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WHITE NOISE DELTA FUNCTIONS AND CONTINUOUS VERSION THEOREM

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Introduction

The recently developed Hida calculus of white noise [5] is an infinite dimensional analogue of Schwartz' distribution theory besed on the Gelfand triple $(E) \subset (L^2) = L^2(E^*, \mu) \subset (E)^*$, where (E^*, μ) is Gaussian space and (L^2) is (a realization of) Fock space. It has been so far discussed aiming at an application to quantum physics, for instance [1], [3], and infinite dimensional harmonic analysis [7], [8], [13], [14], [15]. During the development an important milestone was Kubo-Yokoi's continuous version theorem [11] which asserts that every test white noise functional $\phi \in (E)$ admits a unique continuous version and, therefore, the test functionals constitute a space of continuous functions on E^*. This theorem is very fundamental and indispensable for many arguments. For example, it allows us to introduce a delta function on Gaussian space, which is one of the most important generalized functions. Furthermore, the continuous version theorem is effectively applied to description of positive generalized white noise functionals [19].

The motivation of this paper is to give an alternative proof of the continuous version theorem by means of a direct use of defining Hilbertian seminorms of E^*. In fact, this approach yields a sharp estimate of white noise delta functions $\delta_x \in (E)^*$, $x \in E^*$, from which the continuous version theorem follows. Moreover, with this method we may prove the continuity of $x \mapsto \delta_x \in (E)^*$, $x \in E^*$, which guarantees that the n-fold (topological) tensor product $(E) \otimes \cdots \otimes (E)$ is again a space of continuous functions on the product of the Gaussian space $E^* \times \cdots \times E^*$ (n times).

Here we remark some closely related works. In [12] Lee proved that each test functional $\phi \in (E)$ admits an analytic version on each Hilbert space E_{-p}, where $E^* = \text{ind}\lim_{p\to\infty} E_{-p}$. However, since the inductive system $\{E_{-p}\}_{p\geq 0}$ is not strict, our continuous version theorem does not follow from his result. In [9] Kondrat'ev

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and Samoylenko studied smoothness of test functionals on $\mathbf{R}^\infty$ equipped with the product Gaussian measure. Although based on a different framework, their discussion can be translated into our language. However, it turns out that their results have little in common with ours but some with Lee's. Finally, within the framework of Malliavin calculus the unique existence of quasi-continuous version has been discussed in many contexts, see e.g., [17] and references cited therein.

The paper is organized as follows. In Section 1 we recapitulate a well known construction of Gelfand triples under the name of standard construction. Section 2 is devoted to a brief review of test and generalized white noise functionals. In Section 3 we formulate the main results. In Section 4 we introduce a set of defining Hilbertian seminorms of E^* and in Section 5 we prove the main results. Section 6 contains some results on a tensor product of white noise test functionals.

NOTATION. If $\mathfrak{X}$ is a real vector space, we denote by $\mathfrak{X}_{\mathbf{C}}$ the complexification. For two vector spaces $\mathfrak{X}$ and $\mathfrak{Y}$ we denote by $\mathfrak{X}\otimes_{\mathrm{alg}}\mathfrak{Y}$ their algebraic tensor product. If $\mathfrak{X}=H$ and $\mathfrak{Y}=K$ are Hilbert spaces, we denote by $H\otimes K$ the Hilbert space tensor product. For nuclear spaces $\mathfrak{X}$ and $\mathfrak{Y}$, we denote simply by $\mathfrak{X}\otimes\mathfrak{Y}$ the completion of $\mathfrak{X}\otimes_{\mathrm{alg}}\mathfrak{Y}$ with respect to the π-topology, i.e., the strongest locally convex topology on $\mathfrak{X}\otimes_{\mathrm{alg}}\mathfrak{Y}$ such that the canonical map $\mathfrak{X}\times\mathfrak{Y}\to\mathfrak{X}\otimes_{\mathrm{alg}}\mathfrak{Y}$ is continuous. Although the π-tensor product of Hilbert spaces is different from the Hilbert space tensor product, there will be no confusion. We denote by $\mathfrak{X}^{\widehat{\otimes} n}\subset\mathfrak{X}^{\otimes n}$ the closed subspace of symmetric tensor products. We also use $(\mathfrak{X}^{\otimes n})^*_{\mathrm{sym}}$ for the same meaning in case of the strong dual spaces.

1. Standard construction of Gelfand triples

Motivated by the works of Berezansky-Kondrat'ev [2] and Gelfand-Vilenkin [4], we reformulate a useful method of constructing a nuclear Fréchet space or equivalently a Gelfand triple.

Let H be a real separable Hilbert space with norm $|\cdot|_0$ and inner product $\langle\cdot,\cdot\rangle$. Assume we are given a pair $(\{e_j\}_{j=0}^\infty, \{\lambda_j\}_{j=0}^\infty)$ of a complete orthonormal basis of H and a sequence of positive numbers with $\sum_{j=0}^\infty \lambda_j^{-2r}<\infty$ for some $r>0$. We then put

$$|\xi|_p=\left(\sum_{j=0}^\infty \lambda_j^{2p}\langle\xi, e_j\rangle^2\right)^{1/2},\quad \xi\in H,\ p\in\mathbf{R}, \tag{1}$$

though $|\xi|_p=\infty$ can happen. For $p\geq 0$ let E_p denote the subspace of $\xi\in H$ with $|\xi|_p<\infty$. Obviously, E_p becomes a Hilbert space with norm $|\cdot|_p$. Again for

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$p \geq 0$ let E_{-p} denote the completion of H with respect to the Hilbertian norm $|\cdot|_{-p}$ on H. We have thus obtained a chain of Hilbert spaces:

$$\cdots \subset E_q \subset \cdots \subset E_p \subset \cdots \subset E_0 = H \subset \cdots \subset E_{-p} \subset \cdots \subset E_{-q} \subset \cdots, \qquad q \geq p \geq 0,$$

where every canonical injection $E_q \to E_p$, $q \geq p$, is continuous and has dense image. As is easily seen, the inner product $\langle \cdot, \cdot \rangle$ of H is naturally extended to the canonical bilinear form on $E_{-p} \times E_p$, $p \geq 0$, through which E_{-p} is identified with the strong dual of E_p.

THEOREM 1.1. *Equipped with the Hilbertian norms* $|\cdot|_p$, $p \geq 0$, $E = \cap_{p \geq 0} E_p$ *becomes a nuclear Frèchet space, which is isomorphic to the projective limit* $\operatorname{proj\,lim}_{p \to \infty} E_p$. *Moreover, the strong dual* E^* *is isomorphic to the inductive limit* $\operatorname{ind\,lim}_{p \to \infty} E_{-p}$ *and is identified with* $\cup_{p \geq 0} E_{-p}$ *as vector space.*

The proof is straightforward, see [4: Chap. I] and [16: Chap. IV]. We have thus obtained a Gelfand triple $E \subset H \subset E^*$ from the pair $(\{e_j\}_{j=0}^{\infty}, \{\lambda_j\}_{j=0}^{\infty})$. This construction will be called *standard*.

