0$ and the source-free Maxwell equations are perfectly valid. Equation (10.12) are just the Euclidean, dual, Abelian Maxwell equations with magnetic sources. As these are just the dual Maxwell equations, there exists a gauge potential $\tilde{a}_\mu$ such that

$$\tilde{f}_{\mu\nu} = \partial_\mu\tilde{a}_\nu - \partial_\nu\tilde{a}_\mu. \tag{10.13}$$

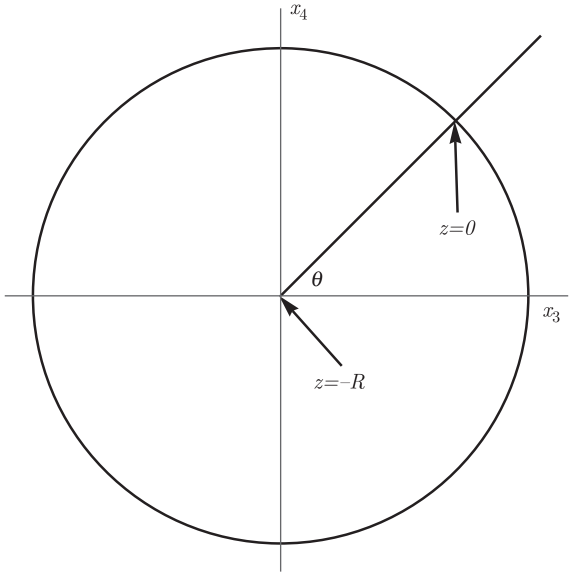

Figure 10.1. Circularly symmetric monopole–anti-monopole instanton

Exploiting the circular symmetry of the point-like solution we write

$$\tilde{a}_\mu = (0,0,-\sin\theta,\cos\theta)\psi(x,y.z)$$
$$\tilde{j}_\mu = (0,0,-\sin\theta,\cos\theta)\rho(x,y.z), \tag{10.14}$$

where x,y are the normal cartesian coordinates, but z,θ are polar coordinates in the x_3,x_4 plane, with the radius shifted so that $z=0$ corresponds to the radius of the circular point-like monopole instanton, *i.e.* the usual radial coordinate is $r=z+R$, as shown in Figure 10.1. Thus $z=-R$ is the origin, and we will expand the action about $z=0$. Then for the first term of the decomposition in Equation (10.11) we write

$$S_E^1 = 2\pi\int dxdydz(R+z)\left(\mathcal{L}-\mathcal{L}_{Abelian}\right), \tag{10.15}$$

where $\mathcal{L}_{Abelian}=\frac{1}{4}\tilde{f}_{\mu\nu}\tilde{f}_{\mu\nu}$ which can be evaluated from Equation (10.14)

$$\mathcal{L}_{Abelian} = \left(\frac{1}{2}(\partial_i\psi\partial_i\psi + \frac{1}{R+z}\psi\partial_z\psi + \frac{1}{2(R+z)^2}\psi^2\right), \tag{10.16}$$

where the index i goes over x,y,z and $\mathcal{L}$ is of course the full Lagrange density given in Equation (10.5). Away from $z=0$, we expect that the solution is exponentially zero, $D_i\phi\approx V(\phi)\approx e^{-M_W|\vec{x}|}$ and $F^a_{ij}F^a_{ij}\to f_{ij}f_{ij}$, exponentially fast, and consequently $(\mathcal{L}-\mathcal{L}_{Abelian})$ also vanishes exponentially.

We make no great error by changing the range of z from $-R\le z\le\infty$ to $-\infty\le z\le\infty$, as long as all fields and densities are exponentially small away from $z = 0$, thus we get

$$S_E^1 = 2\pi R \int d^3x \,(\mathcal{L} - \mathcal{L}_{Abelian}) + \frac{z}{R}\,(\mathcal{L} - \mathcal{L}_{Abelian}), \tag{10.17}$$

where now the integral is over an entire three-dimensional Euclidean space. We expect that we can perform an expansion in powers of $1/R$. The Maxwell equation for the Abelian fields is

$$\partial_i \partial_i \psi + \frac{1}{R+z}\partial_z \psi - \frac{1}{(R+z)^2}\psi = \rho \tag{10.18}$$

then if

$$\psi(\vec{x}) = \sum_{n=0}^{\infty} \psi_n(\vec{x}) \frac{1}{R^n} \tag{10.19}$$

the density $\rho(\vec{x})$ must also admit a similar expansion

$$\rho(\vec{x}) = \sum_{n=0}^{\infty} \rho_n(\vec{x}) \frac{1}{R^n} \tag{10.20}$$

as well as the Lagrange density $\mathcal{L}$. The terms in the expansion must be of alternating parity as $z \to -z$. The second term in Equation (10.17) vanishes to lowest order. The limit, as $R \to \infty$, *i.e.* $B \to 0$, which is the order $n = 0$ term, the solution is simply a static monopole at rest, the circle has infinite radius and thus becomes effectively a straight, world line. Then the first term of Equation (10.17) just gives

$$S_E^1 = 2\pi R(M - M_{Abelian}), \tag{10.21}$$

where M is the mass of the monopole and $M_{Abelian}$ is just the contribution to the Coulomb energy from the zeroth order part of the current density $\rho_0(\vec{x})$, while the second term must give vanishing contribution due to parity. Thus, due to parity, the next correction only comes at $o\left(1/R^2\right)$.

