8

Dissipation and noise in mean field dynamics

In Chapter 6 we presented the main computational schemes to derive the dynamical laws for the mean field, including the back-reaction from quantum fluctuations. These equations may be derived from the variation of the CTPEA. The result of this approach is a semiclassical theory of a c-number condensate interacting with a quantized fluctuation field.

This approach developed at this level of sophistication is limited as it offers no description of the fluctuations themselves. In most applications the magnitude of the fluctuations can be comparable and at times dominates the effects of the mean field in the semiclassical description. One possible way to incorporate fluctuations is to use the 2PI formalism, where the propagators describing the fluctuations are considered as dynamical variables evolving along with the mean fields.

In this chapter we shall explore a different strategy, which is to allow for a stochastic component in the mean field. This component arises from both the uncertainty of the initial configuration of the mean field, and from the fluctuations in the back-reaction from the quantized excitations. Both sources of randomness combine so that stochastic averages in the noisy theory reproduce suitable quantum averages in the underlying quantum field theory.

Formally, this approach lifts the seemingly overladen CTPEA. So far in this generally complex object, only the real part is enlisted in the derivation of the relevant equations of motion of the mean field. By regarding the CTPEA as a kind of influence functional, we shall see that the imaginary part contains the information about the stochastic sources.

The material in this chapter also clarifies the relationship between the CTP and the influence functional approaches. This issue has been addressed by Su *et al.* [SCYC88] and the authors [CalHu94]. In Chapter 5, we derived a Langevin equation for the long-wavelength modes of a quantum field, viewed as an open system interacting with the environment made up of short-wavelength modes. The system–environment divide we shall assume in this chapter is more elusive, since it depends on the c vs. q-number nature rather than on the value of a "hard" observable such as wavelength. In the end, as we shall discuss in detail, the physics is very much the same in one or the other approach. The stochastic mean field approach we shall discuss in this chapter has the redeeming feature that it does not force us to choose an a priori separation between modes which go into the system and which are relegated to the environment. In this sense, it is more pliable to the demands of a particular application: for example, if higher modes are generated through nonlinear effects, we run no risk of them crossing into the environment. This versatility will allow the stochastic mean field approach to retain full information about certain quantum correlations, as opposed to only their long-wavelength components.

An equivalent approach is to write down a Fokker–Planck equation describing the probability density function for the stochastic mean field. We will discuss only the Langevin equation approach; the translation to other formalisms is straightforward with the tools presented in the early chapters of this book.

Also, to facilitate comparison with the illustrated groundwork laid down in Chapter 6, we shall continue with the example of a $g\Phi^3$ relativistic quantum theory at zero temperature. The addition of statistical fluctuations over and above the quantum ones, as well as applications to more realistic theories, will be discussed in the forthcoming chapters.

The themes we shall develop in this chapter are:

(a) The complex terms in the retarded propagator in frequency domain $G_{\mathrm{ret}}(\omega)$ imply dissipation.
(b) Underlying the dissipation of the mean field is the effect of particle creation arising from the amplification of quantum fluctuations by the time-dependent mean field. Dissipation results from the back-reaction of particle creation on the mean field. We shall see this to order g^2 by a direct derivation of the number of created particles.
(c) There are fluctuations in the number of created particles, which brings forth fluctuations in the back-reaction effect. These fluctuations may be incorporated into the dynamics of the mean field or condensate by introducing a stochastic source in the right-hand side of the equation of motion. We shall show that the noise autocorrelation is given precisely by the noise kernel in the 1PI CTPEA. The stochastic c-number field $\phi(t)$ does not represent the expectation value of the Heisenberg field anymore; we shall refer to it as the stochastic condensate. In the linearized theory, the stochastic average of the condensate gives back the quantum average which is the mean field.
(d) The resulting stochastic theory is a nontrivial extension of mean field dynamics, in the sense that, at least for linear theories, the stochastic formulation reproduces some quantum correlation functions of the full theory. This result, which is similar to one already proven for quantum open systems, shows that the identification of the CTPEA as an influence functional – and therefore of the condensate as an effectively open system – is not merely a formal device.
(e) It is clear from their perturbative expressions that the noise and dissipation kernels are closely related to each other. We may now show that, if we allow the condensate to equilibrate under the effect of the noise, then the relationship between the noise and dissipation kernels becomes the fluctuation–dissipation theorem. Alternatively, one may use the fluctuation–dissipation relation to find the noise kernel given the dissipation kernel, and vice versa.

(f) While one can envisage many situations where a quantum field may be split into a system field and an environment, it is not obvious that it is justified to treat the former as classical. We will show that particle creation is also central to this issue, by deriving an expression for the decoherence functional between two system histories in terms of the Bogoliubov coefficients describing particle creation in the environment. In short, system and environment get entangled through particle creation, and decoherence occurs when it is efficient.

(g) From the linear theory results it may seem that these effects are restricted to high frequencies $\omega > 2m$. We shall see that this limitation is lifted by nonlinear effects. In particular, we shall show that a coherent condensate oscillation will create particles even if the frequency is below threshold, through the process of parametric amplification. The difference is that parametric amplification is an essentially nonperturbative phenomenon, and it is exponentially suppressed as we move away from the threshold. So dissipation and fluctuation are generic properties of condensate dynamics.

Of course, a simple oscillation will not in general be a solution of the free equations of motion, precisely because it will dissipate through particle creation. The problem of evolution under back-reaction from quantum fluctuations is rather complex. It involves not only finding long-time solutions to the equations, but also the harder problem of making sure that the equations contain the relevant physics in the different time ranges. For example, fluctuation–fluctuation interactions, which are totally ignored in the one-loop or leading order $1/N$ approximations, are crucial on scales of the order of the thermalization time. We shall discuss these issues in later chapters.

Since dissipation and noise are central elements in nonequilibrium evolutions, a complete set of references for this chapter would be coextensive with the literature on nonequilibrium field theory itself. Our discussion will loosely follow [CalHu89, CalHu94, CalHu95, CalHu97]. See also [Hu89, HuSin95, CamVer96]. The latter two papers, when read as a sequel to [CalHu87] give a clear example of how dissipation and noise can be identified from the CTPEA with the help of the influence action, and Langevin equations (in that context, the Einstein–Langevin equations) can be derived for the stochastic mean field (semiclassical) dynamics. Stochastic equations for classical systems arising from the decoherence functional formalism have been discussed by Gell-Mann and Hartle [HarGel93]. The formal analysis of the Einstein–Langevin equations developed by Hu and Matacz [HuMat95], Lombardo and Mazzitelli [LomMaz97], Martin and Verdaguer [MarVer99a, MarVer99b, MarVer99c, MarVer00] and Roura and Verdaguer [RouVer99, Rou02] (see reviews [HuVer02, HuVer03, HuVer04]) could be adapted (or rather, simplified) to provide a foundation for the stochastic equations of scalar field theory below. The computation of full quantum correlations from the stochastic formulation is elaborated by Calzetta *et al.* [CaRoVe03].

A partial list of references for further reading on this subject is [Law89, Law92, Law99, LawKer00, BerRam01, RamNav00, BeGlRa98, HosSak84, MorSas84, Mor86, Mor90, Paz90a, Paz90b, Bet01, GleRam94, GreMul97, ABBCFJ99, LeeBoy93, Mos02]. See also those mentioned in the chapters on applications to atom–optical physics (13), nuclear–particle physics (14) and gravitation–cosmology (15).