While, it is sometimes more convenient to start with a densely defined operator on H instead of a pair $(\{e_j\}_{j=0}^{\infty}, \{\lambda_j\}_{j=0}^{\infty})$. A linear operator A with dense domain $\mathrm{Dom}(A) \subset H$ is called *standard* if there is a complete orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for H contained in $\mathrm{Dom}(A)$ such that

(S1) $Ae_j = \lambda_j e_j$ with $\lambda_j > 0$;

(S2) $\sum_{j=0}^{\infty} \lambda_j^{-2r} < \infty$ for some $r > 0$.

The relation between a standard operator A and a pair $(\{e_j\}_{j=0}^{\infty}, \{\lambda_j\}_{j=0}^{\infty})$ is described as

$$A\xi = \sum_{j=0}^{\infty} \lambda_j \langle \xi, e_j \rangle e_j, \quad \xi \in \mathrm{Dom}(A).$$

Given a standard operator A, we construct a Gelfand triple in the standard manner.

LEMMA 1.2. *If* A *is a standard operator on* H, *so is* A^s *for any* $s > 0$ *and the resultant Gelfand triples are isomorphic.*

The proof is straightforward. By virtue of Lemma 1.2 we may assume without loss of generality that $r = 1$ in (S2), when we discuss standard construction of Gelfand triples.

Let Ω be a topological space equipped with a Borel measure ν. If A is a standard operator on $H = L^2(\Omega, \nu; \mathbf{R})$, the Gelfand triple constructed from A is explicitly written as

$$\mathcal{S}_A(\Omega) \subset L^2(\Omega, \nu; \mathbf{R}) \subset \mathcal{S}_A^*(\Omega).$$

By construction each element of $\mathcal{S}_A(\Omega)$ is a function on Ω which is determined up to ν-null functions. For many practical reasons it is desired that $\mathcal{S}_A(\Omega)$ can be identified with a space of continuous functions on Ω. In this connection we propose the following hypothesis:

(H1) For each function $\xi \in \mathcal{S}_A(\Omega)$ there exists a unique continuous function $\tilde{\xi}$ on Ω such that $\xi(\omega) = \tilde{\xi}(\omega)$ for ν-a.e. $\omega \in \Omega$.

Once this condition is satisfied, we always regard $\mathcal{S}_A(\Omega)$ as a space of continuous functions on Ω and we do not use the symbol $\tilde{\xi}$. Under (H1) we consider two more hypotheses:

(H2) For each $\omega \in \Omega$ the evaluation map $\delta_\omega : \xi \mapsto \xi(\omega)$, $\xi \in \mathcal{S}_A(\Omega)$, is a continuous linear functional, i.e., $\delta_\omega \in \mathcal{S}_A^*(\Omega)$.

(H3) The map $\omega \mapsto \delta_\omega \in \mathcal{S}_A^*(\Omega)$, $\omega \in \Omega$, is continuous with respect to the strong dual topology of $\mathcal{S}_A^*(\Omega)$.

The above hypotheses are motivated by the work of Kubo and Takenaka [10]. While, in [4: Chap. I] the evaluation map δ_ω is discussed without topological structure of Ω. A sufficient condition for (H1)-(H3) is presented in Appendix.

2. Generalized white noise functionals

We keep to the same notation as in §1. Let A be a standard operator on a real Hilbert space H satisfying

(A1) $Ae_j = \lambda_j e_j$ with $\lambda_j \in \mathbf{R}$;

(A2) $\sum_{j=0}^{\infty} \lambda_j^{-2} < \infty$;

(A3) $1 < \lambda_0 \le \lambda_1 \le \cdots \to \infty$.

The last condition is indispensable to our white noise calculus setup. Let $E \subset H \subset E^*$ be the Gelfand triple constructed from A in the standard manner.

The Gaussian measure μ on E^* is defined by

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$$\exp\left(-\frac{|\xi|_0^2}{2}\right) = \int_{E^*} e^{i\langle x,\xi\rangle}\,\mu(dx), \quad \xi \in E. \tag{2}$$

We consider the Hilbert space $L^2(E^*, \mu; \mathbf{R})$ and its complexification (L^2). Their norms are denoted by $\|\cdot\|_0$.

In order to introduce a standard operator on $L^2(E^*, \mu; \mathbf{R})$ we need a variant of Wiener-Itô decomposition. For $x \in E^*$ and $n = 0,1,2,\dots$ we define $:x^{\otimes n}: \in (E^{\otimes n})^*_{\mathrm{sym}}$ inductively as follows:

$$\begin{cases} :x^{\otimes 0} := 1, \\ :x^{\otimes 1} := x, \\ :x^{\otimes n} := x \widehat{\otimes} :x^{\otimes(n-1)}: - (n-1)\tau \widehat{\otimes} :x^{\otimes(n-2)}:, \quad n \geq 2, \end{cases}$$

where $\tau \in (E \otimes E)^*_{\mathrm{sym}}$ is the *trace* uniquely determined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in E. \tag{3}$$

Let $f_n \in E_{\mathbf{C}}^{\widehat{\otimes} n}$. Then $\phi_n(x) = \langle :x^{\otimes n}:, f_n \rangle$ becomes a continuous function on E^* which satisfies $\|\phi_n\|_0^2 = n!\,|f_n|_0^2$. Using this isometry, we may define $\langle :x^{\otimes n}:, f_n \rangle$ for $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$ in L^2-sense. With these notations we have the following

PROPOSITION 2.1. *For each $\phi \in (L^2)$ there exists a unique sequence $(f_n)_{n=0}^\infty$, $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$, such that*

$$\phi(x) = \sum_{n=0}^\infty \langle :x^{\otimes n}:, f_n \rangle, \quad x \in E^*, \tag{4}$$

where the right hand side is an orthogonal direct sum in (L^2). In that case it holds that

$$\|\phi\|_0^2 = \sum_{n=0}^\infty n!\,|f_n|_0^2. \tag{5}$$

We now define a second quantized operator $\Gamma(A)$. Let $\mathrm{Dom}(\Gamma(A))$ be the space of functions ϕ of the form (4) such that $f_n \in E^{\widehat{\otimes} n}$ and $f_n = 0$ except finitely many n. For $\phi \in \mathrm{Dom}(\Gamma(A))$ put

$$\Gamma(A)\phi(x) = \sum_{n=0}^\infty \langle :x^{\otimes n}:, A^{\otimes n} f_n \rangle. \tag{6}$$

Then, $\Gamma(A)$ becomes a standard operator on $L^2(E^*, \mu; \mathbf{R})$, see [15]. Employing the standard construction, we obtain a new Gelfand triple:

$$\mathcal{S}_{\Gamma(A)}(E^*) \subset L^2(E^*, \mu; \mathbf{R}) \subset \mathcal{S}^*_{\Gamma(A)}(E^*),$$

of which complexification is denoted by

$$(E) \subset (L^2) \subset (E)^*.$$

Elements $\phi \in (E)$ and $\Phi \in (E)^*$ are called a *test* (*white noise*) *functional* and a *generalized* (*white noise*) *functional,* respectively. The canonical bilinear form on $(E)^* \times (E)$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