10.5 The Coulomb Energy

The second term in the action, Equation (10.11), contains simply the energy in the Euclidean Abelian gauge fields, $\tilde{f}_{\mu\nu} = \tilde{f}^{loop}_{\mu\nu} + \tilde{f}^{ext.}_{\mu\nu}$, where $\tilde{f}^{loop}_{\mu\nu}$ comes from the monopole loop, and $\tilde{f}^{ext.}_{\mu\nu}$ comes from the fields outside of the loop. Then

$$S_E^2 = \frac{1}{4}\left(\tilde{f}_{\mu\nu}\tilde{f}_{\mu\nu} - f^{ext.}_{\mu\nu} f^{ext.}_{\mu\nu}\right) = \frac{1}{4}\tilde{f}^{loop}_{\mu\nu}\tilde{f}^{loop}_{\mu\nu} + \frac{1}{2} f^{loop}_{\mu\nu} f^{ext.}_{\mu\nu} \equiv S_E^{2,loop} + S_E^{2,int.}. \tag{10.22}$$

We will find

$$S_E^{2,loop} = \int d^4x \frac{1}{4}\tilde{f}^{loop}_{\mu\nu}\tilde{f}^{loop}_{\mu\nu} = \int d^4x d^4x' \frac{1}{8\pi^2}\frac{\tilde{j}_\mu(x)\tilde{j}_\mu(x')}{|x-x'|^2}. \tag{10.23}$$

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This can be shown by first observing that in the gauge $\partial_\mu \tilde{a}_\mu = 0$, we can solve the dual Maxwell field equation (10.12) for the dual gauge field simply as $\tilde{a}_\mu = \frac{1}{\Box}\tilde{j}_\mu$ where the Green's function is

$$\frac{1}{\Box} = -\frac{1}{4\pi^2}\frac{1}{|x-x'|^2} \tag{10.24}$$

and the dual field strength is as usual

$$\tilde{f}_{\mu\nu} = \partial_\mu \frac{1}{\Box}\tilde{j}_\nu - \partial_\nu \frac{1}{\Box}\tilde{j}_\mu. \tag{10.25}$$

Then it is straightforward to evaluate the contribution to the action

$$\begin{aligned} S_E^{2,loop} &= \frac{1}{2}\int d^4x \left(\partial_\mu \frac{1}{\Box}\tilde{j}_\nu\right)\left(\partial_\mu \frac{1}{\Box}\tilde{j}_\nu\right) - \left(\partial_\mu \frac{1}{\Box}\tilde{j}_\nu\right)\left(\partial_\nu \frac{1}{\Box}\tilde{j}_\mu\right) \\ &= \frac{1}{2}\int d^4x - \left(\frac{1}{\Box}\tilde{j}_\nu\right)(\tilde{j}_\nu) + \left(\frac{1}{\Box}\tilde{j}_\nu\right)\left(\partial_\mu\partial_\nu \frac{1}{\Box}\tilde{j}_\mu\right). \end{aligned} \tag{10.26}$$

The second term in the first line vanishes after integration by parts, the second term in the last line vanishes since ∂_μ commutes with $1/\Box$ and $\partial_\mu \tilde{j}_\mu = 0$ by current conservation, which is necessary for the consistency of the dual Maxwell equations and is assumed to be verified by the current. Then

$$S_E^{2,loop} = \frac{1}{2}\int d^4x - \left(\frac{1}{\Box}\tilde{j}_\nu\right)(\tilde{j}_\nu) = \frac{1}{8\pi^2}\int d^4x d^4x' \frac{\tilde{j}_\mu(x)\tilde{j}_\mu(x')}{|x-x'|^2} \tag{10.27}$$

as desired. To calculate it explicitly is not too difficult. First of all, $\tilde{j}_\mu(x)\tilde{j}_\mu(x') = (\sin\theta\sin\theta' + \cos\theta\cos\theta')\rho(x)\rho(x') = \cos(\theta-\theta')\rho(x)\rho(x')$, thus we get, writing $d^2x = dx_1 dx_2$ and $d^2x' = dx_1' dx_2'$

$$S_E^{2,loop} = \frac{1}{8\pi^2}\int d^2x d^2x \left(\frac{rr'\cos(\theta-\theta')\rho(x)\rho(x')drd\theta dr'd\theta'}{(x_1-x_1')^2+(x_1-x_1')^2+r^2+r'^2-2rr'\cos(\theta-\theta')}\right). \tag{10.28}$$

