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Roots of Simple Modules

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Abstract. We introduce roots of indecomposable modules over group algebras of finite groups, and we investigate some of their properties. This allows us to correct an error in Landrock's book which has to do with roots of simple modules.

The main motivation for this paper is an error in Section III.9 of Landrock's book [7]. In order to explain the error, we fix an algebraically closed field F of characteristic $p > 0$, a finite group G, and a simple module U over the group algebra FG with vertex D. Then U is relatively K-projective where $K := DC_G(D)$, so there is an indecomposable FK-module Y with vertex D such that $Y \mid \operatorname{Res}^G_K(U)$ and $U \mid \operatorname{Ind}^G_K(Y)$. Let $J := N_G(D, Y)$ denote the inertial group of Y in $N_G(D)$. In Theorem III.9.8 of [7], Landrock claims that in this situation J/K is a p'-group.

However, this is not correct. In fact, it is easy to construct a counterexample: Let $p = 2$, let $G = S_4$ denote the symmetric group of degree 4, and let U be the simple FG-module of dimension 2. Then the Klein four-group $D = O_2(G)$ is the unique vertex of U, and $K = DC_G(D) = D$ acts trivially on U, so Y is the trivial FD-module F. Thus $J = N_G(D, Y) = G$, and $|J/K| = 6$ is even.

Landrock's error is contained in the proof of Proposition III.9.3 in [7]. As a substitute for the invalid Theorem III.9.8 in [7], we will show in this paper that, in the situation described above, we always have $O_p(J/K) = 1$. This result is still strong enough to imply Erdmann's Theorem [5] on simple modules with cyclic vertex.

Our result is similar to a property of weights (in the sense of Alperin [1]): If (Q, W) is a weight for FG (*i.e.,* if Q is a p-subgroup of G and W a projective simple $F[N_G(Q)/Q]$-module) then $O_p(N_G(Q, V)/QC_G(Q)) = 1$ where V is a constituent of $\operatorname{Res}^{N_G(Q)}_{QC_G(Q)}(W)$ and W is regarded as an $FN_G(Q)$-module via inflation.

The proof of our result makes use of *roots* of indecomposable modules: Let U be an indecomposable FG-module with vertex D and source Z. Then U (together with D and Z) defines an indecomposable projective $F[DC_G(D)/D]$-module $\tilde{R}$ which we call a *root* of U. Any other $F[DC_G(D)/D]$-module which is a root of U is conjugate to $\tilde{R}$ by an element in $N_G(D, Z)/D$.

These roots appear, under a different name, in the work by Barker [3] on simple modules for p-solvable groups; he calls the pair $(D, \tilde{R})$, a *defect pair* of U. The defect pair of an indecomposable module U is a finer invariant than its *vertex pair* (D, b_D) considered by Sibley in [11] (which builds upon earlier work by Knörr [6]). More precisely, b_D is the block of $FDC_G(D)$ containing the inflation $\hat{R}$ of $\tilde{R}$. It is known that b_D is in Brauer correspondence with the block B of FG containing U: $(b_D)^G = B$. So the vertex pair (D, b_D) of U is a B-subpair in the sense of Alperin–Broué [2].

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We prefer to talk about *roots* of indecomposable modules instead of *defect pairs* because this terminology is analogous to the terminology introduced for blocks by Brauer, *cf.* [7, p. 214]. If U is a simple FG-module (or, more generally, if U is an indecomposable FG-module with simple *multiplicity module*, *cf.* [12]) then, by Knörr's Theorem [6], its root $\tilde{R}$ is a projective simple $F[DC_G(D)/D]$-module. A consequence of this fact is that $C_\Delta(D) \subseteq D \subseteq \Delta$ for a defect group Δ of the block B of FG containing U. This implies that, in a block with abelian defect group Δ, every simple module has vertex Δ.

This paper is organized as follows: In Section 1 we recall some facts about endomorphism algebras of modules. In Section 2 we prove a key lemma on blocks of defect zero (or equivalently: on projective simple modules) and normal subgroups. In Section 3 we recall some facts concerning vertices, sources, multiplicity modules and the Green correspondence. In Section 4 we define roots of indecomposable modules and prove some of their elementary properties. Finally, in Section 5 we prove our main result, mention some consequences, and give examples.

1 The Fitting Correspondence

Let A be a finite-dimensional algebra over a field F. In the following, all modules will be finitely generated. With this convention, every right A-module V has a finite-dimensional endomorphism algebra $E := \mathrm{End}_A(V)$, and V becomes an E-A-bimodule via

$$fva := f(va) = f(v)a \quad (f \in E, v \in V, a \in A).$$

For any idempotent e in E, V has a decomposition into A-submodules:

$$V = eV \oplus (1-e)V.$$

Here, e is primitive if and only if eV is indecomposable. Furthermore, for idempotents $e, e' \in E$, the right A-modules eV and $e'V$ are isomorphic if and only if the (projective) right E-modules eE and $e'E$ are isomorphic. In this case the map

$$eE \mapsto eEV = eV$$

induces a bijection between the set of isomorphism classes of indecomposable projective right E-modules and the set of isomorphism classes of indecomposable direct summands of the A-module V. We will refer to this bijection as the *Fitting correspondence* (defined by V).