It follows from (5), (6) and the definition of norms that

$$\|\phi\|_p^2 = \|\Gamma(A)^p \phi\|_0^2 = \sum_{n=0}^{\infty} n! \, |(A^{\otimes n})^p f_n|_0^2 = \sum_{n=0}^{\infty} n! \, |f_n|_p^2. \tag{7}$$

We then obtain

PROPOSITION 2.2. *Let $\phi \in (L^2)$ be expressed as in* (4). *Then, $\phi \in (E)$ if and only if $f_n \in E_{\mathbf{C}}^{\hat{\otimes} n}$ for all $n = 0,1,2,\dots$ and $\sum_{n=0}^{\infty} n! \, |f_n|_p^2 < \infty$ for all $p \geq 0$.*

3. Continuous version theorem

In this section we formulate the main assertions. Recall that by construction each $\phi \in (E)$ is determined only up to μ-null functions.

THEOREM 3.1. *For each $\phi \in (E)$ there exists a unique continuous function $\tilde{\phi}$ on E^* such that $\phi(x) = \tilde{\phi}(x)$ for μ-a.e. $x \in E^*$. Moreover, $\tilde{\phi}(x)$ is given by the absolutely convergent series:*

$$\tilde{\phi}(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \quad x \in E^*,$$

where $(f_n)_{n=0}^{\infty}$ corresponds to the given ϕ as in Proposition 2.1.

The above assertion should be carefully compared with Proposition 2.2 which asserts that the series converges with respect to the norms $\|\cdot\|_p$, $p \geq 0$. By virtue of Theorem 3.1 we always understand (E) to be the space of continuous functions ϕ on E^* of the form:

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle,$$

where

(i) $f_n \in E_{\mathbf{C}}^{\hat{\otimes} n}$ for $n = 0,1,2,\dots$;

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(ii) $\sum_{n=0}^{\infty} n! \, |f_n|_p^2 < \infty$ for all $p \geq 0$;

(iii) the series is absolutely convergent at every $x \in E^*$.

For $x \in E^*$ a linear function δ_x on (E) is defined by

$$\delta_x : \phi \mapsto \phi(x), \quad \phi \in (E).$$

This is called a *white noise delta function.*

THEOREM 3.2. *$\delta_x \in (E)^*$ for all $x \in E^*$. Moreover,*

$$\| \delta_x \|_{-p} \leq \{1 - (|\tau|_{-p}^{1/2} + |x|_{-p})^2\}^{-1/2}, \tag{8}$$

whenever $|\tau|_{-p}^{1/2} + |x|_{-p} < 1$.

THEOREM 3.3 *The map $x \mapsto \delta_x \in (E)^*$, $x \in E^*$, is continuous.*

In short, Theorems 3.1-3.3 say that the space of test functionals (E) satisfies the hypotheses (H1)-(H3) introduced in §1, The proof of Theorem 3.1 being somehow long, it is devided into three steps (Propositions 3.4-3.6) and will be completed in §5. The estimate (8) in Theorem 3.2 will be sharpened in (33) in §5.

PROPOSITION 3.4. *Let $\tilde{\phi}$ be a continuous function on E^*. If $\tilde{\phi}(x) = 0$ for μ-a.e. $x \in E^*$, then $\tilde{\phi}(x) = 0$ for all $x \in E^*$.*

Proof. Note first that $\mu(E_{-p}) = 1$ for $p \geq 1$. In fact, this follows from (A2) and a general result (e.g., [18: Chap. 3]). We prove the assertion by contradiction. Suppose that $\tilde{\phi}(x_0) > 0$ for some point $x_0 \in E^*$. Take $p \geq 1$ such that $x_0 \in E_{-p}$ and consider the restriction of $\tilde{\phi}$ to E_{-p} which is denoted by the same symbol. Obviously, $\tilde{\phi}$ is a continuous function on E_{-p} with $\tilde{\phi}(x_0) > 0$. It then follows from the assumption that there exists a non-empty open subset $U \subset E_{-p}$ such that $\mu(U) = 0$. We now take a countable subset $\{\xi_k\}_{k=1}^{\infty} \subset H$ such that

$$E_{-p} = \bigcup_{k=1}^{\infty} (U + \xi_k).$$

But since the Gaussian measure μ is quasi-invariant under translations by H, we have $\mu(U + \xi_k) = 0$ and therefore $\mu(E_{-p}) = 0$. This is contradiction. Q.E.D.

We now introduce two basic constant numbers in white noise calculus:

$$\delta = \|A^{-1}\|_{\mathrm{HS}} = \Big(\sum_{j=0}^{\infty} \lambda_j^{-2}\Big)^{1/2}, \quad \rho = \|A^{-1}\| = \lambda_0^{-1},$$

where $\|A^{-1}\|_{\mathrm{HS}}$ and $\|A^{-1}\|$ stand for the Hilbert-Schmidt norm and the operator norm of A^{-1}, respectively. These are frequently used together with the obvious inequalities:

$$(9) \qquad 0 < \rho < 1, \quad \rho < \delta,$$

which follow from (A3).

Again by $|\cdot|_p$ we denote the norm of the Hilbert space $E_p^{\otimes n}$, $p \in \mathbf{R}$. Then, in view of (1) we obtain

$$(10) \quad |\omega|_p = |(A^{\otimes n})^p \omega|_0$$

$$= \Big(\sum_{j_1,\cdots,j_n=0}^{\infty} \lambda_{j_1}^{2p} \cdots \lambda_{j_n}^{2p} \langle \omega, e_{j_1} \otimes \cdots \otimes e_{j_n}\rangle^2\Big)^{1/2}, \quad \omega \in E_p^{\otimes n}.$$

Note also that

$$(11) \qquad |\omega|_p \le \rho^n |\omega|_{p+1}, \quad \omega \in E_p^{\otimes n}, \quad p \in \mathbf{R},$$

and therefore

$$(12) \qquad \lim_{p\to\infty} |F|_{-p} = 0, \quad F \in (E^{\otimes n})^*.$$

PROPOSITION 3.5. *For $n = 0,1,2,\ldots$ let $E_{\mathbf{C}}^{\hat{\otimes} n}$ and assume $\sum_{n=0}^{\infty} n!\,|f_n|_p^2 < \infty$ for all $p \ge 0$. Then the series*

$$\sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle$$

converges absolutely at each $x \in E^*$.