The integral over θ and θ' can be done explicitly, we leave the reader to work it out or find it in tables, giving

$$S_E^{2,loop} = \int d^2x d^2x dr dr' \frac{1}{4}\rho(x)\rho(x')\left(\frac{W}{\sqrt{W^2-1}} - 1\right), \tag{10.29}$$

where, writing $(x_1-x_1')^2+(x_1-x_1')^2+(z-z')^2 = |\vec{x}-\vec{x}'|^2$

$$\begin{aligned} W &= \frac{(x_1-x_1')^2+(x_1-x_1')^2+r^2+r'^2}{2rr'} \\ &= \frac{(x_1-x_1')^2+(x_1-x_1')^2+(R+z)^2+(R+z')^2}{2(R+z)(R+z')} \\ &= 1 + \frac{|\vec{x}-\vec{x}'|^2}{2R^2} - \frac{|\vec{x}-\vec{x}'|^2(z+z')}{2R^3} + o\left(\frac{1}{R^4}\right) \end{aligned} \tag{10.30}$$

and intriguingly the terms $1/R$ exactly cancel. Then expanding carefully

$$\frac{W}{\sqrt{W^2-1}} - 1 = \frac{R}{|\vec{x}-\vec{x}'|} - 1 + \frac{z+z'}{2|\vec{x}-\vec{x}'|} + o\left(\frac{|\vec{x}|,z,z'}{R}\right) \tag{10.31}$$

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and we note that actually the numerator only contributes to the the terms that have been neglected. Then in the evaluation of the contribution of this term to the action, the third term in Equation (10.31) vanishes because of parity when the lowest-order, spherically symmetric monopole charge density is put in for ρ, and the net remaining is simply

$$S_E^{2,loop} = \int d^2x d^2x dr dr' \frac{1}{4}\rho_0(x)\rho_0(x')\left(\frac{R}{|\vec{x}-\vec{x}'|}-1\right)+o\left(\frac{1}{R}\right). \tag{10.32}$$

The first term is exactly the Coulomb energy in the magnetic field while the second is proportion to the magnetic charge squared,

$$S_E^{2,loop} = 2\pi R M_{Abelian} - \frac{1}{4}g^2 + o\left(\frac{1}{R}\right), \tag{10.33}$$

where g is the magnetic charge. The first term exactly cancels against the identical term found in S_E^1, which is expected, since it arises solely because of the somewhat artificial Abelian magnetic charge density that was invented to extend the Abelian integration into the core. No physical phenomenon should depend on it. Thus

$$S_E^1 + S_E^{2,loop} = -\frac{1}{4}g^2 + 2\pi R M. \tag{10.34}$$

The interaction part of S_E^2, which we will call $S_E^{2,int.}$, is, integrating by parts and using the equation of motion,

$$S_E^{2,int.} = \int d^4x \frac{1}{2}\tilde{f}^{loop}_{\mu\nu}\tilde{f}^{ext.}_{\mu\nu} = -\int d^4x \tilde{j}_\mu \tilde{a}^{ext.}_\mu. \tag{10.35}$$

The external gauge potential can be taken with circular symmetry as

$$a^{ext.}_\mu = (0,0,-\frac{1}{2}B(R+z)\sin\theta, \frac{1}{2}B(R+z)\cos\theta) \tag{10.36}$$

and the current is

$$j_\mu = (0,0,-\sin\theta,\cos\theta)\rho(\vec{x}). \tag{10.37}$$

Then using $d^4x = dx_1dx_2drd\theta r = dx_1dx_2dzd\theta(R+z) = d^3x(R+z)d\theta$ and integrating over θ gives a factor of 2π so that we get

$$S_E^{2,int.} = -\int d^3x \pi B(R+z)^2\rho(\vec{x}) = -g\pi BR^2 + \cdots. \tag{10.38}$$

Thus the total action is

$$S_E = 2\pi M - gB\pi R^2 - \frac{1}{4}g^2 + o\left(\frac{1}{R^2}\right). \tag{10.39}$$

We vary the action with respect to R and demand that it be stationary to find the radius of the loop,

$$0 = \frac{\delta S_E}{\delta R} = 2\pi M - 2g\pi BR, \tag{10.40}$$

which gives $R = M/gB$. This is exactly the same value as in the case of the point-like monopoles, therefore we see that the inclusion of the Coulomb energy does not affect the radius of the loop. Inserting the value of R back into the action yields

$$S_E = \frac{\pi M^2}{g^2 B} - \frac{1}{4}g^2, \tag{10.41}$$

and we observe that the Coulomb energy is $\sim 1/R$ integrated over a circle of circumference $2\pi R$, which yields $g^2/4$ which is independent of R. Finally, if we take the second variation we find

$$\frac{\delta^2 S_E}{\delta R^2} = -2gB\pi < 0, \tag{10.42}$$

which means that the action has at least one negative mode and hence is at a saddle point. The negative mode is expected and gives rise to the decay width of the magnetic field.