For any idempotent e in E, the multiplication map

$$eE \otimes_E V \to eV, \quad f \otimes v \mapsto f(v),$$

is an isomorphism of A-modules. Conversely, for every direct summand W of the right A-module V, the restriction map induces an isomorphism of right E-modules $eE \to \mathrm{Hom}_A(V, W)$ where e is the projection map onto W. This gives a somewhat more functorial interpretation of the Fitting correspondence.

The Fitting correspondence can be used in order to reduce questions about *arbitrary* indecomposable modules to questions about *projective* indecomposable modules. For example, it can be used in order to reduce the proof of the Krull–Schmidt Theorem to the Jordan–Hölder Theorem.

Let us introduce some more notation: For A-modules V and V', we write $V'|V$ if V' is isomorphic to a direct summand of V. If B is a (unitary) subalgebra of A then $V_B = \mathrm{Res}_B^A(V)$ denotes the corresponding restricted B-module. Conversely, for a right B-module W, we denote the corresponding induced A-module by $W^A = \mathrm{Ind}_B^A(W) = W \otimes_B A$. If H is a subgroup of a finite group G and if $A = FG$ and $B = FH$ denote the corresponding group algebras, then we write simply $V_H = \mathrm{Res}_H^G(V)$ and $W^G = \mathrm{Ind}_H^G(W)$. Also, we use similar abbreviations for modules over twisted group algebras.

2 Blocks and Normal Subgroups

In this section we will prove an elementary, but important fact concerning inertial subgroups of blocks. We fix an algebraically closed field F of characteristic $p > 0$ and a finite group G.

Proposition 2.1 *Let M and N be normal subgroups of G such that $N \subseteq M$ and M/N is a p-group. Moreover, let B be a block of FG with defect group $D \subseteq N$, and let b be a block of FN covered by B. Then the inertial group of b in M equals N.*

Proof Let I be the inertial group of b in M. Since I/N is a p-group, there is a unique block β of I covering b. Let Δ be a defect group of β. Then, as is well known, we have $I = \Delta N$. Moreover, the induced block β^M is defined and has defect group Δ. Since M/N is a p-group, β^M is the only block of M covering b. Thus β^M is covered by B. Hence the defect groups of β^M are also contained in N. It follows that $I = \Delta N = N$.

We obtain the following consequence.

Proposition 2.2 *Let M and N be normal subgroups of G such that $N \subseteq M$ and M/N is a p-group. Moreover, let P be a simple projective FG-module, and let Q be an indecomposable direct summand of P_N. Then Q is simple and projective, and the inertial group of Q in M equals N.*

Proof Since P_N is semisimple and projective, Q is simple and projective. Let B and b be the blocks of FG and FN containing P and Q, respectively. Then B covers b, and both blocks have defect zero. Moreover, the inertial group I of Q in M equals the inertial group of b in M. Now Proposition 2.1 implies that $I = N$.

We need a version of Proposition 2.2 for twisted group algebras. So we fix a 2-cocycle $\gamma\colon G \times G \to F^\times$ and denote the corresponding twisted group algebra by $F_\gamma G$. Thus $F_\gamma G$ has an F-basis $\{u_x : x \in G\}$ such that

$$u_x u_y = \gamma(x, y) u_{xy} \quad (x, y \in G).$$

For any subgroup H of G, $F_\gamma H := \bigoplus_{x \in H} F u_x$ is a (unitary) subalgebra of $F_\gamma G$. Moreover, for any right $F_\gamma H$-module W, $W^x := W \otimes u_x$ is a right $F_\gamma[x^{-1}Hx]$-module, and we denote the inertial group of W in $N_G(H)$ by

$$N_G(H,W) := \{x \in N_G(H) : W^x \cong W\},$$

so that $H \subseteq N_G(H,W) \subseteq N_G(H)$.

Proposition 2.3 *Let M and N be normal subgroups of G such that $N \subseteq M$ and M/N is a p-group. Moreover, let P be a simple projective $F_\gamma G$-module, and let Q be an indecomposable direct summand of P_N. Then Q is a simple projective $F_\gamma N$-module, and the inertial group of Q in M equals N.*

Proof There exists a central group extension,

$$1 \to Z \to \hat{G} \to G \to 1,$$

where Z is a cyclic p'-subgroup of $\hat{G}$ and $Z \to \hat{G}$ is the inclusion map, and there exists a primitive idempotent e in FZ such that $eF\hat{G} \cong F_\gamma G$. We identify G with $\hat{G}/Z$ and write $M = \hat{M}/Z$, $N = \hat{N}/Z$ with normal subgroups $\hat{M}$, $\hat{N}$ of $\hat{G}$ containing Z. Then $\hat{N} \subseteq \hat{M}$, and $\hat{M}/\hat{N}$ is a p-group. We can view P as a (simple projective) $F\hat{G}$-module via $F\hat{G} \to eF\hat{G} \cong F_\gamma G$. Similarly, we can view Q as an (indecomposable) $F\hat{N}$-module, and then Q is a direct summand of $P_{\hat{N}}$. By Proposition 2.2, Q is a simple projective $F\hat{N}$-module, and the inertial group $\hat{I}$ of Q in $\hat{M}$ equals $\hat{N}$. Thus Q is a simple projective $F_\gamma N$-module. Moreover, $Z \subseteq \hat{I}$, and $I := \hat{I}/Z$ is the inertial group of Q in M. We conclude that $I = \hat{I}/Z = \hat{N}/Z = N$.