8.1 Preliminaries

We return to the $g\varphi^3$ theory to illustrate the ideas highlighted above. The classical action with a cubic potential as in Chapter 6, equation (6.43) is

$$S[\Phi] = \int d^4x \left\{ -\frac{1}{2} (\partial\Phi)^2 - V[\Phi(x)] \right\} \tag{8.1}$$

We shall begin by considering the regression of the mean field towards its equilibrium value. To this end it is enough to consider the linearized equations of motion. The quadratic effective action is given in Chapter 6; see Sections 6.3.1 and 6.4.3 there. We have already seen that, after ultraviolet singularities have been disposed of, and assuming the initial conditions are laid out in the distant past to avoid initial time singularities, the free linearized evolution of the mean field is described by an equation of the form

$$\left[-\frac{d}{dt^2}^2 - m^2 \right] \phi(t) + \int^t ds - \Sigma_{\text{ret}}(t-s)\,\phi(s) = 0 \tag{8.2}$$

where we are assuming a spatially homogeneous mean field, and

$$\Sigma_{\text{ret}}(t) = \int d^3x\, \Sigma_{\text{ret}}(t, \mathbf{x}) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \Sigma_{\text{ret}}(\omega, \mathbf{p} = 0) \tag{8.3}$$

From now on, we shall omit writing the $\mathbf{p}$ argument when it is zero.

The fundamental solution of the equation of motion is the (space averaged) retarded propagator

$$G_{\text{ret}}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} G_{\text{ret}}(\omega) \tag{8.4}$$

$$G_{\text{ret}}(\omega) = (-1) \left[(\omega + i\varepsilon)^2 - m^2 + -\Sigma_{\text{ret}}(\omega) \right]^{-1} \tag{8.5}$$

The physical mass M^2 is defined by the requirement that the retarded propagator has a simple pole at $\omega = \pm M$,

$$M^2 - m^2 + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{d\sigma^2}{\sigma^2 - M^2} \Pi\left(\sigma^2\right) = 0 \tag{8.6}$$

The function $\Pi\left(\sigma^2\right)$ was introduced in Chapter 6, equation (6.135). We shall assume M^2 is positive. The retarded propagator has a branch cut for $\omega^2 > 4m^2$. If M^2 exists, it must be less than $4m^2$; otherwise the retarded propagator has no first sheet poles.

8.2 Dissipation in the mean field dynamics

Let us begin by showing that the existence of an imaginary component in $G_{\text{ret}}(\omega)$ implies that the dynamics of mean fields is dissipative.

The simplest way to show this is by looking at the response of the mean field to an impulse, that is, adding a source $-\delta(t)$ to the right-hand side of equation (8.2). The solution is

$$\phi(t) = G_{\text{ret}}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} G_{\text{ret}}(\omega) \tag{8.7}$$

As we know, the integrand has poles at $\omega = \pm M$ and branch cuts for $|\omega| > 2m$. Separating these contributions, we get

$$\phi(t) = \frac{1}{ZM}\sin Mt + \frac{1}{\pi}\int_{4m^2}^{\infty} d\sigma^2 \frac{\sin\sigma t}{\sigma}\Pi\left(\sigma^2\right)\left|G_{\text{ret}}(\sigma)\right|^2 \tag{8.8}$$

where Z comes from the residue at the pole. Since the integrand in the second term is regular, this term goes to zero as $t \to \infty$.

A less rigorous argument is based on a Breit–Wigner approximation for $G_{\text{ret}}(\omega)$. We simply approximate $\text{Re}G_{\text{ret}}^{-1}(\omega) \sim \omega^2 - M^2$; for the imaginary part, we write

$$\text{Im}G_{\text{ret}}^{-1}(\omega) = \Pi\left(\omega^2\right)\ \text{sign}(\omega) \sim 2\gamma\omega \tag{8.9}$$

$$\gamma \sim \frac{g^2\hbar}{128\pi m} \tag{8.10}$$

Therefore

$$\phi(t) = \frac{1}{M}\sin Mt\, e^{-\gamma t} \tag{8.11}$$

This approximation, which amounts to writing $\Pi\left(\omega^2\right) \sim$ constant, $\text{sign}(\omega) \sim \omega/2m$, cannot be valid at very short times $t^{-1} \gg m$, nor at very late times $t^{-1} \leq m$, but it does show that there is an approximately exponential decay in between. The decay turns to a power law at later times.

As a final argument, let us regard the nonlocal term in the equation of motion as a friction force acting on the mean field. Suppose we act on the mean field with an external source $j(t)$ such that it follows a given trajectory $\phi(t)$ vanishing both in the distant past and future. Therefore the total energy exchanged with the mean field vanishes. The instantaneous power is of course (minus) the product of force times velocity. The total power extracted from the mean field is

$$0 = \int_{-\infty}^{\infty} dt\, \left\{m^2\phi(t) - \int^t ds - \Sigma_{\text{ret}}(t-s)\,\phi(s) - j(t)\right\}\frac{d\phi}{dt} \tag{8.12}$$

so we must have

$$Q = -\int_{-\infty}^{\infty} dt\, \int^t ds - \Sigma_{\text{ret}}(t-s)\,\phi(s)\,\frac{d\phi}{dt} \tag{8.13}$$

where Q is the work extracted from the source. In terms of Fourier transforms

$$Q = \int \frac{d\omega}{2\pi} - \Sigma_{\text{ret}}(\omega)(-i\omega)\,\phi(\omega)\,\phi(-\omega) \tag{8.14}$$

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It is clear that $\phi(\omega)\phi(-\omega) = |\phi(\omega)|^2$ is an even function of ω, so only the odd part of $\Sigma_{\mathrm{ret}}(\omega)$ may contribute to Q. Since the real part of Σ_{ret} is even, we are left with

$$Q = \frac{1}{\pi}\int_{2m}^{\infty} \omega d\omega \, \Pi\left(\omega^2\right)|\phi(\omega)|^2 \tag{8.15}$$

which is clearly positive. We may think of this as work which is transferred from the external source to the mean field and then transformed into "heat," since it is not returned to the source nor stored in the mean field. We shall show in the next section that this work was transferred to the quantum fluctuations above the condensate.

8.3 Dissipation and particle creation

We have seen in the last section that along its evolution the mean field dissipates an amount of heat Q given by equation (8.15). We shall now show that this energy is actually spent in creating particles in the quantum field above the condensate.