10.6 The Fluctuation Determinant

We must now take into account the Gaussian integration over the fluctuations around the instanton

$$K = \frac{1}{2} \frac{\left| det\left(\left. \frac{\delta^2 S_E}{\delta \phi_i^2} \right|_{\text{inst.}} \right) \right|^{-1/2}}{\left(det\left(\left. \frac{\delta^2 S_E}{\delta \phi_i^2} \right|_{0} \right) \right)^{-1/2}}. \tag{10.43}$$

The factor of one-half occurs since we integrate over only half the Gaussian peak for the negative mode and any Faddeev–Popov factors are assumed to be included in the determinant. We have put the numerator in absolute value signs so that the negative mode does not give an imaginary value when we take the square root, as we explicitly put the i in by hand, in that the energy obtains an imaginary part $E = \mathcal{E} + i\Gamma$ with

$$\Gamma = VKe^{-S_E/\hbar}(1 + o(\hbar)). \tag{10.44}$$

We must separate the zero modes, there are five, coming from four translations and one from internal rotation. The translation modes will give a familiar factor of the square root of the normalization

$$K \to \frac{1}{2} \prod_{\mu=1}^{4} \left(\frac{N_\mu}{2\pi e^2} \right)^{1/2} \frac{\left| det'\left(\left. \frac{\delta^2 S_E}{\delta \phi_i^2} \right|_{\text{inst.}} \right) \right|^{-1/2}}{\left(det\left(\left. \frac{\delta^2 S_E}{\delta \phi_i^2} \right|_{0} \right) \right)^{-1/2}}. \tag{10.45}$$

The internal rotation actually corresponds to the dyonic degree of freedom, internal rotation at a given angular frequency gives rise to a magnetically and

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electrically charged state, called the dyon. The full rate of pair production and consequent decay of the magnetic field must include the production of pairs of dyons. But for the lowest order, we can restrict ourselves to the case of a simple monopole pair production. The internal rotation is intimately connected with gauge fixing and the Faddev–Popov factor.

The translation zero modes naively are not gauge-invariant and must be made so by an accompanying gauge transformation, we find

$$\begin{aligned}(\delta A_\mu)_\nu &= \partial_\nu A_\mu - D_\mu A_\nu = -F_{\mu\nu}\\ (\delta\phi)_\nu &= \partial_\nu \phi - [A_\nu, \phi] = D_\nu \phi\end{aligned} \tag{10.46}$$

and the normalization is (no sum on ν, sum on a assumed)

$$N_\nu = \int d^4x \left(\sum_\mu F^a_{\mu\nu} F^a_{\mu\nu} + (D_\nu)^a (D_\nu)^a \right). \tag{10.47}$$

The calculation of the determinant is possible in the limit $R \to \infty$ $(B \to 0)$. In this limit, the fluctuations separate into those that change the shape of the monopole and those that change the shape of the loop.

Using the circular symmetry and the gauge $A_\theta = 0$, we have

$$\left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{inst.} = \left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{3,inst.} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{10.48}$$

where the first term depends on x, y, r and is essentially a three-dimensional operator, while the second term comes from the kinetic energy, for example,

$$D_\mu D_\mu = D_1 D_1 + D_2 D_2 + D_r D_r + \frac{1}{r} D_r + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{10.49}$$

Eigenfunctions admit a separation of variables as

$$\Psi(x_1, x_2, r, \theta) = \psi(x_1, x_2, r) \begin{cases} \cos(n\theta) & n = 0, 1, 2, \cdots \\ \sin(n\theta) & n = 1, 2, 3, \cdots \end{cases} \tag{10.50}$$

and then in the sector of angular momentum n we get

$$\left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{inst.} = \left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{3,inst.} + \frac{n^2}{r^2}. \tag{10.51}$$

Now we make an expansion in $1/R$, with $z = r - R$, then, for example,

$$\begin{aligned}(D_\mu D_\mu)_3 &= D_{x_1} D_{x_1} + D_{x_2} D_{x_2} + D_z D_z + \frac{1}{R+z} D_z\\ &= D_i D_i + \left(\frac{1}{R} - \frac{z}{R^2} + \cdots \right) D_z.\end{aligned} \tag{10.52}$$

To lowest order $(1/R)^0$ we just get the operator corresponding to the second variation of the Hamiltonian with a static monopole at $\vec{x} = 0$

$$\left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{3,inst.} = \left.\frac{\delta^2 H}{\delta\phi_i^2}\right|_{3,mono.} + o\left(\frac{1}{R}\right). \tag{10.53}$$