Proof. By definition (3) we have $\tau = \sum_{j=0}^{\infty} e_j \otimes e_j$ and $|\tau|_{-p}^2 = \sum_{j=0}^{\infty} \lambda_j^{-4p} < \infty$ whenever $p > 1/2$. We next note the inequality:

$$(13) \qquad |:x^{\otimes n}:|_{-p} \le \sqrt{n!}\,(|\tau|_{-p}^{1/2} + |x|_{-p})^n,$$

which follows from the well-known identity:

$$(14) \qquad :x^{\otimes n}: = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{(n-2k)!\,k!\,2^k}\, \tau^{\hat{\otimes} k} \hat{\otimes} x^{\otimes(n-2k)},$$

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and an obvious inequality:

$$\frac{1}{k!2^k} \le \frac{\sqrt{n!}}{(2k)!}, \quad 0 \le k \le \left[\frac{n}{2}\right].$$

Now let $x \in E^*$ be fixed. It follows from (13) that

$$(15) \quad \sum_{n=0}^{\infty} |\langle : x^{\otimes n} :, f_n \rangle| \le \sum_{n=0}^{\infty} |: x^{\otimes n} :|_{-p} |f_n|_p$$
$$\le \sum_{n=0}^{\infty} \sqrt{n!} (|\tau|_{-p}^{1/2} + |x|_{-p})^n |f_n|_p$$
$$\le \left(\sum_{n=0}^{\infty} n! |f_n|_p^2\right)^{1/2} \left(\sum_{n=0}^{\infty} (|\tau|_{-p}^{1/2} + |x|_{-p})^{2n}\right)^{1/2}.$$

In view of (12) we choose $p > 1/2$ such that $|\tau|_{-p}^{1/2} + |x|_{-p} < 1$. Then (15) becomes

$$(16) \quad \sum_{n=0}^{\infty} |\langle : x^{\otimes n} :, f_n \rangle| \le \left(\sum_{n=0}^{\infty} n! |f_n|_p^2\right)^{1/2} \{1 - (|\tau|_{-p}^{1/2} + |x|_{-p})^2\}^{-1/2} < \infty,$$

as desired. Q.E.D.

We now go back to the proof of Theorem 3.1. According to Proposition 2.2 a given $\phi \in (E)$ admits an expression:

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle, \quad x \in E^*,$$

where $f_n \in E_{\mathbf{C}}^{\hat{\otimes} n}$ for $n = 0,1,2, \ldots$ and $\sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty$ for all $p \ge 0$. The series converges in L^2-sense. On the other hand, it follows from Proposition 3.5 that

$$(17) \qquad \tilde{\phi}(x) \equiv \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$$

converges at every point $x \in E^*$. Therefore, $\phi(x) = \tilde{\phi}(x)$ for μ-a.e. $x \in E^*$. Since the uniqueness of a continuous version follows from Proposition 3.4, the proof of Theorem 3.1 is completed by the following

Proposition 3.6. *For $n = 0,1,2, \ldots$ let $f_n \in E_{\mathbf{C}}^{\hat{\otimes} n}$ and assume $\sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty$ for all $p \ge 0$. Then, $\tilde{\phi}$ defined in (17) is a continuous function on E^*.*

It is much simpler to show that the restriction of $\tilde{\phi}$ to E_{-p} is continuous with respect to the norm $|\cdot|_{-p}$. However, this is not enough to assert the continuity of $\tilde{\phi}$

with respect to the strong dual topology of E^*, because the inductive system $\{E_{-p}\}_{p\geq 0}$ is not strict.

Proposition 3.6 will be shown in Proposition 5.4 together with a precise estimate of $|\tilde{\phi}(x) - \tilde{\phi}(y)|$. While, the proof of Theorem 3.2 has been already established during the proof of Proposition 3.5. The estimate (8) follows immediately from (16).

4. Defining seminorms for the strong dual E^*

Following [18: Chap. 3] we introduce a set of defining Hilbertian seminorms of E^*. Let $\mathscr{C}$ be the set of sequences $C = (C_p)_{p=0}^{\infty}$ such that $C_0 \geq C_1 \geq \cdots > 0$. For $C \in \mathscr{C}$ we put

$$\|\xi\|_C^2 = \sum_{p=0}^{\infty} C_p^2 |\xi|_p^2, \quad \xi \in E, \tag{18}$$

though possibly $\|\xi\|_C = \infty$. Then $E(C) = \{\xi \in E\,;\, \|\xi\|_C < \infty\}$ becomes a Hilbert space with norm $\|\cdot\|_C$. We put

$$|x|_C = \sup\{|\langle x, \xi\rangle|\,;\, \|\xi\|_C \leq 1,\ \xi \in E\}, \quad x \in E^*. \tag{19}$$

Obviously, $|\cdot|_C$ is a Hilbertian seminorm on E^* though it is not necessarily a norm. Note also that for any $C \in \mathscr{C}$

$$|\langle x, \xi\rangle| \leq |x|_C \|\xi\|_C, \quad x \in E^*, \quad \xi \in E, \tag{20}$$

though $\|\xi\|_C = \infty$ can happen.

LEMMA 4.1. *$\{|\cdot|_C\}_{C\in\mathscr{C}}$ is a set of defining Hilbertian seminorms of E^*.*

Proof. The strong dual topology of E^* is defined by the seminorms

$$x \mapsto \sup\{|\langle x, \xi\rangle|\,;\, \xi \in B\}, \quad x \in E^*,$$

where B runs over all bounded subsets of E. It is therefore sufficient to show that for any bounded subset $B \subset E$ there is $C \in \mathscr{C}$ such that $B \subset \{\xi \in E\,;\, \|\xi\|_C \leq 1\}$. But this is easily verified. Q.E.D.

LEMMA 4.2. *For $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ it holds that*

$$|x|_C^2 = \sum_{j=0}^{\infty} \langle x, e_j\rangle^2 \|e_j\|_C^{-2} = \sum_{j=0}^{\infty} \langle x, e_j\rangle^2 \left(\sum_{p=0}^{\infty} C_p^2 \lambda_j^{2p}\right)^{-1}, \quad x \in E^*.$$

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Proof. Recall that $\{e_j\}_{j=0}^{\infty}$ is an orthogonal basis for all E_p, $p \geq 0$. In view of (18) we see that $\{\| e_j \|_C^{-1} e_j\}_{j=0}^{\infty}$ is an orthonormal basis for $E(C)$ and also for $E(C)^*$, where we understand $\| e_j \|_C^{-1} e_j = 0$ if $\| e_j \|_C = \infty$. Then the assertion follows from Fourier expansion $x = \sum_{j=0}^{\infty} \langle x, e_j \rangle e_j$, which converges in E^*.

Q.E.D.