Let us consider the Heisenberg equation of motion as given in Section 4.1.2 of Chapter 4. Split the quantum field Φ into a (c-number) mean field ϕ and a quantum fluctuation field φ

$$\Phi = \phi + \varphi \tag{8.16}$$

where

$$\langle\varphi\rangle = 0 \tag{8.17}$$

The expectation value of the Heisenberg equation yields

$$\partial^2\phi - m^2\phi + \frac{1}{2}g\phi^2 + \frac{1}{2}g\left[\left\langle\varphi^2\right\rangle_\phi - \left\langle\varphi^2\right\rangle_{\phi=0}\right] = 0 \tag{8.18}$$

where $\left\langle\varphi^2\right\rangle_\phi$ denotes the expectation value computed in the presence of the mean field. A linear expansion of (8.18) around $\phi = 0$ takes us back to (8.2). Subtracting (8.18) from the Heisenberg equation we find the equation for the fluctuations

$$\partial^2\varphi - m^2\varphi + g\phi\varphi + \frac{1}{2}g\left[\varphi^2 - \left\langle\varphi^2\right\rangle_\phi\right] = 0 \tag{8.19}$$

The one-loop approximation amounts to leaving out the last term

$$\partial^2\varphi - m^2\varphi + g\phi\varphi = 0 \tag{8.20}$$

We see that, in this model, the one-loop approximation reduces to the Hartree approximation. If the mean field is spatially independent, we may expand the fluctuation field in modes as in Chapter 4

$$\varphi(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{kx}}\varphi_{\mathbf{k}}(t) \tag{8.21}$$

Each mode is a harmonic oscillator with a time-dependent natural frequency

$$\frac{d^2\varphi_{\mathbf{k}}}{dt^2} + \omega_k^2\varphi_{\mathbf{k}} - g\phi(t)\varphi_{\mathbf{k}} = 0; \qquad \omega_k^2 = \mathbf{k}^2 + m^2 \tag{8.22}$$

Given two complex independent solutions f_k, f_k^* of equation (8.22), we may write

$$\varphi_{\mathbf{k}}(t) = f_k(t)\, a_{\mathbf{k}} + f_k^*(t)\, a_{-\mathbf{k}}^\dagger \tag{8.23}$$

where $a_{\mathbf{k}}$ is the usual destruction operator. Let us solve for the modes in powers of g. To zeroth order in g the Minkowski modes $f_k(t)$ introduced in Chapter 4 are the single global positive frequency solution. To first order in g we have a choice: either the *in* positive frequency solution

$$f_k^{in}(t) = f_k(t) + g \int_{-\infty}^{t} ds\, \frac{\sin\omega_k (t-s)}{\omega_k} \phi(s)\, f_k(s) \tag{8.24}$$

or the *out* positive frequency wave

$$f_k^{out}(t) = f_k(t) + g \int_{t}^{\infty} ds\, \frac{\sin\omega_k (s-t)}{\omega_k} \phi(s)\, f_k(s) \tag{8.25}$$

If the mean field is well behaved, then at very late times we have $f_k^{out}(t) \sim f_k(t)$ whereas $f_k^{in}(t)$ is obtained through a Bogoliubov transformation

$$f_k^{in}(t) = \alpha_k f_k^{out}(t) + \beta_k \left[f_k^{out}(t)\right]^* \tag{8.26}$$

Conversely, the destruction operators in the distant past and future are related through

$$a_{\mathbf{k}}^{out} = \alpha_k a_{\mathbf{k}}^{in} + \beta_k^* a_{-\mathbf{k}}^{in\dagger} \tag{8.27}$$

As we saw in Chapter 4, if the initial state is the *in* vacuum, at late times we find a nonzero population density of created particles $|\beta_k|^2$. From the explicit expression, we find

$$\beta_k = \frac{(-ig)}{2\omega_k} \int_{-\infty}^{\infty} ds\, \phi(s)\, e^{-2i\omega_k s} = \frac{(-ig)}{2\omega_k} \left[\phi(2\omega_k)\right]^* \tag{8.28}$$

Since each particle carries an energy $\hbar\omega_k$, the total energy density in the fluctuations is

$$\begin{aligned} \rho &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{g^2\hbar}{4\omega_k} \left|\phi(2\omega_k)\right|^2 \\ &= \frac{g^2\hbar}{4\pi} \int_{2m}^{\infty} \omega d\omega\, \nu(\omega) \left|\phi(\omega)\right|^2 \\ &= Q \end{aligned} \tag{8.29}$$

where

$$\nu(\omega) = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{\omega^2}}\, \theta\left(\omega^2 - 4m^2\right) \tag{8.30}$$

was already introduced in Chapter 5. We see that the energy extracted from the source is being transferred to the fluctuations. For completeness, we observe that the kernel ν in (8.30) is related to the kernel Π in (8.6) through

$$\Pi\left(\omega^2\right) = \frac{g^2\hbar}{4} \nu(\omega) \tag{8.31}$$

8.4 Particle creation and noise

We have seen in the last section that the mean field loses energy which is spent in exciting the quantum fluctuations of the vacuum into particles. The back-reaction from this process is experienced by the mean field as dissipation. We now observe that particle creation from the vacuum has an intrinsic stochastic character: there are always fluctuations in the number of created particles. These fluctuations affect the mean field through its back-reaction. The dynamics of the mean field thus acquires a stochastic element. Of course, at this point it ceases to be the "mean" field: it is a c-number field which represents the evolution of the condensate component of the full Heisenberg field.

To obtain a measure of the fluctuations in particle creation, let us consider the correlations between particles created in different modes

$$\begin{aligned}\langle N_{\mathbf{p}} N_{\mathbf{q}}\rangle &= \left\langle 0in \left| a_{\mathbf{p}}^{out\dagger} a_{\mathbf{p}}^{out} a_{\mathbf{q}}^{out\dagger} a_{\mathbf{q}}^{out} \right| 0in \right\rangle \\ &= V^2 \left|\beta_p\right|^2 \left|\beta_q\right|^2 + V \left|\beta_p\right|^2 \left|\alpha_p\right|^2 \left[\delta\left(\mathbf{p}-\mathbf{q}\right) + \delta\left(\mathbf{p}+\mathbf{q}\right)\right] \end{aligned} \tag{8.32}$$

It follows that the fluctuations in the energy density are

$$\begin{aligned}\left\langle \delta\rho^2 \right\rangle &= \left\langle \rho^2 \right\rangle - \left\langle \rho \right\rangle^2 \\ &= \frac{2\hbar^2}{V} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \omega_p^2 \left|\beta_p\right|^2 \left|\alpha_p\right|^2 \end{aligned} \tag{8.33}$$

To lowest order we may approximate $\left|\alpha_p\right|^2 = 1$. Using the explicit expression for the Bogoliubov coefficients, we get,

$$\begin{aligned}\left\langle \delta\rho^2 \right\rangle &= \frac{g^2\hbar^2}{2V} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \; \left|\phi\left(2\omega_k\right)\right|^2 \\ &= \frac{g^2\hbar^2}{4\pi V} \int_{2m}^{\infty} \omega^2 d\omega \; \nu\left(\omega\right) \left|\phi\left(\omega\right)\right|^2 \\ &= \delta Q^2 \end{aligned} \tag{8.34}$$

We may account for these fluctuations by adding a stochastic term $\zeta\left(t, \mathbf{x}\right)$ to the right-hand side of the mean field equations of motion. For a homogeneous condensate, they reduce to

$$\left[-\frac{d}{dt^2}^{2} - m^2\right]\phi\left(t\right) + \int^t ds - \Sigma_{\mathrm{ret}}\left(t-s\right)\phi\left(s\right) = -\frac{g}{2}\Xi\left(t\right) \tag{8.35}$$

where

$$\phi\left(t\right) = \frac{1}{V}\int d^3\mathbf{x}\; \phi\left(t, \mathbf{x}\right) \tag{8.36}$$

$$\Xi\left(t\right) = \frac{1}{V}\int d^3\mathbf{x}\; \zeta\left(t, \mathbf{x}\right) \tag{8.37}$$