3 Vertices, Sources, Multiplicity Modules

In this section we recall some results on vertices, sources and multiplicity modules of indecomposable modules. We fix an algebraically closed field F of characteristic $p > 0$, a finite group G and an indecomposable right FG-module U. We recall that a subgroup D of G is called a *vertex* of U if D is minimal subject to the condition $U|(U_D)^G$. It is well known that the vertices of U form a conjugacy class of p-subgroups of G.

We fix a vertex D of U and denote by Z a *source* of U. Thus Z is an indecomposable right FD-module with vertex D such that $Z|U_D$ and $U|Z^G$. Moreover, Z is unique up to isomorphism and conjugation with elements in $N_G(D)$.

In the following, we set $H := N_G(D)$ and denote by V the *Green correspondent* of U in H. Thus V is an indecomposable right FH-module with vertex D and source Z. Moreover, $V|U_H$ with multiplicity 1, $U|V^G$ with multiplicity 1, and V is uniquely determined by U and D, up to isomorphism. Conversely, U is uniquely determined by V and D, up to isomorphism.

Next, let $I = N_G(D,Z) = \{x \in N_G(D) : Z^x \cong Z\}$ denote the inertial group of Z in $N_G(D)$. By Clifford theory, there is an indecomposable right FI-module W with vertex D and source Z such that $V \cong W^H$. Moreover, W is uniquely determined, up to isomorphism, by V and Z. We will refer to W as the *Clifford correspondent* of V with respect to Z.

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The endomorphism algebra $E := \mathrm{End}_{FI}(Z^I)$ is a finite-dimensional F-algebra *graded* by the factor group $\bar{I} := I/D$. This means that E has a decomposition into F-subspaces,

$$E = \bigoplus_{\bar{x} \in \bar{I}} E_{\bar{x}},$$

such that $E_{\bar{x}} E_{\bar{y}} \subseteq E_{\bar{x}\bar{y}}$ for $\bar{x}, \bar{y} \in \bar{I}$. In our situation, we have

$$E_{\bar{x}} = \{ f \in E : f(Z \otimes 1) \subseteq Z \otimes x \} \cong \mathrm{Hom}_{FD}(Z, Z^x)$$

for $x \in I$ and $\bar{x} = xD \in I/D = \bar{I}$. The *identity component* E_1 of E is a unitary local subalgebra of E isomorphic to $\mathrm{End}_{FD}(Z)$, *cf.* [4, 9].

Since $Z \cong Z^x$ for $x \in I$, the $\bar{I}$-graded F-algebra E is in fact a *crossed product*. This means that every *$\bar{x}$-component* $E_{\bar{x}}$ of E contains a unit of E. Such a unit can be constructed in the following way: For $x \in I$, let $u_x \colon Z \otimes 1 \to Z \otimes x$ be an FD-isomorphism. Then u_x extends uniquely to an FI-automorphism of Z^I which we denote by u_x again. Now u_x is a unit of E and contained in $E_{\bar{x}}$.

The induced left E-module $Z^E := E \otimes_{E_1} Z$ becomes an E-FI-bimodule in the following way: For $z \in Z$ and $x \in I$, we write $u_x(z \otimes 1) = v_x(z) \otimes x$ with $v_x(z) \in Z$. This defines an F-linear map $v_x \colon Z \to Z$ satisfying

$$v_x(zd) = v_x(z) x d x^{-1} \quad (z \in Z, d \in D).$$

The right FI-module structure of Z^E is then defined by

$$(f \otimes z)x := f u_x \otimes v_x^{-1}(z) \quad (f \in E, z \in Z, x \in I).$$

In this way, Z^E becomes an E-FI-bimodule.

In fact, Z^E is an $\bar{I}$-graded E-FI-bimodule. (Recall that an A-B-bimodule M, where A and B are G-graded F-algebras, is called *G-graded* if M has a decomposition into F-subspaces, $M = \bigoplus_{y \in G} M_y$, such that $A_x M_y B_z \subseteq M_{xyz}$ for $x, y, z \in G$.) Note that in our situation both FI and Z^E are naturally $\bar{I}$-graded.

In a similar way, $Z^I = Z \otimes_{FD} FI$ is an $\bar{I}$-graded E-FI-bimodule, and the map

$$\Phi \colon Z^E \to Z^I, \quad f \otimes z \mapsto f(z \otimes 1),$$

is an isomorphism of $\bar{I}$-graded E-FI-bimodules. This isomorphism shows also that the right FI-module structure on Z^E is independent of the choice of the elements u_x ($x \in I$). In the following, we will often identify Z^I and Z^E via Φ.