It is noted that

$$E^{\otimes n} = \bigcap_{p \geq 0} E_p^{\otimes n} \cong \operatorname{proj\,lim}_{p \to \infty} E_p^{\otimes n}, \quad (E^{\otimes n})^* = \bigcup_{p \geq 0} E_{-p}^{\otimes n} \cong \operatorname{ind\,lim}_{p \to \infty} E_{-p}^{\otimes n}.$$

Therefore the topology of $(E^{\otimes n})^*$ is defined in a similar way. Namely, for $C \in \mathscr{C}$ put

$$\| \omega \|_C^2 = \sum_{p_1, \cdots, p_n = 0}^{\infty} C_{p_1}^2 \cdots C_{p_n}^2 | \omega |_{p_1, \cdots, p_n}^2, \quad \omega \in E^{\otimes n}, \tag{21}$$

where

$$\begin{aligned} | \omega |_{p_1, \cdots, p_n}^2 &= | (A^{p_1} \otimes \cdots \otimes A^{p_n}) \omega |_0^2 \\ &= \sum_{j_1, \cdots, j_n = 0}^{\infty} \lambda_{j_1}^{2p_1} \cdots \lambda_{j_n}^{2p_n} \langle \omega, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle^2, \end{aligned} \tag{22}$$

see also (10). Then for $F \in (E^{\otimes n})^*$ we put

$$| F |_C = \sup \{ | \langle F, \omega \rangle | \, ; \, \| \omega \|_C \leq 1, \omega \in E^{\otimes n} \}.$$

It is proved in a similar way to Lemma 4.1 that $\{| \cdot |_C\}_{C \in \mathscr{C}}$ is a set of defining Hilbertian seminorms of $(E^{\otimes n})^*$. We next note the following

LEMMA 4.3. *Let* $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$. *Then*

$$| F |_C \leq C_p^{-n} | F |_{-p}, \quad F \in (E^{\otimes n})^*,$$

though $| F |_{-p} = \infty$ *may happen. Moreover,* $| \cdot |_C$ *is a cross norm, i.e.,*

$$| x_1 \otimes \cdots \otimes x_n |_C = | x_1 |_C \cdots | x_n |_C, \quad x_1, \cdots, x_n \in E^*.$$

Proof. A similar argument as in Lemma 4.1 yields

$$| F |_C^2 = \sum_{j_1, \cdots, j_n = 0}^{\infty} \langle F, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle^2 \| e_{j_1} \|_C^{-2} \cdots \| e_{j_n} \|_C^{-2}. \tag{23}$$

It is then obvious that $| \cdot |_C$ is a cross norm. Since

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$$|||e_j|||_C^{-2} = \Big(\sum_{p=0}^{\infty} C_p^2 \lambda_j^{2p}\Big)^{-1} \le C_p^{-2}\lambda_j^{-2p}$$

for any $p \ge 0$, it follows from (23) and (10) that

$$|F|_C^2 \le C_p^{-2n} \sum_{j_1,\cdots,j_n=0}^{\infty} \lambda_{j_1}^{-2p}\cdots\lambda_{j_n}^{-2p} \langle F,\, e_{j_1}\otimes\cdots\otimes e_{j_n}\rangle^2 = C_p^{-2n} |F|_{-p}^2$$

as desired. Q.E.D.

5. Proof of the main results

For any $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ we put

$$|||\phi|||_C^2 = \sum_{p=0}^{\infty} C_p^2 \|\phi\|_p^2, \quad \phi \in (E), \tag{24}$$

though $|||\phi|||_C = \infty$ may happen.

LEMMA 5.1. *Let $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$. Then for any $\phi \in (E)$ it holds that*

$$n!\, |||f_n|||_C^2 \le C_0^{-2}\, |||\phi|||_C^2 \left(\frac{C_0^2}{1-\rho^2}\right)^n,$$

where $(f_n)_{n=0}^{\infty}$ is given as in Proposition 2.1.

Proof. Recall the definition (21) to obtain

$$|||f_n|||_C^2 = \sum_{p_1,\cdots,p_n=0}^{\infty} C_{p_1}^2 \cdots C_{p_n}^2 |f_n|_{p_1,\cdots,p_n}^2. \tag{25}$$

In view of (10) and (22) we obtain

$$C_{p_1}^2 \cdots C_{p_n}^2 |f_n|_{p_1,\cdots,p_n}^2 \le C_0^{2(n-1)} C_p^2 \rho^{2(p-p_1)+\cdots+2(p-p_n)} |f_n|_p^2,$$

where $p = \max\{p_1,\cdots,p_n\}$. We then see that

$$\begin{aligned}\sum_{\max\{p_1,\cdots,p_n\}=p} C_{p_1}^2\cdots C_{p_n}^2 |f_n|_{p_1,\cdots,p_n}^2 &\le C_0^{2(n-1)} C_p^2 |f_n|_p^2 \Big(\sum_{q=0}^{p}\rho^{2(p-q)}\Big)^n \\ &\le C_0^{2(n-1)} C_p^2 |f_n|_p^2 (1-\rho^2)^{-n}.\end{aligned}$$

Thereby (25) becomes

$$|||f_n|||_C^2 = \sum_{p=0}^{\infty} \sum_{\max\{p_1,\cdots,p_n\}=p} C_{p_1}^2\cdots C_{p_n}^2 |f_n|_{p_1,\cdots,p_n}^2$$

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$$\leq C_0^{-2}\left(\frac{C_0^2}{1-\rho^2}\right)^n \sum_{p=0}^{\infty} C_p^2 \,|\, f_n \,|_p^2.$$

Finally with an obvious inequality: $n! \,|\, f_n \,|_p^2 \leq \| \phi \|_p^2$, which follows from (7), we obtain

$$n! \,\|\, f_n \,\|_C^2 \leq C_0^{-2}\left(\frac{C_0^2}{1-\rho^2}\right)^n \sum_{p=0}^{\infty} C_p^2 \| \phi \|_p^2 = C_0^{-2} \,\|\, \phi \,\|_C^2 \left(\frac{C_0^2}{1-\rho^2}\right)^n.$$

This completes the proof. Q.E.D.

LEMMA 5.2. *For $k = 0,1,2,\dots$ it holds that*

$$\sum_{n=0}^{\infty} \frac{(n+k)!}{n!n!} t^n \leq (t+k)^k e^t, \quad t \geq 0.$$

Proof. We put

$$P_k(t) = e^{-t} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!n!} t^n.$$

As is easily shown, $P_k(t)$ is a polynomial of degree k. Actually, it is related to the Laguerre polynomial (e.g. [6: Appendix]) as $P_k(t) = k!L_k(-t)$. We thus put $P_k(t) = \sum_{l=0}^{k} a_{kl} t^l$. Then, by induction we may prove

$$0 \leq a_{kl} \leq \binom{k}{l} k^{k-l}, \quad 0 \leq l \leq k,$$

from which the assertion follows immediately. Q.E.D.

LEMMA 5.3. *Put*

$$\lambda(z, w) = \sum_{k=0}^{\infty} \frac{(z+k)^{k+1} w^k}{k!}.$$

Then the series converges in $\mathbf{C} \times \{|\,\omega\,| < e^{-1}\}$ *and* $\lambda(z, w)$ *becomes a holomorphic function in two variables.*

Proof. We only need to apply Cauchy-Hadamard formula. Q.E.D.