The angular momentum term also admits an expansion

$$\frac{n^2}{r^2} = \frac{n^2}{R^2}\left(1 - \frac{2z}{R} + \cdots\right) \tag{10.54}$$

so that to lowest order we have

$$\left(\left.\frac{\delta^2 H}{\delta\phi_i^2}\right|_{3,mono.} + o\left(\frac{1}{R}\right) + \frac{n^2}{R^2} + o\left(\frac{1}{R}^3\right)\right)\psi_i^{(n)} = \lambda_i^{(n)}\psi_i^{(n)} \tag{10.55}$$

and we note that the angular momentum term is a constant. The eigenvalues are then simply

$$\lambda_i^{(n)} = \omega_i^2 + \frac{n^2}{R^2}, \tag{10.56}$$

where ω_i^2 are the eigenvalues of $\left.\frac{\delta^2 H}{\delta\phi_i^2}\right|_{3,mono.}$. The $\lambda_i^{(n)}$ admit an expansion in $1/R$ as

$$\lambda_i^{(n)} = \omega_i^2 + \frac{n^2}{R^2} + \frac{a_i}{R^2} + \frac{b_i n^2 + c_i}{R^4} + \cdots, \tag{10.57}$$

where the odd powers vanish as the order zero eigenfunctions have definite parity under $z \to -z$. The correction a_i is difficult to compute, but it is expected to give a small correction for the non-zero eigenmodes. To calculate them in principle, we must find the correction to the instanton to order $o(1/R^2)$ and then compute the correction to the eigenvalues to second order in perturbation theory. However, for the zero modes the correction is important, but easily calculable.

There are three translational zero modes; first, consider the modes for translation in the x_1 and x_2 directions. These are out of the plane of the loop and correspond to $\omega_{x_1}^2 = 0$ and $\omega_{x_2}^2 = 0$. For these $n = 0$ and $\lambda_{x_1}^{(0)} = 0 = \lambda_{x_2}^{(0)}$. Thus for these to remain zero modes to order $1/R$ we must have $a_{x_1} = a_{x_2} = 0$. For translation in the z direction, we see these are translational zero modes of the monopole in the plane of the loop. These must come with multiplicity two as there are two independent directions for the translation. Furthermore, they must deform the loop, hence they must correspond to $n \neq 0$. Indeed, the first deformation of the loop occurs for $n = 1$ and the two independent angular eigenmodes give the two independent directions of the deformation. Thus we require that $\lambda_z^{(1)} \equiv \lambda_{x_3}^{(1)} = \lambda_{x_4}^{(1)} = 0$. For the zero-order Hamiltonian, we already have $\omega_{x_3}^2 = 0$ and $\omega_{x_4}^2 = 0$, hence to order $1/R$ we must have

$$\lambda_z^{(1)} = 0 = 0 + \left.\frac{n^2}{R^2}\right|_{n=1} + \frac{a_z}{R^2} + \cdots = \frac{1}{R^2} + \frac{a_z}{R^2} \quad \Rightarrow \quad a_z = -1. \tag{10.58}$$

We perform exactly the same separation of variables and analysis for the denominator in Equation (10.45)

$$\left.\frac{\delta^2 S_E}{\delta\phi_i^2}\right|_{3,0} = \left.\frac{\delta^2 H^0}{\delta\phi_i^2}\right|_3 + o\left(\frac{1}{R}\right), \tag{10.59}$$

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which gives

$$\lambda_{i,0}^{(n)} = \omega_{i,0}^2 + \frac{n^2}{R^2} + \frac{a_{i,0}}{R^2} + \cdots . \tag{10.60}$$

The determinant corresponds to the product of the eigenvalues, thus the angular momentum family corresponding to eigenmode i contributes as

$$\ln K_i = -\frac{1}{2}\left(\ln\lambda_i^{(0)} - \ln\lambda_{i,0}^{(0)} + 2\sum_{n=1}^{\infty}\left(\ln\lambda_i^{(n)} - \ln\lambda_{i,0}^{(n)}\right)\right), \tag{10.61}$$

where the factor of 2 is because all the $n \neq 0$ modes come with multiplicity two while the mode $n = 0$ is solitary. To perform the summation we use the Euler–Maclaurin formula [2]

$$\begin{aligned} f(0) + 2\sum_{n=1}^{N} f(n) = 2\left(\int_0^N dx f(x)\right) + f(N) + B_1(f'(N) - f'(0)) \\ - \frac{1}{12}B_2(f'''(N) - f'''(0)) + \cdots , \end{aligned} \tag{10.62}$$

where the B_is are the Bernoulli numbers and $f(n) = \ln(\omega_i^2 + \frac{n^2}{R^2} + \frac{a_i}{R^2} + \cdots) - \ln(\omega_{i,0}^2 + \frac{n^2}{R^2} + \frac{a_{i,0}}{R^2} + \cdots)$. For large n, we expect that $\lambda_i^{(n)} \to \lambda_{i,0}^{(n)}$, hence $f(N), f'(N), f'''(N), \cdots$ all vanish. Also since $\lambda_i^{(n)}$ is actually a function of n^2 the odd derivatives vanish at $n = 0$, and only the first term contributes, giving (letting $Ry = x$)

$$\begin{aligned} \ln K_i &= -R\int_0^{\infty} dy\left(\ln(\omega_i^2 + y^2 + \frac{a_i}{R^2} + \cdots) - \ln(\omega_{i,0}^2 + y^2 + \frac{a_{i,0}}{R^2} + \cdots)\right) \\ &= -R\int_0^{\infty} dy\left(\ln\left(\frac{\omega_i^2 + y^2}{\omega_{i,0}^2 + y^2}\right) + o\left(\frac{1}{R^2}\right)\right) \\ &= -R\pi(\omega_i^2 - \omega_{i,0}^2) + o\left(\frac{1}{R}\right). \end{aligned} \tag{10.63}$$