Since Z is a source of W, W is an indecomposable direct summand of Z^I (as a right FI-module). So W has a Fitting correspondent $P = \mathrm{Hom}_{FI}(Z^I, W)$, which is an indecomposable projective right E-module such that

$$W \cong P \otimes_E Z^I \cong P \otimes_E Z^E \cong P \otimes_{E_1} Z.$$

The Jacobson radical $\mathrm{Rad}(E_1)$ is a nilpotent ideal of E_1. Hence $\mathrm{Rad}(E_1)E = E\,\mathrm{Rad}(E_1)$ is a nilpotent graded ideal of E. Thus $\tilde{E} := E/E\,\mathrm{Rad}(E_1)$ is an $\bar{I}$-graded F-algebra, with $\bar{x}$-component

$$\tilde{E}_{\bar{x}} = E_{\bar{x}} + E\,\mathrm{Rad}(E_1)/E\,\mathrm{Rad}(E_1) \cong E_{\bar{x}}/E_{\bar{x}}\,\mathrm{Rad}(E_1),$$

for $\bar{x} \in \bar{I}$. Since E is a crossed product, so is $\tilde{E}$. Moreover, since E_1 is a local F-algebra, we have $\tilde{E}_1 \cong E_1/\operatorname{Rad}(E_1) \cong F$. This means that $\tilde{E}$ is in fact a twisted group algebra of $\bar{I}$ over F. We denote a corresponding 2-cocycle by $\gamma\colon \bar{I} \times \bar{I} \to F^\times$ and write $\tilde{E} = F_\gamma \bar{I}$.

Now $P\operatorname{Rad}(E_1)$ is an E-submodule of P, and $\tilde{P} := P/P\operatorname{Rad}(E_1)$ becomes an indecomposable projective right $\tilde{E}$-module which is called the *multiplicity module* of U. It is uniquely determined, up to isomorphism, by U, D and Z. Conversely, P is the projective cover of $\tilde{P}$, considered as an E-module via inflation. We illustrate the situation by the following diagram:

$$\begin{tikzcd}[column sep=small]
G, U \arrow[d, dash] & & \\
H = N_G(D), V = W^H \arrow[d, dash] & & \\
I = N_G(D, Z), W \arrow[d, dash] & E, P \arrow[d, dash] & \tilde{E} = F_\gamma \bar{I}, \tilde{P} \arrow[d, dash] \\
D, Z & E_1, E_1 & \tilde{E}_1 = F, F
\end{tikzcd}$$

4 Roots of Indecomposable Modules

We keep the setup of the previous section. Thus U is an indecomposable FG-module with vertex D, source Z and multiplicity module $\tilde{P}$. We set $K := DC_G(D)$, so that K is a normal subgroup of $H = N_G(D)$, and

$$D \subseteq K = DC_G(D) \subseteq I = N_G(D, Z) \subseteq H = N_G(D) \subseteq G.$$

Since W is relatively K-projective, there is an indecomposable right FK-module Y such that $Y|W_K$ and $W|Y^I$. Then Y has vertex D and source Z; moreover, we have $Y|U_K$ and $U|Y^G$. Also, Y is uniquely determined up to isomorphism and conjugation with elements in I.

Lemma 4.1 *Let Y' be an indecomposable FK-module with vertex D such that $Y'|U_K$ and $U|(Y')^K$. Then Y' is conjugate to Y in H.*

Proof We write $U_H = V \oplus V_1 \oplus \cdots \oplus V_n$ with indecomposable FH-modules $V_1, \ldots, V_n$. Then, for $i = 1, \ldots, n$, no indecomposable direct summand of $(V_i)_K$ has vertex D. Since $Y'|(U_H)_K$, this implies that $Y'|V_K$. But any two indecomposable direct summands of V_K are conjugate in H. Since $Y|W_K|(V_I)_K = V_K$, the result follows.

The endomorphism algebra $\operatorname{End}_{FK}(Z^K)$ is graded by the factor group

$$\bar{K} := K/D = DC_G(D)/D \cong C_G(D)/Z(D).$$

Moreover, $\operatorname{End}_{FK}(Z^K)$ is isomorphic to the subalgebra

$$E_{\bar{K}} := \bigoplus_{\bar{x} \in \bar{K}} E_{\bar{x}}$$

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of E via the restriction map

$$E_{\bar{K}} \to \operatorname{End}_{FK}(Z^K), \quad f \mapsto f|Z^K,$$

where we consider Z^K as a subset of Z^I. We will usually identify $E_{\bar{K}}$ and $\operatorname{End}_{FK}(Z^K)$ in this way.

Now $E_{\bar{K}} \operatorname{Rad}(E_1) = \operatorname{Rad}(E_1)E_{\bar{K}}$ is a nilpotent ideal of $E_{\bar{K}}$, and the F-algebra $\tilde{E}_{\bar{K}} = E_{\bar{K}}/E_{\bar{K}} \operatorname{Rad}(E_1)$ coincides with the twisted subgroup algebra $F_\gamma \bar{K}$.