We assume ζ is a Gaussian noise with zero average $\left\langle \zeta\left(t, \mathbf{x}\right)\right\rangle_s = 0$, where, hereafter, the subscript s will denote stochastic averages over the noise distribution function, and (possibly colored) autocorrelation $\langle \zeta(t,\mathbf{x})\,\zeta(s,\mathbf{y})\rangle_s = \nu_s(t-s,\mathbf{x}-\mathbf{y})$. For a prescribed trajectory $\phi(t)$, the work done by the random source is

$$Q_s = \frac{g}{2}\int dt\;\Xi(t)\,\frac{d}{dt}\phi(t) \tag{8.38}$$

Assuming independence of $\phi(t)$ and $\Xi(t)$, $\langle Q_s\rangle_s = 0$ and

$$\left\langle Q_s^2\right\rangle_s = \frac{g^2}{4V}\int dtds\;\frac{d}{dt}\phi(t)\,\frac{d}{ds}\phi(s)\int d^3\mathbf{x}\;\nu_s(t-s,\mathbf{x}) \tag{8.39}$$

Introducing the Fourier transform

$$\nu_s(t,\mathbf{x}) = \int\frac{d^4k}{(2\pi)^4}\;e^{ikx}\nu_s(k) \tag{8.40}$$

$$\left\langle Q_s^2\right\rangle_s = \frac{g^2}{4V}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\;\omega^2\left|\phi(\omega)\right|^2\nu_s(\omega) = \frac{g^2}{4V}\int_0^{\infty}\frac{d\omega}{\pi}\;\omega^2\left|\phi(\omega)\right|^2\nu_s(\omega) \tag{8.41}$$

where as usual we write $\nu_s(\omega) = \nu_s(\omega,\mathbf{p}=0)$ and we have used the obvious symmetry condition that ν_s is even. If we request that $\left\langle Q_s^2\right\rangle_s$ accounts for the fluctuations δQ^2, equation (8.34), then $\nu_s = \hbar^2\nu$.

Since we are discussing a Lorentz invariant theory, this result determines $\nu(k)$ everywhere. We of course recognize the noise kernel introduced in Chapters 5 and 6. In other words, we could arrive at the same Langevin type equation for the mean field simply by arguing that the CTPEA may be regarded as a Feynman–Vernon influence functional for an open system (the condensate) interacting with an environment (the quantum fluctuations) and adopting the usual interpretation that the imaginary part of the influence action (IA) describes noise.

This point of view is validated by the fact that the stochastic formulation allows us to compute certain quantum expectation values in the original theory. Before developing this point further, let us show briefly yet another way to arrive at the same stochastic equation. If we consider the full Heisenberg equation and subtract the equation for the fluctuations we see that the Langevin equation (8.35) amounts to the replacement

$$\left[\varphi^2 - \left\langle\varphi^2\right\rangle_\phi\right] \leftrightarrow \zeta \tag{8.42}$$

Of course we cannot simply interpose an identity, because we have on the left a Heisenberg quantum operator, and on the right a c-number stochastic field. To give meaning to the connection between the two, we adopt the Landau prescription that the symmetric quantum expectation value of the left-hand side equals (twice) the stochastic expectation value of the right-hand side, or

$$\nu_s(t-s,\mathbf{x}-\mathbf{y}) = \frac{1}{2}\left[\left\langle\left\{\varphi^2(t,\mathbf{x}),\varphi^2(s,\mathbf{y})\right\}\right\rangle_\phi - 2\left\langle\varphi^2(t,\mathbf{x})\right\rangle_\phi\left\langle\varphi^2(s,\mathbf{y})\right\rangle_\phi\right] \tag{8.43}$$

An explicit evaluation at $\phi = 0$ gives again the noise kernel from the CTPEA, as we have seen in Chapter 6. We see how this approach leading to a Langevin equation is an improvement over the usual mean field theory, which simply disregards $\varphi^2 - \left\langle \varphi^2 \right\rangle_\phi$ entirely.

8.5 Full quantum correlations from the Langevin approach

As a simple application of the Langevin approach, we shall show how it may be used to compute the Hadamard propagator for the underlying field theory. This is the field theory counterpart of a method applicable more generally to quantum open systems, and therefore reinforces the view of the CTPEA as the IA for the mean field.

Let us begin by connecting the propagators of the theory to the CTPEA. In the condensed notation from Chapter 6, the full propagators $G^{AB} = \left\langle \varphi^A \varphi^B \right\rangle$ in the equilibrium state are given by

$$G^{AB} = -i\hbar \left. \frac{\delta^2 W}{\delta J_A \delta J_B} \right|_{J=0} \tag{8.44}$$

where W is the CTP generating functional. As usual we identify $G^{11} = G_{\mathrm{F}}$, $G^{12} = G^-$, $G^{21} = G^+$ and $G^{22} = G_{\mathrm{D}}$. On the other hand, $W^{,AB} = \delta\phi^A/\delta J_B = -\left(\Gamma_{,AB}\right)^{-1}$, so we obtain an equation relating the propagators to the second variation of the CTPEA

$$\Gamma_{,AB} G^{BC} = i\hbar \delta_A^C \tag{8.45}$$

Observe that if the field theory is defined only for $t > t_0$, rather than on the whole Minkowski space, then the intermediate integral is equally restricted:

$$\phi_B \psi^B \equiv \int d^3\mathbf{x} \int_{t_0}^{\infty} dt\; \phi_b\left(t, \mathbf{x}\right) \psi^b\left(t, \mathbf{x}\right) \tag{8.46}$$

We have seen in Chapter 6 that the quadratic part of the CTPEA must have the structure of equation (6.97), where the kernels $\mathbf{D}^{\mathrm{full}}$ and $\mathbf{N}$ are real, and $\mathbf{D}^{\mathrm{full}}$ is causal. We may further split $\mathbf{D}^{\mathrm{full}}$ into its symmetric and antisymmetric parts, $\mathbf{D}^{\mathrm{full}} = \mathbf{D}_s^{\mathrm{full}} + \mathbf{\Gamma}$, respectively. The Hessian $\Gamma_{,AB}$ becomes

$$\Gamma_{,AB} = \begin{pmatrix} \mathbf{D}_s^{\mathrm{full}} + i\mathbf{N} & \mathbf{\Gamma} - i\mathbf{N} \\ -\mathbf{\Gamma} - i\mathbf{N} & -\mathbf{D}_s^{\mathrm{full}} + i\mathbf{N} \end{pmatrix} \tag{8.47}$$

Since the equilibrium state is translation invariant, the propagators (as well as the $\mathbf{D}_s^{\mathrm{full}}$, $\mathbf{\Gamma}$ and $\mathbf{N}$ kernels) are functions of the difference variable $x - x'$ alone, and equations (8.45) are algebraic equations for their Fourier transforms. Setting $a = 1$ in equations (8.45) and using the matrix form (8.47), we obtain

$$\left(\mathbf{D}_s^{\mathrm{full}} + i\mathbf{N}\right) G^{11} + \left(\mathbf{\Gamma} - i\mathbf{N}\right) G^{21} = i\hbar \tag{8.48}$$

$$\left(\mathbf{D}_s^{\mathrm{full}} + i\mathbf{N}\right) G^{12} + \left(\mathbf{\Gamma} - i\mathbf{N}\right) G^{22} = 0 \tag{8.49}$$

Subtracting these two equations, and writing the fundamental propagators in terms of G_{ret}, G_{adv} and G_1, we get $\mathbf{D}^{\mathrm{full}} G_{\mathrm{ret}} = -1$. This is just the statement that the retarded propagator is the fundamental solution to the linearized equations of motion for the mean field, $\mathbf{D}^{\mathrm{full}}\phi = -J$.

