A NOTE ON EXACT COLIMITS

John R. Isbell*

(received March 18, 1968)

Introduction. This note proves another special case of a conjecture of U. Oberst. Oberst considered [1], for any small category $\mathfrak{X}$, the abelian category $\mathfrak{A}^{\mathfrak{X}}$ of abelian group-valued functors on $\mathfrak{X}$, and the functor Colim: $\mathfrak{A}^{\mathfrak{X}} \to \mathfrak{A}$ which takes each diagram to its colimit. The question is, when is Colim exact? For its relationships, see [1]. It is a sufficient condition that each component of $\mathfrak{X}$ is upward filtered. Oberst conjectured that it is also necessary, and proved this under some conditions. He mentioned particularly the case that $\mathfrak{X}$ is a monoid, i.e. a category with one object. We shall verify the conjecture in that case.

We formulate a necessary and sufficient condition on $\mathfrak{X}$ for exactness of Colim. It is messy, and evidently implicit in a neater characterization in [1]; but it presents the combinatorial problem which seems to be in the middle of Oberst's conjecture. In a local sense, there are five cases. The assumption that $\mathfrak{X}$ is a monoid permits us to reduce to two cases and permits one other crucial reduction.

1. Characterization. This portion of the paper more or less rephrases work of Oberst-Röhrl [2] and Oberst [1] which transports the property "Colim is exact" into some additive categories canonically constructed from the given category $\mathfrak{X}$; but we go only into the free abelian group $F_{\mathfrak{X}}$ generated by the morphisms of $\mathfrak{X}$.

THEOREM. Colim: $\mathfrak{A}^{\mathfrak{X}} \to \mathfrak{A}$ is exact if and only if the components of $\mathfrak{X}$ have directed classes of objects and for every finite set of pairs of morphisms $f_i : X_i \to X_0$, $g_i : X_i \to X_0$, there are two equipotent finite sets of morphisms $h_k : X_0 \to Y_k$, $j_k : X_0 \to Z_k$, such that in $F_{\mathfrak{X}}$ each $\sigma_i = \sum_k (h_k - j_k)(f_i - g_i)$ is $f_i - g_i$.

The formula for σ_i could be called improper; the meaning is, of course, that the 4n group elements $h_k f_i$, $-h_k g_i$, $-j_k f_i$, $j_k g_i$ $(k = 1, \ldots, n)$ are added. Further, a category has a directed class of objects if its preordered reflection is directed, i.e. for any two objects X, Y there is an object Z such that Hom(X, Z) and Hom(Y, Z) are nonempty.

---

*Supported by the National Science Foundation.

Canad. Math. Bull. vol. 11, no. 4, 1968.

https://doi.org/10.4153/CMB-1968-068-7 Published online by Cambridge University Press

Before proving the theorem let us note how it may be read as offering five cases. It is five cases for each pair (f_i, g_i); and at first there appear to be six. The 2n positive and 2n negative terms must be f_i, $-g_i$, and other terms cancelling in pairs. We might have $f_i = h_1 f_i$, $g_i = h_1 g_i$; $f_i = h_1 f_i$, $g_i = h_2 g_i$; or similarly $h_1 f_i$, $j_1 f_i$; $j_1 g_i$, $h_1 g_i$; $j_1 g_i$, $j_1 f_i$; $j_1 g_i$, $j_2 f_i$. The fifth case reduces to the first on putting $h'_1 = j_1^2 : X_0 \to X_0$, $h'_k = j_k$ for $k > 1$, $j'_k = h_k$.

Now consider the construction of Colim. For nonempty $\mathfrak{X}$, $\mathfrak{G}$ is naturally embedded in $\mathfrak{G}^{\mathfrak{X}}$ as the subcategory of constant functors, and Colim reflects upon the subcategory. Hence (for all $\mathfrak{X}$) Colim is right exact. It is exact if and only if it takes monomorphisms to monomorphisms.

A diagram $\mathfrak{X} \to \mathfrak{G}$ amounts to a system of abelian groups A_α indexed by (or like) the objects X_α of $\mathfrak{X}$, and homomorphisms suitably indexed by the morphisms of $\mathfrak{X}$. A colimit is a quotient of the direct sum group ΣA_α by the subgroup K generated by all the differences $a - f(a)$, a in A_α and $f : A_\alpha \to A_\beta$ in the diagram. It is convenient to generalize the form $a - f(a)$ to $f(a) - g(a)$, and equivalent. $(f(a) - g(a) = a - g(a) + (-a) - f(-a).)$

The general kernel element $k = \Sigma f_i(a_i) - g_i(a_i)$ determines a smallest subfunctor $\{A'_\alpha\}$ such that $k \in \Sigma A'_\alpha$; and for the homomorphism of colimits $\Sigma A'_\alpha / K' \to \Sigma A_\alpha / K$ to be monic, we must have $k \in K'$. Let k_α be the α-th coordinate of k; the subgroups A'_α are generated by elements $t(k_\beta)$, and K'_α by elements $h(k_\beta) - j(k_\beta)$.