We note that the group algebra $FC_G(D)$ is naturally graded by $C_G(D)/Z(D) \cong \bar{K}$. The following result is elementary, but important.

Lemma 4.2 *For $c \in C_G(D)$, the F-linear map*

$$\rho_c\colon Z^K \to Z^K, \quad z \otimes x \mapsto z \otimes cx,$$

is an FK-isomorphism, and the map

$$\rho\colon FC_G(D) \to \operatorname{End}_{FK}(Z^K) = E_{\bar{K}}, \quad \sum_{c \in C_G(D)} \alpha_c c \mapsto \sum_{c \in C_G(D)} \alpha_c \rho_c,$$

is a homomorphism of $\bar{K}$-graded F-algebras. If $\nu\colon E_{\bar{K}} \to \tilde{E}_{\bar{K}}$ denotes the canonical map, then $\nu \circ \rho\colon FC_G(D) \to \tilde{E}_{\bar{K}} = F_\gamma \bar{K}$ induces an isomorphism of $\bar{K}$-graded F-algebras

$$\sigma : F\bar{K} \cong F[C_G(D)/Z(D)] \to \tilde{E}_{\bar{K}} = F_\gamma \bar{K}.$$

Proof It is obvious that ρ_c is an FK-isomorphism, for $c \in C_G(D)$. Moreover, we have $\rho_1 = \mathrm{id}_{Z^K}$ and $\rho_c \circ \rho_{c'} = \rho_{cc'}$ for $c, c' \in C_G(D)$. Thus the map ρ defined above is a homomorphism of F-algebras. For $c \in C_G(D)$ and $z \in Z$, we have $\rho_c(z \otimes 1) = z \otimes c$, so $\rho_c \in E_{\bar{c}}$. This shows that ρ is a homomorphism of $\bar{I}$-graded F-algebras. Since the $\bar{K}$-graded F-algebra $FC_G(D)$ is a crossed product, $\nu \circ \rho$ is surjective. Also, for $c \in Z(D)$, the map $\rho_c - 1 \in E_1$ is nilpotent since $(\rho_c - 1)(z \otimes x) = z \otimes (c-1)x$ for $z \in Z$ and $x \in K$. Hence $\rho_c - 1 \in \operatorname{Rad}(E_1)$ and $(\nu \circ \rho)(c-1) = 0$. Thus $\nu \circ \rho$ induces an epimorphism of $\bar{K}$-graded F-algebras $\sigma\colon F\bar{K} \to F_\gamma \bar{K}$. Comparing dimensions we see that σ is an isomorphism.

In the following, we will often identify $\tilde{E}_{\bar{K}} = F_\gamma \bar{K}$ and $F\bar{K}$ via σ. Since Y has source Z, Y is an indecomposable direct summand of Z^K (as an FK-module). Hence Y has a Fitting correspondent $R = \operatorname{Hom}_{FK}(Z^K, Y)$, which is an indecomposable projective right $E_{\bar{K}}$-module. Moreover, $R \operatorname{Rad}(E_1)$ is an $E_{\bar{K}}$-submodule of R, and $\tilde{R} := R/R \operatorname{Rad}(E_1)$ is an indecomposable projective module over

$$\tilde{E}_{\bar{K}} = F\bar{K} = F[DC_G(D)/D].$$

We call $\tilde{R}$ a *root* of U. It is uniquely determined by U, D, and Z, up to conjugation with elements in I.

Lemma 4.3 *In the situation above, we have $R|P_{E_{\bar{K}}}$ and $P|R^E$. This implies that $\tilde{R}|\tilde{P}_{\tilde{E}_{\bar{K}}}$ and $\tilde{P}|\tilde{R}^{\tilde{E}}$.*

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Proof Since $Y|W_K$ we can write $W_K = Y \oplus Y'$ with an FK-submodule Y' of W_K. Then

$$\begin{aligned} P_{E_{\bar{K}}} &= \mathrm{Hom}_{FI}(Z^I, W)_{E_{\bar{K}}} \cong \mathrm{Hom}_{FK}(Z^K, W_K)_{E_{\bar{K}}} \\ &\cong \mathrm{Hom}_{FK}(Z^K, Y) \oplus \mathrm{Hom}_{FK}(Z^K, Y') = R \oplus R' \end{aligned}$$

with a left $E_{\bar{K}}$-module $R' := \mathrm{Hom}_{FK}(Z^K, Y')$. This shows that $R|P_{E_{\bar{K}}}$.

Since $W|Y^I$ we can write $Y^I = W \oplus W'$ with an FI-submodule W' of Y^I. There is an isomorphism of right E-modules

$$\mathrm{Hom}_{FK}(Z^K, Y)^E \to \mathrm{Hom}_{FI}(Z^I, Y^I)$$

sending an element of the form $h \otimes f$, with $h \in \mathrm{Hom}_{FK}(Z^K, Y)$ and $f \in E$, to $h^I \circ f$ where $h^I \colon Z^I \cong (Z^K)^I \to Y^I$ is induced from $h \colon Z^K \to Y$. We conclude that

$$\begin{aligned} R^E &= \mathrm{Hom}_{FK}(Z^K, Y)^E \cong \mathrm{Hom}_{FI}(Z^I, Y^I) \\ &\cong \mathrm{Hom}_{FI}(Z^I, W) \oplus \mathrm{Hom}_{FI}(Z^I, W') = P \oplus P' \end{aligned}$$

with a right E-module $P' := \mathrm{Hom}_{FI}(Z^I, W')$. This shows that $P|R^E$.