Let us go back to equation (8.48) to get

$$\mathbf{D}^{\mathrm{full}} G_1 + 2\hbar \mathbf{N} G_{\mathrm{adv}} = 0 \tag{8.50}$$

Since the equation $\mathbf{D}^{\mathrm{full}}\phi = 0$ admits plane waves of momentum p as homogeneous solutions, provided $\left(-p^2\right) = M^2$, the solution to this equation reads

$$G_1 = C\delta\left(-p^2 - M^2\right) + 2\hbar G_{\mathrm{ret}} \mathbf{N} G_{\mathrm{adv}} \tag{8.51}$$

We are using the fact that G_1 must be Lorentz invariant, so C must be a simple constant.

In the Langevin approach we postulate an equation for the stochastic condensate (absorbing coupling constants into the stochastic source, i.e. $\xi = g\zeta/2$)

$$\mathbf{D}^{\mathrm{full}}\phi = -\xi \tag{8.52}$$

where

$$\left\langle \xi\left(x\right)\xi\left(y\right)\right\rangle_s = \hbar \mathbf{N}\left(x,y\right) \tag{8.53}$$

Suppose we set the initial conditions for this equation at some time t_0. Then

$$\phi\left(x\right) = \phi_{\mathrm{hom}}\left(x\right) + \int_{y^0>t_0} d^4y\; G_{\mathrm{ret}}\left(x,y\right)\xi\left(y\right) \tag{8.54}$$

where $\phi_{\mathrm{hom}}\left(x\right)$ is determined by the Cauchy data at t_0

$$\phi_{\mathrm{hom}}\left(t,\mathbf{x}\right) = \int d^3\mathbf{y}\; \left\{ G_{\mathrm{ret}}\left(t-t_0,\mathbf{x}-\mathbf{y}\right)\frac{d}{dt_0}\phi\left(t_0,\mathbf{y}\right) + \frac{d}{dt}G_{\mathrm{ret}}\left(t-t_0,\mathbf{x}-\mathbf{y}\right)\phi\left(t_0,\mathbf{y}\right)\right\} \tag{8.55}$$

The stochastic average, assuming independence between the initial conditions and the noise sources, becomes

$$\left\langle \phi\left(x\right)\phi\left(y\right)\right\rangle_s = \left\langle \phi_{\mathrm{hom}}\left(x\right)\phi_{\mathrm{hom}}\left(y\right)\right\rangle_s + \hbar\int_{z^0,z'^0>t_0,} d^4z d^4z'\; G_{\mathrm{ret}}\left(x,z\right)\mathbf{N}\left(z,z'\right)G_{\mathrm{adv}}\left(z',y\right) \tag{8.56}$$

Twice this is a solution of equation (8.50), and therefore if the Cauchy data for $2\left\langle \phi\left(x\right)\phi\left(y\right)\right\rangle_s$ and $G_1\left(x,y\right)$ are chosen to be the same, they will remain equal everywhere.

This shows that the stochastic approach may reproduce the Hadamard propagator of the underlying quantum theory. Observe that both random initial conditions and noise sources are required.

8.6 The fluctuation–dissipation theorem

Before we show how the above analysis may be generalized to the nonlinear regime, it is interesting to pause for the following observation. We have just shown that the Hadamard propagator for quantum fluctuations may be obtained as a stochastic average over a random c-number field. This field may be decomposed into a homogeneous solution of the linearized mean field equations of motion plus an extra term, induced by the effect of a particular Gaussian noise.

We have seen at the beginning of this chapter that solutions of the mean field equations are partially dissipated away as they evolve. But the Hadamard propagator is time-translation invariant. So the noise sources must be injecting the precise amount of fluctuations necessary to compensate for the dissipation of the free part. The quantitative statement of this observation is the (zero temperature) fluctuation dissipation relation [CalWel51, LaLiPi80a, Ma76, BooYip91]. This is a simple application of a deeper, generic relationship between noise and dissipation in the CTPEA, whose origin is that both arise from particle creation in the fluctuation field. Here we are using the term "particle creation" also to denote such phenomena as the transfer of atoms from a condensate to higher modes, as in the Bose–Nova experiment [Don01].

To quantify this statement, let us return to the expression for the heat dissipated during the whole evolution of the field

$$Q = \int d^4x \left\{ \left[-\nabla^2 + m^2\right] \phi(x) - \int_{y^0<x^0} d^4y - \Sigma_{\mathrm{ret}}(x,y)\,\phi(y) - \xi(x) \right\} \frac{d\phi}{dt}(x) \tag{8.57}$$

Since the spectrum of fluctuations is stationary, we must have $\langle Q \rangle_s = 0$. The first terms have been analyzed in the beginning of this chapter, with the only difference that now we do not assume a homogeneous condensate. Using the results of the last section to replace field averages by the Hadamard propagator, we get

$$\int d^4x \left\langle \xi(x) \frac{d\phi}{dt}(x) \right\rangle_s = VT \int \frac{d^4k}{(2\pi)^4}\, k^0 \Pi\left(-k^2\right) G_1(k)\,\theta\left(k^0\right) \tag{8.58}$$

where VT is the 4-volume of spacetime. Since we are assuming a Gaussian noise and a linearized equation of motion,

$$\begin{aligned}\int d^4x \left\langle \xi(x) \frac{d\phi}{dt}(x) \right\rangle_s &= \int d^4x d^4y\ \langle \xi(x)\,\xi(y) \rangle_s \frac{d}{dt}\frac{\delta\phi(x)}{\delta j(y)} \\ &= -iVT\hbar \int \frac{d^4k}{(2\pi)^4}\, k^0 \mathbf{N}(k)\, G_{\mathrm{ret}}(k)\end{aligned} \tag{8.59}$$

Since $\mathbf{N}(k)$ is even, this becomes

$$-i\frac{VT\hbar}{2} \int \frac{d^4k}{(2\pi)^4}\, \left|k^0\right| \mathbf{N}(k) \left[G_{\mathrm{ret}}(k) - G_{\mathrm{ret}}(-k)\right] \theta\left(k^0\right) \tag{8.60}$$

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But $G_{\rm ret}(k) - G_{\rm ret}(-k) = iG(k)$, where G is the Jordan propagator. Therefore, defining

$$G_1(k) = \rho(k)\, G(k)\, \mathrm{sign}\left(k^0\right) \tag{8.61}$$

then

$$\hbar \mathbf{N}(k) = 2\Pi\left(-k^2\right)\rho(k) \tag{8.62}$$

This is the fluctuation–dissipation theorem at zero temperature (cf. Einstein's relation from Chapter 2). By the way, for free fields $\rho = 1$, as we saw in Chapter 6.

8.7 Particle creation and decoherence

At this point it is interesting to go back to the beginning and question whether it is consistent to treat the system field ϕ as classical. One possible answer is to consider two different histories for the ϕ field, leaving the environment field φ unspecified, and to compute their decoherence functional $\mathcal{D}$ (introduced in Chapter 3). If $|\mathcal{D}| \ll 1$, the classical approximation is warranted.