Assuming that $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ satisfies the condition $C_0^2 < 1 - \rho^2$, we put

$$\Lambda_C(z, w) = \frac{2}{\sqrt{1-\rho^2-C_0^2}} \exp\left(\frac{z^2}{2}\right) \lambda\left(\frac{z^2}{2}+1, \frac{C_0^2 w}{1-\rho^2}\right).$$

Obviously, $\Lambda_C(z, w)$ is holomorphic in $\mathbf{C} \times \{|w| < e^{-1} C_0^{-2}(1-\rho^2)\}$. With these notations we have

PROPOSITION 5.4. *For $n = 0,1,2, \ldots$ let $f_n \in E_{\mathbf{C}}^{\widehat{\otimes} n}$ and assume $\sum_{n=0}^{\infty} n! \, |f_n|_p^2 < \infty$ for all $p \geq 0$. Put*

$$\tilde{\phi}(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \quad x \in E^*.$$

If $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ satisfies

(26) $$C_0^2 < 1 - p^2 \quad \text{and} \quad C_0^2 |\tau|_C < \frac{1-\rho^2}{e},$$

then

(27) $$|\tilde{\phi}(x) - \tilde{\phi}(y)| \leq |x-y|_C \, \|\tilde{\phi}\|_C \, \Lambda_C(|x|_C + |y|_C, |\tau|_C)$$

for all $x, y \in E^*$. In particular, $\tilde{\phi}$ is a continuous function on E^*.

Proof. Let $x, y \in E^*$ and suppose $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ satisfies (26). Then, in view of (14), (20) and Lemma 4.3 we observe

(28) $$\begin{aligned} &|\langle :x^{\otimes n}: - :y^{\otimes n}:, f_n \rangle| \\ &\leq \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!2^k} \left| \langle \tau^{\widehat{\otimes} k} \widehat{\otimes} (x^{\otimes(n-2k)} - y^{\otimes(n-2k)}), f_n \rangle \right| \\ &\leq \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!2^k} |\tau|_C^k |x^{\otimes(n-2k)} - y^{\otimes(n-2k)}|_C \, \|f_n\|_C. \end{aligned}$$

Using an obvious inequality:

$$|x^{\otimes m} - y^{\otimes m}|_C \leq |x-y|_C (|x|_C + |y|_C)^{m-1}, \quad m \geq 1,$$

and summing up both sides of (28) with n, we obtain

(29) $$\begin{aligned} |\tilde{\phi}(x) - \tilde{\phi}(y)| \leq |x-y|_C \sum_{k=0}^{\infty} \frac{|\tau|_C^k}{k!2^k} \sum_{n=0}^{\infty} \frac{(n+2k+1)!}{(n+1)!} \\ \times (|x|_C + |y|_C)^n \, \|f_{n+2k+1}\|_C. \end{aligned}$$

Applying the Schwarz inequality, we have

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$$\begin{aligned}
(30) \qquad & \sum_{n=0}^{\infty} \frac{(n+2k+1)!}{(n+1)!} (|x|_C + |y|_C)^n \| f_{n+2k+1} \|_C \\
& \leq \left(\sum_{n=0}^{\infty} (n+2k+1)! \| f_{n+2k+1} \|_C^2 \right)^{1/2} \\
& \quad \times \left(\sum_{n=0}^{\infty} \frac{(n+2k+1)!}{(n+1)!(n+1)!} (|x|_C + |y|_C)^{2n} \right)^{1/2}.
\end{aligned}$$

We now estimate the last two series. By Lemma 5.1 we have

$$\begin{aligned}
(31) \qquad \sum_{n=0}^{\infty} (n+2k+1)! \| f_{n+2k+1} \|_C^2 & \leq C_0^{-2} \| \tilde{\phi} \|_C^2 \sum_{n=0}^{\infty} \left(\frac{C_0^2}{1-\rho^2} \right)^{n+2k+1} \\
& = \frac{\| \tilde{\phi} \|_C^2}{1-\rho^2-C_0^2} \left(\frac{C_0^2}{1-\rho^2} \right)^{2k},
\end{aligned}$$

where we used the assumption $C_0^2 < 1-\rho^2$. On the other hand, application of Lemma 5.2 yields

$$\begin{aligned}
(32) \qquad & \sum_{n=0}^{\infty} \frac{(n+2k+1)!}{(n+1)!(n+1)!} (|x|_C + |y|_C)^{2n} \\
& \leq \sum_{n=0}^{\infty} \frac{(n+2k+2)!}{n!n!} (|x|_C + |y|_C)^{2n} \\
& \leq \{ (|x|_C + |y|_C)^2 + 2k + 2 \}^{2k+2} \exp\{ (|x|_C + |y|_C)^2 \}.
\end{aligned}$$

Therefore (30) is estimated by (31) and (32) as

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(n+2k+1)!}{(n+1)!} (|x|_C + |y|_C)^n \| f_{n+2k+1} \|_C \\
& \leq \frac{2 \| \tilde{\phi} \|_C}{\sqrt{1-\rho^2-C_0^2}} \left(\frac{2C_0^2}{1-\rho^2} \right)^k \exp\left(\frac{(|x|_C + |y|_C)^2}{2} \right) \\
& \qquad \times \left\{ \frac{(|x|_C + |y|_C)^2}{2} + k + 1 \right\}^{k+1}.
\end{aligned}$$

Consequently, (29) becomes

$$\begin{aligned}
| \tilde{\phi}(x) - \tilde{\phi}(y) | & \leq \frac{2 \| \tilde{\phi} \|_C |x-y|_C}{\sqrt{1-\rho^2-C_0^2}} \exp\left(\frac{(|x|_C + |y|_C)^2}{2} \right) \\
& \quad \times \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{C_0^2 |\tau|_C}{1-\rho^2} \right)^k \left\{ \frac{(|x|_C + |y|_C)^2}{2} + k + 1 \right\}^{k+1}
\end{aligned}$$

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$$= \frac{2 \| \tilde{\phi} \|_C |x-y|_C}{\sqrt{1-\rho^2-C_0^2}} \exp\left(\frac{(|x|_C+|y|_C)^2}{2}\right)$$
$$\times \lambda\left(\frac{(|x|_C+|y|_C)^2}{2}+1, \frac{C_0^2|\tau|_C}{1-\rho^2}\right)$$
$$= |x-y|_C \| \tilde{\phi} \|_C \Lambda_C(|x|_C+|y|_C, |\tau|_C).$$

This completes the proof of (27). Take $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ with the properties (26) (such a C exists certainly). Since $z \mapsto \Lambda_C(z, |\tau|_C)$ is continuous (in fact, analytic), we conclude from (27) that $\tilde{\phi}$ is continuous on E^*. Q.E.D.