It follows easily that $\tilde{R}|\tilde{P}_{\tilde{E}_{\bar{K}}}$ and $\tilde{P}|\tilde{R}^{\tilde{E}}$.

We illustrate the situation by the following diagram:

$$\begin{array}{ccc} G, U & & \\ \Big| & & \\ H = N_G(D), V = W^H & & \\ \Big| & & \\ I = N_G(D,Z), W & E, P & \tilde{E} = F_\gamma \bar{I}, \tilde{P} \\ \Big| & \Big| & \Big| \\ K = DC_G(D), Y & E_{\bar{K}}, R & \tilde{E}_{\bar{K}} = F\bar{K}, \tilde{R} \\ \Big| & \Big| & \Big| \\ D, Z & E_1, E_1 & \tilde{E}_1 = F, F \end{array}$$

We have now attached to the indecomposable FG-module U with vertex D and source Z an indecomposable projective $F[DC_G(D)/D]$-module $\tilde{R}$, unique up to conjugation with elements in $I = N_G(D, Z)$. The pair $(D, \tilde{R})$ appears in the work of Barker [3] where it is called a *defect pair* of U.

Let B be the block of FG containing U, and let b_D be the block of FK containing Y. By Nagao's Lemma, *cf.* [7], b_D and B are in Brauer correspondence: $(b_D)^G = B$. Thus (D, b_D) is a B-subpair in the sense of Alperin–Broué [2]. It is uniquely determined by U, up to conjugation in G. Sibley [11] has called (D, b_D) a *vertex pair* of U.

It is well known that the image $\bar{b}_D$ of b_D in $F[DC_G(D)/D] = F\bar{K}$ is a block of $F\bar{K}$, and it is easy to verify that the indecomposable projective right $F\bar{K}$-module $\tilde{R}$ belongs to $\bar{b}_D$. Indeed, let

$$a = \sum_{c \in C_G(D)} \alpha_c c \in Z(FC_G(D)) \subseteq Z(FDC_G(D)) = Z(FK).$$

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Then, for $z \in Z$ and $x \in K$, we have

$$(\rho(a))(z \otimes x) = \Big(\sum_{c \in C_G(D)} \alpha_c \rho_c\Big)(z \otimes x) = \sum_{c \in C_G(D)} \alpha_c (z \otimes cx)$$
$$= z \otimes \sum_{c \in C_G(D)} \alpha_c cx = z \otimes ax = z \otimes xa = (z \otimes x)a.$$

This shows that, for $a \in Z(FC_G(D))$, $\rho(a) \in E_{\bar{K}} = \mathrm{End}_{FK}(Z^K)$ is right multiplication with a. Thus, for $f \in \mathrm{End}_{FK}(Z^K)$, $z \in Z$ and $x \in K$, we have

$$\big(f \circ \rho(a)\big)(z \otimes x) = f(z \otimes xa) = f(z \otimes x)a = \big(\rho(a) \circ f\big)(z \otimes x).$$

This means that $\rho\big(Z(FC_G(D))\big) \subseteq Z(\mathrm{End}_{FK}(Z^K))$. Now let e be an idempotent in $\mathrm{End}_{FK}(Z^K)$ such that $Y \cong eZ^K$. Then

$$0 \neq Y = Y 1_{b_D} \cong eZ^K 1_{b_D} = \rho(1_{b_D}) eZ^K,$$

so $0 \neq \rho\big(1_{b_D}\big)\, e$ and $0 \neq \nu(\rho(1_{b_D} e)) = 1_{\bar{b}_D} \tilde{e}$ with $\tilde{e} = \nu(e) \in \tilde{E}_{\bar{K}} = F[DC_G(D)/D]$. This shows that indeed the projective indecomposable $F\bar{K}$-module $\tilde{R} = \tilde{e}F\bar{K}$ belongs to the block $\bar{b}_D$. This implies that the defect pair $(D, \tilde{R})$ determines the vertex pair $(D, 1_{b_D})$, so in general the defect pair $(D, \tilde{R})$ of U is a finer invariant than the vertex pair (D, b_D).

The situation above is already rather involved but we need to complicate it further by introducing the inertial group $J := N_G(D, Y)$ of Y in $N_G(D)$. Since $Y|W_K$, the restriction Y_D is isomorphic to a direct sum of copies of Z. Thus we have

$$K = DC_G(D) \subseteq J = N_G(D, Y) \subseteq I = N_G(D, Z).$$

The Fitting correspondence implies easily that $\bar{J} := J/D$ is the inertial group of R in $\bar{I}$. Hence, by lifting theorems for idempotents, $\bar{J}$ is also the inertial group of $\tilde{R}$ in $\bar{I}$.