The basic observation is that to compute the decoherence functional we must perform a CTP path integral over all histories of the field φ^a, adding in each branch an external source to enforce the constraint that $\langle \varphi^a \rangle = 0$. The result is that the path integral defining $\mathcal{D}$ is identical to the one defining the CTPEA, and we find the relationship

$$\mathcal{D}\left[\phi^1, \phi^2\right] = \exp\left\{\frac{i\Gamma\left[\phi^1, \phi^2\right]}{\hbar}\right\} \tag{8.63}$$

It is clear that the classical part of the CTPEA does not contribute to decoherence. Let us consider the one-loop term Γ_1 (cf. Chapter 6). In canonical terms, Γ_1 measures the overlap between the state which evolves from the *in* vacuum under the influence of the external field $\phi^1(x)$ and the state which evolves under $\phi^2(x)$, as measured in the far future. For simplicity, let us assume that the background fields are homogeneous, in which case we may decompose the fluctuation fields in plane waves, to find

$$\Gamma_1\left[\phi^A\right] = \sum_{k^0>0} \Gamma_{1k}\left[\phi^A\right]$$

$$\Gamma_{1k}\left[\phi^A\right] = -i\hbar \ln \int D\varphi^a_k D\varphi^a_{-k} \times \exp\left\{-\frac{i}{\hbar}\int dt\; \varphi^a_{-k}\left[c_{ab}\left(\frac{d^2}{dt^2} + \omega_k^2\right) + g\, c_{abc}\phi^c\right]\varphi^b_k\right\} \tag{8.64}$$

Interposing a complete set of *out* modes, we may write

$$\Gamma_{1k}\left[\phi^A\right] = -i\hbar \ln \sum_n \left\langle 0in \left| n_k, n_{-k}, out \right.\right\rangle_{\phi^2} \left\langle n_k, n_{-k}, out \right| 0in \left.\right\rangle_{\phi^1} \tag{8.65}$$

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where we are using the fact that particles may be created in pairs only; the subscript ϕ indicates the external field under which the quantum field evolves. Since the quantum field on each branch is a free Klein–Gordon field with a time-dependent frequency, the *in* and *out* destruction operators are related through a Bogoliubov transform. The relevant brackets are given in Chapter 4, and after a simple summation, we arrive at

$$\Gamma_{1k}\left[\phi^{A}\right] = i\hbar\ln\left[\alpha_k^2\alpha_k^{1*} - \beta_k^2\beta_k^{1*}\right] \tag{8.66}$$

where α_k^i, β_k^i denote the Bogoliubov coefficients for the corresponding branch.

One can check that this expression complies with the basic expectations regarding the CTPEA. It is clear that Γ_{1k} vanishes if $\phi^1 = \phi^2$. If the two fields are exchanged, the real part changes sign, while the imaginary part is unchanged.

To clarify the meaning of equation (8.66) let us observe that it is invariant if we subject both pairs of Bogoliubov coefficients to the same Bogoliubov transformation. In other words, the effective action is independent of the choice of *out* particle model in equation (8.65). Therefore we may assume without loss of generality that $\beta_k^2 = 0$. This implies $\left|\alpha_k^2\right| = 1$, and so, in this representation,

$$|\mathcal{D}| = \frac{1}{|\alpha_k^1|} = \frac{1}{\sqrt{1+|\beta_k^1|^2}} \tag{8.67}$$

As expected, particle creation is necessary to suppress coherence. Of course, the physical mechanism behind this result is the entanglement of the system and environment fields through the particle creation process.

The relation between particle creation and decoherence was given in [CalMaz90]. The expression of the CTP effective action or the influence functional in terms of the Bogoliubov coefficients was given in [CalHu94, HKMP96, RaStHu98].

8.8 The nonlinear regime

So far we have demonstrated the presence of noise and dissipation for far off-shell modes ($-p^2 > 4M^2$). We shall now see that particle creation, and therefore dissipation and noise, is restricted by a lower threshold only in the linearized theory. In the nonlinear regime, the possibilities are much richer: Schwinger proved the existence of particle creation from static electric fields, shown as an example in Chapter 4. So dissipation and noise in particle creation are the rule rather than the exception.

Let us attempt a nonperturbative evaluation of the one-loop effective action as given in Chapter 6, equation (6.120). Observe that

$$\phi^1\left(\varphi^1\right)^2 - \phi^2\left(\varphi^2\right)^2 = \phi_+\left(\left(\varphi^1\right)^2 - \left(\varphi^2\right)^2\right) + \frac{\phi_-}{2}\left(\left(\varphi^1\right)^2 + \left(\varphi^2\right)^2\right) \tag{8.68}$$

This suggests expanding

$$\Gamma_1 = \left(-\frac{g}{2}\right)\int d^4x\langle\varphi^2\rangle_{\phi_+}(x)\phi_- + \Delta S\left[\phi_+,\phi_-\right] \tag{8.69}$$

where $\Delta S\left[\phi_+,\phi_-\right] \sim O\left(\phi_-^2\right)$. To compute $\langle\varphi^2\rangle_{\phi_+}(x)$ we consider the fluctuation field φ as a free quantum field with the equation of motion

$$\left[\partial^2 - m^2 - g\phi_+\right]\varphi = 0 \tag{8.70}$$

In other words, φ is a quantum field propagating on the dynamic background ϕ_+, a situation we have already analyzed in Chapter 4.

To see the effect of ΔS on the equation of motion for ϕ, we perform a functional Fourier transform

$$\exp\left\{i\Delta S\left[\phi_+,\phi_-\right]/\hbar\right\} = \int D\xi\; e^{i\hbar^{-1}\int\xi\phi_-}P\left[\xi,\phi_+\right] \tag{8.71}$$

Calling $\langle\ldots\rangle_s = \int D\xi\;\ldots P\left[\xi,\phi_+\right]$, we find

$$\langle\xi(x)\rangle_s = \left.\frac{\delta\Delta S}{\delta\phi_-}\right|_{\phi_-=0} = 0 \tag{8.72}$$

$$\begin{aligned}\langle\xi(x)\xi(x')\rangle_s &\equiv \hbar\mathbf{N}(x,x')\\ &= \left(\frac{g^2}{8}\right)\left[\left\langle\left\{\varphi^2(x),\varphi^2(x')\right\}\right\rangle_{\phi_+} - 2\langle\varphi^2\rangle_{\phi_+}(x)\langle\varphi^2\rangle_{\phi_+}(x')\right]\end{aligned} \tag{8.73}$$

This is to be contrasted with the result in the perturbative treatment.

The functional $P\left[\xi,\phi_+\right]$ must be real (as follows from $\Delta S\left[\phi_+,-\phi_-\right] = -\Delta S\left[\phi_+,\phi_-\right]^*$) and it is nonnegative to the one-loop approximation. We may think of it as a functional Wigner transform of the effective action [Hab92], and thereby as a probability density "for all practical purposes." Observe that P will not be Gaussian in general.

In the limit $\phi_-\to 0$, $\phi_+\to\phi$, we obtain the equation of motion for the mean field

$$\left(-\partial^2 + m^2\right)\phi(x) + \frac{1}{2}g\phi^2 + \left(\frac{g}{2}\right)\left[\langle\varphi^2\rangle_\phi - \langle\varphi^2\rangle_0\right](x) = \xi(x) \tag{8.74}$$

A linear expansion of (8.74) around $\phi = 0$ and the assumption of a homogeneous condensate give back (8.35). Our goal is to show that the noise $\xi(x)$ is not restricted to modes above threshold. To this end, we shall assume a simple mean field configuration, homogeneous in space and harmonic in time, i.e.

$$\phi(t) = \phi_0\cos\gamma t \tag{8.75}$$

where $\gamma \le 2M$, so we are below threshold. We shall see that in spite of this the noise is nonzero. Moreover, the noise itself is not restricted to the high-frequency domain, but it has low-frequency components as well.