Given morphisms $f: X_\alpha \to X_\beta$, $g: X_\alpha \to X_\gamma$ in $\mathfrak{X}$, there must exist X_δ with $X_\beta \to X_\delta$, $X_\gamma \to X_\delta$. For if not, taking a constant Z-valued functor and $a = 1$, $\Sigma(h_i - j_i)(f(a) - g(a))$ could be $f(a) - g(a)$ only with $\Sigma(h_i - j_i)f(a) = f(a)$, an odd number of units adding (in several copies of Z) to 0. It follows easily (see [1] if necessary) that $\mathfrak{X}$ has directed components.

In the same way we may split any $k \in K$ into its parts in various components. For k in one component, all k_α may be sent into some A_0 by morphisms $c_\alpha : X_\alpha \to X_0$; since $k_\alpha - c_\alpha(k_\alpha)$ is certainly in K', we need only consider $k^* = \Sigma c_\alpha(k_\alpha) = k^*_0$. Finally, the condition of the theorem, not mentioning a_i's, is seen to be necessary by considering

https://doi.org/10.4153/CMB-1968-068-7 Published online by Cambridge University Press

a constant free group-valued functor and different generators a_i.

Sufficiency is evident.

2. Monoids. A category is upward filtered if it has a directed class of objects and for every pair $f: X \to Y$, $g: X \to Y$, there exists $j: Y \to Z$ such that $jf = jg$.

THEOREM. *A monoid $\mathfrak{X}$ has $\mathrm{Colim}: \mathfrak{A}^{\mathfrak{X}} \to \mathfrak{A}$ exact if and only if it is upward filtered.*

Proof. Sufficiency (known [1]) is easy; one has jf_i = constant over any finite set of f_i's by induction, and one puts $h = 1$ in the previous theorem.

For necessity, it will suffice to show that given morphisms a and x there exists y such that $yxa = yx$. Then, given a and b, we have $y1a = y1$ and $y'yb = y'y = y'ya$.

For xa and x the previous theorem gives us h_k and j_k $(k \le n)$ with $\Sigma(h_k x + j_k xa) - \Sigma(h_k xa + j_k x) = x - xa$. One of the positive terms is x, one of the negative ones is $-xa$, and the others cancel. We may assume there is no cancellation $h_k x = j_\ell x$, for then $h_k xa = j_\ell xa$ and we may delete that h_k and j_ℓ. Consider first the case that x is $h_1 x$. Here $h_1 xa$ is xa, $\Sigma' h_k xa = \Sigma' h_k x$ (summed over $k > 1$), $\Sigma j_k xa = \Sigma j_k x$. Form an oriented graph whose vertices are the indices of the $h_k (k > 1)$, with an edge from k to ℓ when $h_k xa = h_\ell x$. From the equal sums, there is a sum of disjoint oriented loops partitioning the vertices, and likewise for the indices of the j's. Now a loop on r vertices, including, say, h_m, yields $h_m x = h_m xa^r = h_m xa^{cr}$ for any natural number c. Select an h or j (say h_m) so that r has a minimum number of distinct prime factors; and assume that any $x' - x'a$ yielding a relation $\mu x' = \mu x' a^t$ where t has fewer prime factors satisfies a relation $yx'a = yx'$. Then we consider $x' = h_m x$. If it falls in this case, it yields as above looped graphs on some n' indices of morphisms j'_k and $n' - 1$ indices of morphisms h'_k. For any prime p dividing r, one loop must have order s prime to p, giving (say) $j'_\ell h_m xa^{cr+ds} = j'_\ell h_m x$ for any integers c, d such that $cr + ds > 0$. $t = (r, s)$ is such a $cr + ds$, so we have $(yh_m)xa = (yh_m)x$.

https://doi.org/10.4153/CMB-1968-068-7 Published online by Cambridge University Press

In the other case $x = j_1 xa$, we get relations $\mu xa^r = \mu x$ with loop lengths r adding to n and $n + 1$ instead of n and $n - 1$, as follows. Since the terms $j_1 x, \ldots, j_n x$ are all different from all $h_k x$, they can cancel only against terms $j_k xa$. One of them is left over and must be xa. If $j_1 x = xa$, then $xa^2 = j_1 xa = x$. Otherwise we may index so that $j_2 x = xa$. Then $j_2 xa = xa^2$ must be cancelled, so some $j_k x$ is xa^2. If $k = 1$, we have $xa^3 = j_1 xa = x$; in any case this procedure must close a loop with $j_q xa = xa^q = j_1 x$, $xa^{q+1} = x$. As $n - q$ j's and n h's remain, with $\Sigma h_k xa = \Sigma h_k x$ and $\Sigma' j_k xa = \Sigma' j_k x$, the proof may be concluded as in the other case.

REFERENCES

1. U. Oberst, Homology of categories and exactness of direct limits, to appear.

2. U. Oberst and H. Rőhrl, Singular homology with sheaf coefficients, to appear.

Case Western Reserve University,
Cleveland,
Ohio