By Clifford theory, there is a unique indecomposable right FJ-module X, up to isomorphism, such that $Y|X_K$ and $W \cong X^I$. Then X has vertex D and source Z.

The $\bar{J}$-graded F-algebra $\mathrm{End}_{FJ}(Z^J)$ is isomorphic to the subalgebra

$$E_{\bar{J}} := \bigoplus_{\bar{x} \in \bar{J}} E_{\bar{x}}$$

of E. An isomorphism is given by the restriction map

$$E_{\bar{J}} \to \mathrm{End}_{FJ}(Z^J), \quad f \mapsto f|Z^J,$$

where we consider Z^J as a subset of Z^I. We identify both algebras in this way.

Since $X|Z^J$, the Fitting correspondent $Q := \mathrm{Hom}_{FJ}(Z^J, X)$ of X is an indecomposable projective module over $\mathrm{End}_{FJ}(Z^J) = E_{\bar{J}}$. Moreover, $Y|X_K$ and $W \cong X^I$ imply that $R|Q_{E_{\bar{K}}}$ and $P \cong Q^E$, by a proof similar to that of Lemma 4.3.

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Now $\operatorname{Rad}(E_1)E_{\bar J} = E_{\bar J}\operatorname{Rad}(E_1)$ is a nilpotent graded ideal of $E_{\bar J}$, and the $\bar J$-graded F-algebra $\tilde E_{\bar J} = E_{\bar J}/E_{\bar J}\operatorname{Rad}(E_1)$ is isomorphic to the twisted group algebra $F_\gamma \bar J$. Moreover, $Q\operatorname{Rad}(E_1)$ is a submodule of Q, and $\tilde Q := Q/Q\operatorname{Rad}(E_1)$ becomes an indecomposable projective module over $\tilde E_{\bar J} = F_\gamma \bar J$. It follows easily that $\tilde R | \tilde Q_{\tilde E_{\bar K}}$ and $\tilde P \cong \tilde Q^{\tilde E}$. We illustrate the whole situation by the following diagram:

$$
\begin{tikzcd}
G, U \arrow[d, dash] & & \\
H = N_G(D), V = W^H \arrow[d, dash] & & \\
I = N_G(D,Z), W \arrow[d, dash] & E, P \arrow[d, dash] & \tilde E = F_\gamma \bar I, \tilde P \arrow[d, dash] \\
J = N_G(D,Y), X \arrow[d, dash] & E_{\bar J}, Q \arrow[d, dash] & \tilde E_{\bar J} = F_\gamma \bar J, \tilde Q \arrow[d, dash] \\
K = DC_G(D), Y \arrow[d, dash] & E_{\bar K}, R \arrow[d, dash] & \tilde E_{\bar K} = F\bar K, \tilde R \arrow[d, dash] \\
D, Z & E_1, E_1 & \tilde E_1 = F, F
\end{tikzcd}
$$

5 Roots of Simple Modules

We keep the notation of the previous section but assume, in addition, that the FG-module U is simple (not just indecomposable). In this situation, Knörr [6] has proved the following important fact.

Theorem 5.1 *If U is a simple FG-module then, in the notation above, the multiplicity module $\tilde P$ of U is a simple projective $F_\gamma \bar I$-module.*

Now Proposition 2.3 implies that the $F\bar K$-module $\tilde R$ which is a root of U, is also simple and projective. Hence the block $\bar b_D$ of $F\bar K$ containing $\tilde R$ has defect zero, and $\tilde R$ is the only simple $F\bar K$-module in $\bar b_D$, up to isomorphism. As is well-known, this implies that the block b_D of FK has defect group D; so the vertex pair (D, b_D) is a *self-centralizing* subpair. (We recall that b_D is the block of FK containing Y.)

We know that $\bar J$ is the inertial group of $\tilde R$ in $\bar I$. Since $\tilde R$ is the only simple module in $\bar b_D$, $\bar J$ is also the inertial group of $\bar b_D$ in $\bar I$. Thus J is the inertial group of b_D in I. We can now prove our main result.

Theorem 5.2 *Let U be a simple FG-module. Then, with the notation above, we have $O_p(J/K) = 1$.*

Proof We write $O_p(J/K) = L/K$ with a normal subgroup L of J containing K. Then $\bar K = K/D$ and $\bar L = L/D$ are normal subgroups of $\bar J = J/D$ with $\bar K \subseteq \bar L$, and $\bar L/\bar K \cong L/K$ is a p-group. Since $\tilde Q^{\tilde E} \cong \tilde P$ is simple by Theorem 5.1, so is $\tilde Q$. Thus $\tilde Q$ is a simple projective $F_\gamma \bar J$-module, and $\tilde R$ is an indecomposable direct summand of $\tilde Q_{\bar K}$. By Proposition 2.3, $\bar K$ is the inertial group of $\tilde R$ in $\bar L$. On the other hand, $\tilde R$ is $\bar L$-stable, so $\bar K = \bar L$, $K = L$ and $O_p(J/K) = L/K = 1$.