As was already explained in §3, Proposition 5.4 completes the proof of the continuous version theorem (Theorem 3.1). Theorem 3.2 was proved at the end of §3. We now give

Proof of Theorem 3.3. Since $(E)^*$ is constructed in a similar way to E^* by the standard construction, the topology of $(E)^*$ is defined by the Hilbertian seminorms:

$$\Phi \mapsto \sup\{|\langle\langle \Phi, \phi \rangle\rangle| ; \| \phi \|_C \le 1\}, \quad \Phi \in (E)^*,$$

where $\| \phi \|_C$ is defined as in (24) and C runs over $\mathscr{C}$. While, it follows from Proposition 5.4 that

$$\lim_{y\to x} \sup\{|\langle\langle \delta_x - \delta_y, \phi \rangle\rangle| ; \| \phi \|_C \le 1\} = 0,$$

for all $C = (C_p)_{p=0}^{\infty} \in \mathscr{C}$ satisfying the conditions in (26). It is therefore sufficient to show that all $C \in \mathscr{C}$ with (26) constitute a set of defining seminorms of E^*. Note that $|x|_C \le |x|_{C'}$ for any $x \in E^*$ if $C' \le C$, namely, if $C_p' \le C_p$ for all $p = 0,1,2,\ldots$. Thus it is sufficient to show that for a given $C \in \mathscr{C}$ there is $C' \in \mathscr{C}$ with (26) such that $C' \le C$. Choose $q \ge 0$ such that $|\tau|_{-p} < e^{-1}(1-\rho^2)$. Define $C' = (C_p')_{p=0}^{\infty} \in \mathscr{C}$ by

$$\begin{cases} 0 < C_0' = \cdots = C_q' < \min\{|\tau|_{-q}, C_q, \sqrt{1-\rho^2}\}, \\ C_p' = \min\{C_{p-1}', C_p\}, \quad p > q. \end{cases}$$

Then, by construction, $C' \le C$ and $C_0'^2 < 1-\rho^2$. Moreover, since $|\tau|_{C'} \le C_q'^{-2}|\tau|_{-q}$ by Lemma 4.3, we have

$$C_0'^2 |\tau|_{C'} \le C_0'^2 C_q'^{-2} |\tau|_{-q} = |\tau|_{-q} \le \frac{1-\rho^2}{e}.$$

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This completes the proof. Q.E.D.

Remark. By a similar (but much simpler) computation as in the proof of Propositon 5.4, we obtain a somehow better estimate of a white noise delta function δ_x, $x \in E^*$. Let $\phi \in (E)$ be given by

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle, \quad x \in E^*,$$

as usual. Then, for $p \geq 0$ we have

$$\begin{aligned} |\phi(x)| &\leq \sum_{n=0}^{\infty} |\langle : x^{\otimes n} :, f_n \rangle| \\ &\leqq \|\phi\|_p \exp\left(\frac{|x|_{-p}^2}{2}\right) \sum_{k=0}^{\infty} \frac{|\tau|_{-p}^k}{k!} \left(\frac{|x|_{-p}^2}{2} + k\right)^k, \end{aligned}$$

and therefore,

$$\|\delta_x\|_{-p} \leq \exp\left(\frac{|x|_{-p}^2}{2}\right) \sum_{k=0}^{\infty} \frac{|\tau|_{-p}^k}{k!} \left(\frac{|x|_{-p}^2}{2} + k\right)^k. \tag{33}$$

The last series converges whenever $|\tau|_{-p} < e^{-1}$ and $|x|_{-p} < \infty$. This condition may be compared with $|\tau|_{-p}^{1/2} + |x|_{-p} < 1$ in Theorem 3.2.

6. Tensor product

The standard construction of a Gelfand triple is well suited to tensor products. We begin with the following

PROPOSITION 6.1. *For $i = 1,2$ let A_i be a standard operator on H_i and let $E_i \subset H_i \subset E_i^*$ be the Gelfand triple constructed in the standard manner. Then, $A_1 \otimes A_2$ is a standard operator on $H_1 \otimes H_2$ with domain* $\mathrm{Dom}(A_1 \otimes A_2) = \mathrm{Dom}(A_1) \otimes_{\mathrm{alg}} \mathrm{Dom}(A_2)$ *and the Gelfand triple obtained from $A_1 \otimes A_2$ is given by $E_1 \otimes E_2 \subset H_1 \otimes H_2 \subset (E_1 \otimes E_2)^*$.*

Proof. It follows from Theorem 1.1 that $E_i = \mathrm{proj\,lim}_{p\to\infty} E_{ip}$, $i = 1,2$, where E_{ip} is the Hilbert space obtained by completing E_i with respect to the norm $|\xi|_p = |A_i^p \xi|_0$, $\xi \in E_i$. Then a simple observation implies that

$$E_1 \otimes E_2 = \mathop{\mathrm{proj\,lim}}_{p,q\to\infty} E_{1p} \otimes E_{2q} = \mathop{\mathrm{proj\,lim}}_{p\to\infty} E_{1p} \otimes E_{2p}. \tag{34}$$

On the other hand, it is easily verified that $A_1 \otimes A_2$ is a standard operator on

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$H_1 \otimes H_2$. Let F be the nuclear Fréchet space constructed from $A_1 \otimes A_2$ in the standard manner. Then, $F = \text{proj lim}_{p\to\infty} F_p$ with F_p being the completion of F with respect to the norm $|\zeta|_p = |(A_1 \otimes A_2)^p \zeta|_0$. Note here that $F_p = E_{1p} \otimes E_{2p}$. It then follows from (34) that $F = E_1 \otimes E_2$. Q.E.D.

PROPOSITION 6.2. *For $i = 1,2$ let Ω_i be a topological space with a Borel measure ν_i and let A_i be a standard operator on $L^2(\Omega_i, \nu_i; \mathbf{R})$. Then*

$$\mathcal{S}_{A_1 \otimes A_2}(\Omega_1 \times \Omega_2) = \mathcal{S}_{A_1}(\Omega_1) \otimes \mathcal{S}_{A_2}(\Omega_2)$$

under the identification: $L^2(\Omega_1 \times \Omega_2, \nu_1 \times \nu_2; \mathbf{R}) = L^2(\Omega_1, \nu_1; \mathbf{R}) \otimes L^2(\Omega_2, \nu_2; \mathbf{R})$.

Proof. Immediate from Proposition 6.1. Q.E.D.