This follows from using the integral,

$$\begin{aligned} &= R\int_0^{N/R} dy \ln(\omega^2 + y^2)\; Ry\ln(\omega^2 + y^2) - 2Ry + 2R\omega \arctan\frac{y}{\omega}\Big|_0^{N/R} \\ &= N\ln\left(\omega^2 + \frac{N^2}{R^2}\right) - 2N + 2R\omega\arctan\left(\frac{N}{\omega R}\right) \\ &= N\ln\left(\omega^2 + \frac{N^2}{R^2}\right) - 2N + R\pi\omega \end{aligned} \tag{10.64}$$

taking $N \to \infty$. This approximation is fine for all the angular momentum families that do not have exact zero modes. For $n = 0, 1$ we would get a vanishing result and singularities in the amplitude.

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We can, of course, still apply the method to the comparison theory of the true vacuum without the monopole. Here we get

$$
\begin{aligned}
&= \frac{1}{2}\ln\lambda_{i,0}^{(0)} + \sum_{n=1}^{N}\ln\lambda_{i,0}^{(n)}\frac{1}{2}\ln\omega_{i,0}^2 + \sum_{n=1}^{N}\left(\omega_{i,0}^2 + \frac{n^2}{R^2}\right) \\
&= R\int_0^{N/R} dy \ln(\omega_{i,0}^2 + y^2) + \frac{1}{2}\ln\left(\omega_{i,0}^2 + \frac{N^2}{R^2}\right) + o\left(\frac{1}{N}\right) \\
&= R\pi\omega_{i,0} + (2N+1)\ln\left(\frac{N}{R}\right) - 2N + o\left(\frac{1}{N}\right). \qquad (10.65)
\end{aligned}
$$

This follows from the integral Equation (10.64) after adding $\frac{1}{2}\ln\left(\omega_{i,0}^2 + \frac{N^2}{R^2}\right)$ and expanding for large N.

For the three zero modes, the sum over $\lambda_a^{(n)}$ for $a = x_1, x_2, z$ is done explicitly excluding $\lambda_{x_1}^{(0)}$, $\lambda_{x_2}^{(0)}$ and $\lambda_z^{(1)}$ (with multiplicity two). We will use the Stirling approximation $\ln N! \approx N\ln N - N + \frac{1}{2}\ln(2\pi N)$. For $a = x_1, x_2$ we get, noting $\omega_a^2 = 0$

$$
\begin{aligned}
-\sum_{n=1}^{N}\ln\lambda_a^{(n)} &= -\sum_{n=1}^{N}\ln\left(\left(\frac{n^2}{R^2}\right) + o\left(\frac{1}{R}\right)\right) \\
&\approx -2\ln\left(\frac{1}{R^N}\prod_{n=1}^{N} n\right) = -2\ln\left(\frac{N!}{R^N}\right) \\
&= -2\left(N\ln N - N + \frac{1}{2}\ln(2\pi N)\right) + 2N\ln R \\
&= -2N\ln\left(\frac{N}{R}\right) + 2N - \ln(2\pi N) + o\left(\frac{1}{N}\right). \qquad (10.66)
\end{aligned}
$$

Then subtracting the true vacuum result, Equation (10.65), from the result in the presence of the instanton, Equation (10.66), we get

$$
\ln K_{x_1} = -R\pi(\omega_{x_1} - \omega_{x_1.0}) - \ln(2\pi R) \quad \text{and} \quad \ln K_{x_2} = -R\pi(\omega_{x_2} - \omega_{x_1.0}) - \ln(2\pi R) \qquad (10.67)
$$

keeping in mind that $\omega_{x_1}^2 = \omega_{x_1}^2 = 0$. For $a = z$ we have $\lambda_z^{(1)} = 0$, thus we must perform the sum

$$
-\frac{1}{2}\ln|\lambda_z^{(0)}| - \sum_{n=2}^{N}\ln\lambda_z^{(n)}, \qquad (10.68)
$$

where we have put absolute value signs around $\lambda_z^{(0)}$ as it is negative (and the i is taken out explicitly in the Equations (10.43) and (10.44)). As $\omega_z^2 = 0$ and $a_1 = -1$, we get $\lambda_z^{(0)} = -1/R$. Furthermore, putting $\lambda_z^{(n)} = (n^2-1)/R$, we get

$$
-\frac{1}{2}\ln|\lambda_z^{(0)}| - \sum_{n=2}^{N}\ln\lambda_z^{(n)} = -\frac{1}{2}\ln\left|\frac{-1}{R^2}\right| - \sum_{n=2}^{N}\ln\left(\frac{n^2-1}{R^2}\right). \qquad (10.69)
$$