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We note that Theorem 5.2 holds, more generally, for an arbitrary indecomposable module U with a simple multiplicity module $\tilde{P}$. Let us briefly indicate that Theorem 5.2 is still strong enough to prove Erdmann's result on simple modules with cyclic vertex, *cf.* [5].

Corollary 5.3 *Let U be a simple FG-module with cyclic vertex D in the block B of FG. Then D is a defect group of B.*

Proof Let Y be as above. By the remarks following Theorem 5.1, the block b_D of $FK = FC_G(D)$ has defect group D and satisfies $(b_D)^G = B$. The group algebra FD has exactly one indecomposable module of dimension i, up to isomorphism, for $i = 1, \dots, |D|$. Thus $I = N_G(D, Z) = N_G(D) = H$. Since D is cyclic, its automorphism group is abelian. Hence $J/K = N_G(D, Y)/C_G(D)$ is abelian. Now Theorem 5.2 implies that $O_p(J/K) = 1$, so J/K is a p'-group. But $J/K = N_G(D, b_D)/C_G(D)$, so Brauer's Extended First Main Theorem (see [7]) implies that D is a defect group of $(b_D)^G = B$.

We may therefore view Theorem 5.2 as a generalization of Erdmann's Theorem. Let us return to the general situation where U is indecomposable (not necessarily simple) with vertex D, but suppose also that the trivial FD-module F_D is a source of U. Then $I = N_G(D, Z) = H$, and

$$E = \mathrm{End}_{FI}(Z^I) = \mathrm{End}_{FH}\big((F_D)^H\big) \cong F[H/D].$$

The corresponding left $F[H/D]$-module structure on $(F_D)^H$ is given by

$$xD(\alpha \otimes y) = \alpha \otimes xy \quad (x, y \in H, \alpha \in F).$$

It follows that $\tilde{E} \cong F[H/D]$ also, so the multiplicity module $\tilde{P}$ can be viewed as an indecomposable projective $F[H/D]$-module. In fact, the inflation of $\tilde{P}$ to FH is just the Green correspondent V of U. Thus Knörr's Theorem 5.1 and Theorem 5.2 imply the following result of Okuyama [10].

Proposition 5.4 *Let U be a simple FG-module with vertex D and trivial source F_D. Then the Green correspondent V of U is a simple projective $F[N_G(D)/D]$-module. If Y denotes a constituent of $V_{DC_G(D)}$, then we have $O_p\big(N_G(D, Y)/DC_G(D)\big) = 1$.*

We would like to finish with an example, *cf.* [8]. Let $p = 2$, and let $G = J_1$ be the smallest Janko group, of order 175560. A Sylow 2-subgroup D of G is elementary abelian of order 8, and $N_G(D)/D$ is a non-abelian group of order 21 acting faithfully on D. There are 5 simple FG-modules in the principal block B of FG, of dimensions 1, 20, 56, 56, 76. By Knörr's Theorem, they all have vertex D. The Green correspondents of the simple FG-modules in B have dimensions 1, 12, 8, 8, 12, respectively, and their sources have dimensions 1, 4, 8, 8, 12. The respective inertial groups have orders 168, 56, 168, 168, 168. In each case, the twisted group algebra $F_\gamma \bar{I}$ is isomorphic to $F\bar{I}$ (where $|\bar{I}| = 21$ in four cases and $|\bar{I}| = 7$ in one case. The corresponding multiplicity modules all have dimension 1.

On the other hand, the weights of B are just the simple modules in the principal block of $FN_G(D)$, and their dimensions are 1, 1, 1, 3, 3. Although they are also modules for the non-abelian group $N_G(D)/D$ of order 21, there does not seem to be a direct connection to the multiplicity modules.

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References

[1] J. Alperin, *Weights for finite groups*. Proc. Symp. Pure Math. **47**(1987), 369–379.
[2] J. Alperin and M. Broué, *Local methods in block theory*. Ann. Math. (2) **110**(1979), 143–157.
[3] L. Barker, *On p-soluble groups and the number of simple modules associated with a given Brauer pair*. Quart. J. Math. Oxford Ser. (2) **48**(1997), 133–160.
[4] E. C. Dade, *Group-graded rings and modules*. Math. Z. **174**(1980), 241–262.
[5] K. Erdmann, *Blocks and simple modules with cyclic vertices*. Bull. London Math. Soc. **9**(1977), 216–218.
[6] R. Knörr, *On the vertices of irreducible modules*. Ann. Math. **110**(1979), 487–499.
[7] P. Landrock, *Finite group algebras and their modules*. Cambridge University Press, Cambridge, 1983.
[8] P. Landrock and G. O. Michler, *Block structure of the smallest Janko group*. Math. Ann. **232**(1978), 205–238.
[9] A. Marcus, *Representation theory of group-graded algebras*. Nova Science Publishers, Inc., Commack, New York, 1999.
[10] T. Okuyama, *Module correspondence in finite groups*. Hokkaido Math. J. **10**(1981), 299–318.
[11] D. A. Sibley, *Vertices, blocks and virtual characters*. J. Algebra **132**(1990), 501–507.
[12] J. Thévenaz, *G-algebras and modular representation theory*. Oxford University Press, Oxford, 1995.

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