To compute the nonperturbative noise kernel, we expand the quantum field φ in normal modes. The amplitude functions of the normal modes are complex, with

$$\varphi_{-\mathbf{k}} = \varphi_{\mathbf{k}}^\dagger \tag{8.76}$$

They obey a mode equation where the time-dependent natural frequency of the $\mathbf{k}$th mode is

$$\omega_k^2 = \mathbf{k}^2 + m^2 - g\phi(t) \tag{8.77}$$

Here we shall disregard the possibility of ω becoming imaginary through a large negative light field, i.e. we assume $g\phi_0 \leq m^2$. We assume the fluctuation field is in the vacuum state at some initial time $t = 0$ (we assume the coupling constant g is switched on adiabatically, so initial time singularities do not arise). Since φ is a free field, Wick's theorem holds, and our problem is to relate the field at arbitrary times to the initial creation and destruction operators.

The general relationship we seek is

$$\varphi_k(t) = f_k(t)a_k(0) + f_k^*(t)a_{-k}^\dagger(0) \tag{8.78}$$

where f_k is the positive frequency mode associated with the *in* particle model. For the given mean field evolution (8.75) the mode equation is in the form of Mathieu's equation, with a periodically driven field in the narrow resonance regime. The results of Chapter 4 will apply here if we identify $\omega_{k0}^2 = \mathbf{k}^2 + m^2 - g\phi_0$, $\omega_1^2 = g\phi_0$. The mode function f_k may be written as a linear combination of WKB solutions.

$$f_k(t) = \alpha_k\,(t)\,f_k^+(t) + \beta_k\,(t)\,f_k^-(t) \tag{8.79}$$

Let us consider the case where we are in the neighborhood of the ℓth resonance band, namely $\omega_{k0} = \gamma\,(\ell + \delta_k)$ (remember in Chapter 4 we set $\gamma = 1$, so now we must re-insert γ in all the equations). Then

$$\begin{aligned} \alpha_k\,(t) &= \left[\alpha_{k0}^{(+)}e^{\mu_k\gamma t} + \alpha_{k0}^{(-)}e^{-\mu_k\gamma t}\right]e^{i\sigma_k\gamma t} \\ \beta_k\,(t) &= \left[\beta_{k0}^{(+)}e^{\mu_k\gamma t} + \beta_{k0}^{(-)}e^{-\mu_k\gamma t}\right]e^{-i\sigma_k\gamma t} \end{aligned} \tag{8.80}$$

where

$$\begin{aligned} \sigma_k &= \frac{\omega_1^2}{2\gamma\omega_{0k}} + \delta_k \\ \kappa_k &\sim \frac{1}{(2\ell-1)!}\frac{\gamma}{4\omega_{k0}}\left(\frac{\omega_1^2}{2\gamma\omega_{0k}}\right)^{2\ell} \\ \mu_k &= \sqrt{\kappa_k^2 - \sigma_k^2} \\ \beta_{k0}^{(\pm)} &= \frac{1}{\kappa_k}\left[\sigma_k \mp i\mu_k\right]\alpha_{k0}^{(\pm)} \end{aligned} \tag{8.81}$$

We are particularly interested in the case where μ_k is real. Let us write

$$\frac{\sigma_k + i\mu_k}{\kappa_k} = e^{i\vartheta_k} \tag{8.82}$$

(outside of the resonant region, ϑ_k becomes imaginary). Imposing the boundary conditions $\alpha_k\,(0) = 1$, $\beta_k\,(0) = 0$, we find

$$\begin{aligned} \alpha_k\,(t) &= \frac{\sinh\left[\mu_k\gamma t + i\vartheta_k\right]}{i\sin\vartheta_k}e^{i\sigma_k\gamma t} \\ \beta_k\,(t) &= \frac{\sinh\left[\mu_k\gamma t\right]}{i\sin\vartheta_k}e^{-i\sigma_k\gamma t} \end{aligned} \tag{8.83}$$

Finally, let us define

$$f_k^+(t)e^{i\sigma_k\gamma t} = \hbar^{1/2}\frac{e^{-i\ell\gamma t}}{\sqrt{2\ell\gamma}}g_k(t) \tag{8.84}$$

where $g_k(t) \sim 1$ with great accuracy. We can now write the mode functions as

$$f_k(t) = \hbar^{1/2}\left\{\frac{\sinh[\mu_k\gamma t + i\vartheta_k]}{i\sin\vartheta_k}\frac{e^{-i\ell\gamma t}}{\sqrt{2\ell\gamma}}g_k(t) + \frac{\sinh[\mu_k\gamma t]}{i\sin\vartheta_k}\frac{e^{i\ell\gamma t}}{\sqrt{2\ell\gamma}}g_k^*(t)\right\} \tag{8.85}$$

Three features stand out, namely (1) the generation of the negative frequency components, which is the physical basis for vacuum particle creation; (2) the exponential amplification due to ongoing particle creation; and (3) the phase-locking of a whole range of wavelengths at the resonance frequency $\ell\gamma$. As we shall now see, phase locking allows the generation of a low-frequency, inhomogeneus stochastic field. This is the main physical indication of the new features of dissipation and fluctuation below threshold we want to highlight.

In order to find the noise kernel, let us decompose the Heisenberg operator φ^2 into a c-number, a diagonal (D) and a nondiagonal (ND) (in the particle number basis) part

$$\varphi^2 = \left\langle\varphi^2\right\rangle_\phi + \varphi^2_{\rm D} + \varphi^2_{\rm ND} \tag{8.86}$$

where

$$\left\langle\varphi^2\right\rangle_\phi = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\;|f_k(t)|^2 \tag{8.87}$$

and the (D) and (ND) components are

$$\varphi^2_{\rm D} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\;e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}}\left\{f_k(t)f^*_{k'}(t)a^\dagger_{-\mathbf{k}'}a_{\mathbf{k}} + f^*_k(t)f_{k'}(t)a^\dagger_{-\mathbf{k}}a_{\mathbf{k}'}\right\} \tag{8.88}$$

$$\varphi^2_{\rm ND} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\;e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}}\left\{f_k(t)f_{k'}(t)a_{\mathbf{k}}a_{\mathbf{k}'} + f^*_k(t)f^*_{k'}(t)a^\dagger_{-\mathbf{k}}a^\dagger_{-\mathbf{k}'}\right\} \tag{8.89}$$

Observe that, assuming vacuum initial conditions,

$$\left\langle\varphi^2_{\rm D}\right\rangle_\phi = \left\langle\varphi^2_{\rm ND}\right\rangle_\phi = \left\langle\varphi^2_{\rm D}\varphi^2_{\rm ND}\right\rangle_\phi = \left\langle\varphi^2_{\rm ND}\varphi^2_{\rm D}\right\rangle_\phi = \left\langle\varphi^2_{\rm D}\varphi^2_{\rm D}\right\rangle_\phi \equiv 0 \tag{8.90}$$

Therefore

$$\begin{aligned}\hbar\mathbf{N}(x,x') &= \left(\frac{g^2}{8}\right)\left\langle\left\{\varphi^2_{\rm ND}(x),\varphi^2_{\rm ND}(x')\right\}\right\rangle_{\phi_+}\\ &= \left(\frac{g^2}{2}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\;e^{i(\mathbf{k}+\mathbf{k}')(\mathbf{x}-\mathbf{x}')}\mathrm{Re}\left\{f_k(t)f_{k'}(t)f^*_k(t')f^*_{k'}(t')\right\}\end{aligned} \tag{8.91}$$

If no particle creation occurred, the noise kernel would contain frequencies above threshold only. However, in the presence of frequency-locking and a negative