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We evaluate the sum as follows

$$
\begin{aligned}
&= \sum_{n=2}^{N} \ln\left(\frac{n^2-1}{R^2}\right) \sum_{n=2}^{N} \ln\left(\frac{n^2}{R^2}\left(1-\frac{1}{n^2}\right)\right) = \sum_{n=2}^{N} \ln\frac{n^2}{R^2} + \sum_{n=2}^{N} \ln\left(1-\frac{1}{n^2}\right)\\
&= \ln\prod_{n=2}^{N}\frac{n^2}{R^2} + \ln\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right) = 2\ln N! + \ln\prod_{n=2}^{N}\left(\frac{(n+1)(n-1)}{n^2}\right)\\
&= 2\ln N! - 2(N-1)\ln R + \sum_{n=2}^{N}\left(\ln\left(\frac{n+1}{n}\right) - \ln\left(\frac{n}{n-1}\right)\right)\\
&= 2N\ln\left(\frac{N}{R}\right) - 2N + \ln 2\pi N + 2\ln R - \ln(2)
\end{aligned}
\tag{10.70}
$$

as the final sum is telescopic and gives the $-\ln 2$. Adding the $-\frac{1}{2}\ln\left(\frac{|-1|}{R^2}\right) = \ln R$ gives

$$
\begin{aligned}
-\frac{1}{2}\ln|\lambda_z^{(0)}| - \sum_{n=2}^{N}\ln\lambda_z^{(n)} &= \ln R - \left(2N\ln\left(\frac{N}{R}\right) - 2N + \ln 2\pi N + 2\ln R - \ln(2)\right)\\
&= -2N\ln\left(\frac{N}{R}\right) + 2N - \ln(\pi N R).
\end{aligned}
\tag{10.71}
$$

Then subtracting the vacuum result, Equation (10.65), we get

$$
\ln K_z = -R\pi(\omega_z - \omega_{z,0}) - \ln \pi R^2, \tag{10.72}
$$

where of course $\omega_z = 0$. Thus finally adding all the three contributions together we get

$$
\sum_i \ln K_i = -R\pi(\omega_i - \omega_{i,0}) - 2\ln 2\pi R - \ln\pi R^2 = R\pi(\omega_i - \omega_{i,0}) - \ln 4\pi^3 R^4 \tag{10.73}
$$

or equally well

$$
K = \frac{1}{4\pi^3 R^4} e^{-R\pi\sum_i(\omega_i - \omega_{i,0})}. \tag{10.74}
$$

The sum $\frac{1}{2}\sum_i(\omega_i - \omega_{i,0})$ has a perfect physical interpretation as the renormalized energy of the magnetic monopole due to vacuum fluctuations about the monopole configuration. This energy is properly subtracted with the energy of the vacuum fluctuations about the true vacuum. Thus we write

$$
\frac{1}{2}\sum_i(\omega_i - \omega_{i,0}) = \Delta M. \tag{10.75}
$$

The Faddeev–Popov factors, which we have not explicitly dealt with, will also contribute; however, their contribution also simply contributes to the renormalization of the mass of the monopole.

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10.7 The Final Amplitude for Decay

The final thing we must calculate are the normalization factors of the translation zero modes using the explicit expressions for the zero modes given by Equation (10.46). We use the coordinates x_1, x_2, r, θ, but will rather use $r = z + R$. First for the directions $i = x_1, x_2$, circular symmetry gives a factor of 2π. The field strength and covariant derivatives of the scalar field are independent of the θ direction, *i.e.* $F_{\theta,\mu} = 0, D_\theta\phi = 0$. The dominant contribution comes from the regions near $z = R$. We can use spherical symmetry in the three independent coordinates x_1, x_2, z. Then the normalization is given by,

$$
\begin{aligned}
&= N_i \; 2\pi \int dx_1 dx_2 dr r \left(\sum_\mu F^a_{\mu i} F^a_{\mu i} + (D_i\phi)^a (D_i\phi)^a \right) \\
&\approx 2\pi R \int dx_1 dx_2 dr \left(\sum_\mu F^a_{\mu i} F^a_{\mu i} + (D_i\phi)^a (D_i\phi)^a \right) \\
&= \frac{2\pi R}{3} \int d^3x \left(\sum_{ij} F^a_{ij} F^a_{ij} + (D_i\phi)^a (D_i\phi)^a \right)
\end{aligned} \tag{10.76}
$$

as, for example, $F^2_{21} + F^2_{31} = (2/3)(F^2_{21} + F^2_{31} + F^2_{32}) = (1/3)\sum_{jk} F^2_{jk}$.

For the mode $i = 3, 4$ we get a similar expression, but there is angular dependence. Then, for example, $D_3 = \cos\theta D_z$ and we get

$$
\begin{aligned}
&= N_3 \; 2\pi \int dx_1 dx_2 dr r \left(\sum_\mu F^a_{\mu 3} F^a_{\mu 3} + (D_3\phi)^a (D_3\phi)^a \right) \\
&= R \int d^3x \int d\theta \left(\sum_{i=1,2} F^a_{iz} F^a_{iz} + (D_z\phi)^z (D_z\phi)^a \right) \cos^2\theta \\
&= N_i \frac{1}{2\pi} \int_0^{2\pi} d\theta \cos^2\theta = \frac{1}{2} N_i.
\end{aligned} \tag{10.77}
$$

Thus we only have to evaluate the integral $\int d^3x \left(\sum_{ij} F^a_{ij} F^a_{ij} + (D_i\phi)^a (D_i\phi)^a \right)$, which can be related easily to the monopole mass. The monopole mass is given by

$$
M = \frac{1}{e^2} \int d^3x \left(\frac{1}{4} F^a_{jk} F^a_{jk} + \frac{1}{2} (D_j\phi)^a (D_j\phi)^a + V(\phi) \right). \tag{10.78}
$$

However, the expression for mass, which is the energy of the monopole, must be stationary with respect to arbitrary variations for the fields. Making a scale transformation $\phi(x) \to \phi(\alpha x)$ and $A(x) \to aA(\alpha x)$ and demanding the mass be stationary at $\alpha = 1$ gives

$$
\int d^3x \left(\left(\frac{1}{4} F^a_{jk} F^a_{jk} \right) - \frac{1}{2} (D_j\phi)^a (D_j\phi)^a - 3V(\phi) \right) = 0. \tag{10.79}
$$

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Thus

$$\int d^3xV(\phi(x)) = \frac{1}{3}\int d^3x\frac{1}{4}f^2 - \frac{1}{3}(D\phi)^2, \tag{10.80}$$

which gives

$$M = \frac{1}{e^2}\int d^3x\left(\frac{1}{3}F^a_{jk}F^a_{jk} + \frac{1}{3}(D_j\phi)^a(D_j\phi)^a\right) = \frac{1}{e^2}\frac{N_i}{2\pi R}. \tag{10.81}$$

So $N_i = 2\pi Re^2M$ and $N_3 = N_4 = \pi Re^2M$. Thus

$$\left(\frac{N_i}{2\pi e^2}\right)^{1/2} = (RM)^{1/2} \quad i = 1,2, \quad \left(\frac{N_i}{2\pi e^2}\right)^{1/2} = (RM/2)^{1/2} \quad i = 3,4 \tag{10.82}$$

and

$$K = \frac{1}{2}\prod_{i=1}^{4}\left(\frac{N_i}{2\pi e^2}\right)^{1/2}K' = RM\times\frac{RM}{2}\frac{1}{4\pi^3R^4}e^{-R\pi 2\Delta M} = \frac{M^2}{8\pi^3}e^{-R\pi 2\Delta M} \tag{10.83}$$

Then putting in the factor for the classical instanton action we get the final expression for the amplitude of the decay of the magnetic field

$$\Gamma = \frac{M^2}{8\pi^3R^2}e^{-R\pi 2\Delta M}e^{-\left(\pi M^2/g^2B - g^2/4\right)}. \tag{10.84}$$

Using $M/R = gB$, writing $M_{ren.} = M + \Delta M$ and assuming $\Delta M \ll M$

$$\Gamma = \frac{g^2B^2}{8\pi^3}e^{-\left(\pi M^2_{ren.}/g^2B - g^2/4\right)}. \tag{10.85}$$

We have not taken into account the zero mode corresponding to internal rotations. As we have mentioned, this mode corresponds to the dyonic excitation. Without the creation of dyonic pairs, the zero mode will give a factor of

$$\left(\frac{J}{Re^2}\right)^{1/2}, \tag{10.86}$$

where J/R is defined to be the normalization of this zero mode. J is calculable from the exact solutions for the dyons as is the mass of the dyon [66]. There is a whole family of dyon solutions with all possible charges, all of which can be produced in pairs. We will not treat the calculation in detail here and refer the reader to the original article [4]. We simply quote the final result, writing Γ_M for the pure monopole result Equation (10.85)

$$\Gamma = \Gamma_M\left(\frac{J}{Re^2}\right)^{1/2}\sum_{-\infty}^{\infty}e^{-(\pi J/Re^2)n^2} = \Gamma_M\sum_{-\infty}^{\infty}e^{-(\pi M/gB)(e^2n^2/J)} \tag{10.87}$$

using the Poisson summation formula

$$\sum_m f(m) = \sum_m\left(\int dxe^{2\pi imx}f(x)\right) \tag{10.88}$$

and performing the ensuing Gaussian integral and that $M/R = gB$.