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frequency part of the mode functions f, the noise kernel also contains a steady component

$$\hbar \mathbf{N}_{\mathrm{S}}(x,x') = \left(\frac{g^2\hbar^2}{8\ell^2\gamma^2}\right)\int' \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\; e^{i(\mathbf{k}+\mathbf{k}')(\mathbf{x}-\mathbf{x}')}\mathbf{F}_{kk'}(t,t') \tag{8.92}$$

where the integral is restricted to those modes where μ_k is real, and

$$\mathbf{F}_{kk'}(t,t') = \mathrm{Re}\left\{F_{kk'}\left(t\right)F^*_{kk'}\left(t'\right)\right\} \tag{8.93}$$

$$F_{kk'}\left(t\right) = \frac{1}{\sin\vartheta_k \sin\vartheta_{k'}}\left[\sinh\left[\mu_k\gamma t + i\vartheta_k\right]\sinh\left[\mu_{k'}\gamma t\right]g_k\left(t\right)g^*_{k'}\left(t\right) + \left(k\leftrightarrow k'\right)\right] \tag{8.94}$$

It is important to notice that $\mathbf{F}$ is slowly varying not only with respect to the frequency $\ell\gamma$, but also with respect to the background frequency γ itself. Of course we do not observe the noise kernel directly, but only through its effect on the mean field. However, since the steady part of the stochastic source is slowly varying in space and time, to a first approximation it induces a stochastic mean field ϕ_{S} which is simply proportional to it:

$$\phi_{\mathrm{S}} \sim \left(\frac{1}{m^2}\right)\xi_{\mathrm{S}};\qquad \langle\phi_{\mathrm{S}}\phi_{\mathrm{S}}\rangle \sim \left(\frac{1}{m^2}\right)^2 \hbar\mathbf{N}_{\mathrm{S}} \tag{8.95}$$

One can deduce the noise and its auto-correlation in this way.

Since κ_k decays exponentially with l, it is clear that only the lowest possible resonance band makes a meaningful contribution. So we may assume that $\mathbf{k}^2 \ll m^2$ and approximate $\omega_{k0} = \omega_{00} + \left(\mathbf{k}^2/2\omega_{00}\right)$, where $\omega_{00}^2 = m^2 - g\phi_0$. Therefore

$$\begin{aligned}\delta_k &= \delta_0 + \frac{\mathbf{k}^2}{2\gamma\omega_{00}}\\ \sigma_k &\sim \sigma_0 + \frac{\mathbf{k}^2}{2\gamma\omega_{00}}\\ \kappa_k &\sim \kappa_0 e^{-\ell\mathbf{k}^2/\gamma\omega_{00}}\end{aligned} \tag{8.96}$$

The limit of the resonance band is reached at some wavenumber k_0, and we may approximate

$$\mu_k = \mu_0\left(1 - \frac{\mathbf{k}^2}{k_0^2}\right) \tag{8.97}$$

If $\ell\kappa_0^2 \ll 1$, $k_0^2 \sim 2\gamma\omega_{00}\kappa_0 \ll 2\gamma^2$. At short times, we may approximate

$$\frac{\sinh\left[\mu_k\gamma t\right]}{\sin\vartheta_k} \sim \frac{\mu_k\gamma t}{\sin\vartheta_k} = \kappa_k\gamma t \tag{8.98}$$

$$F_{kk'}\left(t\right) = \gamma t\left[\kappa_k + \kappa_{k'}\right] \tag{8.99}$$

Initially the stochastic source grows linearly in time and is coherent over distances

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of order k_0^{-1}. At late times

$$F_{kk'}(t) = \frac{e^{(\mu_k+\mu_{k'})\gamma t}}{2\sin\vartheta_k \sin\vartheta_{k'}}\left(e^{i\vartheta_k} + e^{i\vartheta_{k'}}\right) \sim F_{00}e^{2\mu_0\gamma t}\exp\left[-\frac{\mu_0\gamma t}{k_0^2}\left(\mathbf{k}^2 + \mathbf{k}'^2\right)\right] \tag{8.100}$$

so not only the strength of the stochastic source grows exponentially, with a time constant defined by the Floquet exponent, but also the size of the coherent domains grows as a power of time (in this simple model, $t^{1/2}$).

8.9 Final remarks

In this chapter, we have analyzed dissipation and fluctuations in the mean field by viewing it as an effectively open system, interacting with the environment provided by the quantum fluctuations of the same fundamental field. We shall conclude by mentioning some concrete problems where this way of thinking is fruitful in understanding their behaviors.

Physically, a quantum field develops a nontrivial expectation value through the process of condensation. By including fluctuations in its dynamics, we see the distinction between a condensate field and a mean field. The condensate is now regarded as a classical subsystem, interacting with a quantum environment and acquiring a stochastic component as a consequence.

Since in practice only long-wavelength–low-frequency modes condensate, one may attempt to draw a sharp distinction between condensate and fluctuations by defining an a priori separation between a long-wavelength condensate band, and a short-wavelength fluctuation band. Then the former may be described as a quantum open system. Eventually, if it actually condensates, the quantum fluctuations in the condensate band may be neglected. This kind of approach to condensate dynamics has been proposed by Gardiner and Anglin [GaAnFu01], Gardiner and Davis [GarDav03], and by Stoof [Sto99] in the context of Bose–Einstein condensates (BEC) (Chapter 13). Another example is in nuclear-particle physics. The effect of high-frequency modes in the quark–gluon plasma (QGP) on the (soft) gluon dynamics can be described by a Langevin equation, the so-called "Boedeker equation" of a similar construct (Chapter 10). Both the BEC and the QGP problems can be described by the coarse-grained effective action (or its equivalent) discussed in Chapter 5, with better built-in self-consistency.

In a truly dynamical setting, any a priori separation between a condensate and a fluctuations band may turn out to be artificial. Nonlinear effects in the condensate will tend to create short-wavelength features even out of smooth initial conditions, as shown dramatically in the phenomena of condensate collapse. Therefore it is better to avoid such a rigid distinction, but stress instead the difference between a c-number (albeit stochastic) component and the environment provided by the remnant q-number fluctuations. This is the approach taken in this chapter. An example we mentioned at the beginning of this chapter is stochastic semiclassical gravity [HuVer03, HuVer04] the arena where many of these ideas were developed and advanced. There, the Einstein–Langevin equation arises from incorporating the fluctuations of the quantum field as a noise term in the semiclassical Einstein equation. By implication this views Einstein's theory as a mean-field theory, a novel conception which can shed some new light on a radically different approach towards quantum gravity, via kinetic theory and stochastic dynamics. For further exposition of these ideas, see [Hu99, Hu02, Hu05].

Although the c-number part is not quite an open system – since no a priori criteria for separation between system and environment have been established – in practice it amounts to very much the same thing. Formally this is reflected by the close analogy between the CTPEA and an influence functional. We therefore say that, by adopting a description based on the fluctuating condensate, we turn the original problem into an effectively open system.

More generally, the fact that the Langevin approach allows us to reproduce the correct quantum Hadamard propagator makes this kind of approach useful in any situation where the amplitudes of fluctuations, rather than their coherence properties, are the main concern. In this light, the decoherence of the mean field is both a subject of theoretical and practical interest – theoretical for reasons stated above, practical because many physical phenomena originate from such processes. Examples are cosmological structure formation and quantum phase transitions. These topics will be treated in later chapters.