PROPOSITION 6.3. *Let notations and assumptions be the same as in Proposition 6.2. If both $\mathcal{S}_{A_1}(\Omega_1)$ and $\mathcal{S}_{A_2}(\Omega_2)$ satisfy the hypotheses* (H1)-(H3), *so does $\mathcal{S}_{A_1 \otimes A_2}(\Omega_1 \times \Omega_2)$.*

Proof. For $\zeta \in \mathcal{S}_{A_1 \otimes A_2}(\Omega_1 \times \Omega_2) = \mathcal{S}_{A_1}(\Omega_1) \otimes \mathcal{S}_{A_2}(\Omega_2)$ we put

$$\tilde{\zeta}(\omega_1, \omega_2) = \langle \delta_{\omega_1} \otimes \delta_{\omega_2}, \zeta \rangle, \quad \omega_1 \in \Omega_1, \quad \omega_2 \in \Omega_2.$$

Then $\tilde{\zeta}$ is a continuous function on $\Omega_1 \times \Omega_2$ because of (H3) and the fact that

$$x, y \mapsto x \otimes y \in \mathcal{S}^*_{A_1}(\Omega_1) \otimes \mathcal{S}^*_{A_2}(\Omega_2), \quad x \in \mathcal{S}^*_{A_1}(\Omega_1), \quad y \in \mathcal{S}^*_{A_2}(\Omega_2),$$

is continuous. Take an approximating sequence $\{\zeta_n\}_{n=1}^{\infty} \subset \mathcal{S}_{A_1}(\Omega_1) \otimes_{\text{alg}} \mathcal{S}_{A_2}(\Omega_2)$ such that

$$\lim_{n\to\infty} |\zeta_n - \zeta|_p = 0 \quad \text{for all } p \geq 0. \tag{35}$$

Then

$$\begin{aligned} |\zeta_n - \tilde{\zeta}|_0^2 &= \int_{\Omega_1 \times \Omega_2} |\zeta_n(\omega_1, \omega_2) - \tilde{\zeta}(\omega_1, \omega_2)|^2 \nu_1(d\omega_1)\nu_2(d\omega_2) \\ &= \int_{\Omega_1 \times \Omega_2} |\langle \delta_{\omega_1} \otimes \delta_{\omega_2}, \zeta_n - \zeta \rangle|^2 \nu_1(d\omega_1)\nu_2(d\omega_2) \\ &\leq |\zeta_n - \zeta|_1^2 \int_{\Omega_1} |\delta_{\omega_1}|_{-1}^2 \nu_1(d\omega_1) \int_{\Omega_2} |\delta_{\omega_2}|_{-2}^2 \nu_2(d\omega_2). \end{aligned} \tag{36}$$

Note that

$$\int_{\Omega_i} |\delta_{\omega_i}|_{-2}^2 \nu_i(d\omega_i) = \sum_{j=0}^{\infty} \lambda_{ij}^{-2} \equiv \delta_i^2 < \infty, \quad i = 1,2, \tag{37}$$

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which is immediate from the identity:

$$|\delta_{\omega_i}|_{-1}^2 = \sum_{j=0}^{\infty} \langle \delta_{\omega_i}, e_{ij}\rangle^2 \lambda_{ij}^{-2} = \sum_{j=0}^{\infty} e_{ij}(\omega_i)^2 \lambda_{ij}^{-2}, \quad i = 1,2.$$

It then follows from (36) and (37) that

$$|\zeta_n - \tilde{\zeta}|_0^2 \leq \delta_1^2 \delta_2^2 |\zeta_n - \zeta|_1^2. \tag{38}$$

Hence, we see from (35) and (38) that

$$\begin{aligned} |\zeta - \tilde{\zeta}|_0 &\leq |\zeta - \zeta_n|_0 + |\zeta_n - \tilde{\zeta}|_0 \\ &\leq |\zeta - \zeta_n|_0 + \delta_1\delta_2 |\zeta_n - \zeta|_1 \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Namely, $\zeta(\omega_1, \omega_2) = \tilde{\zeta}(\omega_1, \omega_2)$ for $\nu_1 \times \nu_2$-a.e. $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$. This proves (H1). The properties (H2) and (H3) are now immediate. Q.E.D.

In view of Theorems 3.1-3.3 and Proposition 6.3 we conclude the following

THEOREM 6.4. *Any function in the n-fold tensor product $(E) \otimes \cdots \otimes (E)$ is continuous on the product space $E^* \times \cdots \times E^*$ (n times), or more precisely, admits a unique continuous version with respect to the product measure $\mu \times \cdots \times \mu$ (n times), $n \geq 1$.*

Appendix. A sufficient condition for the hypotheses (H1)-(H3)

We shall give a sufficient condition for (H1)-(H3) from a different viewpoint.

PROPOSITION. *Let Ω be a topological space with a Borel measure ν and let A be a standard operator on $H = L^2(\Omega, \nu; \mathbf{R})$ with eigenfunctions $\{e_j\}_{j=0}^{\infty}$ and eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ satisfying* (S1) *and* (S2). *Assume the following three conditions:*

(i) $\nu(U) > 0$ *for any non-empty open subset* $U \subset \Omega$;

(ii) *every e_j is a continuous function on Ω* ;

(iii) *Ω admits an open covering $\Omega = \cup_\gamma \Omega_\gamma$ with the property that for each γ there exists $\alpha(\gamma) \geq 0$ such that*

$$M_\gamma \equiv \sup \{\lambda_j^{-\alpha(\gamma)} |e_j(\omega)| ; \omega \in \Omega_\gamma, j = 0,1,2,\ldots\} < \infty.$$

Then $\mathcal{S}_A(\Omega)$ satisfies (H1)-(H3). *Moreover, $\tilde{\phi}$ is given by the absolutely convergent series:*

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(39) $$\tilde{\phi}(\omega) = \sum_{j=0}^{\infty} \langle \phi, e_j \rangle \, e_j(\omega).$$

Proof. By Lemma 1.2 we may assume $r = 1$ in (S2). We first show that for $\phi \in \mathcal{S}_A(\Omega)$ the series (39) converges absolutely at any $\omega \in \Omega$. Choosing Ω_γ containing ω, we observe

$$\sum_{j=0}^{\infty} | \langle \varphi, e_j \rangle \, e_j(\omega) | \le M_\gamma \sum_{j=0}^{\infty} \lambda_j^{\alpha(\gamma)} | \langle \phi, e_j \rangle |$$
$$\le M_\gamma \left(\sum_{j=0}^{\infty} \lambda_j^{2(\alpha(\gamma)+1)} \langle \phi, e_j \rangle^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} \lambda_j^{-2} \right)^{1/2}.$$

Hence

(40) $$\sum_{j=0}^{\infty} | \langle \phi, e_j \rangle \, e_j(\omega) | \le \delta M_\gamma | \phi |_{\alpha(\gamma)+1}, \quad \omega \in \Omega_\gamma, \quad \phi \in \mathcal{S}_A(\Omega),$$

where $\delta = (\sum_{j=0}^{\infty} \lambda_j^{-2})^{1/2} < \infty$. This proves that the series (39) converges absolutely at each $\omega \in \Omega$.

For the continuity of $\tilde{\phi}$ we need only to prove that $\tilde{\phi}$ is continuous on Ω_γ. For $\omega_1, \omega_2 \in \Omega_\gamma$, a similar argument as above yields

(41)
$$| \tilde{\phi}(\omega_1) - \tilde{\phi}(\omega_2) | \le \sum_{j=0}^{n} | \langle \phi, e_j \rangle | \, | e_j(\omega_1) - e_j(\omega_2) | + 2M_\gamma | \phi |_{\alpha(\gamma)+1} \left(\sum_{j>n} \lambda_j^{-2} \right)^{1/2}.$$

Since e_j is continuous and $\delta^2 = \sum_{j=0}^{\infty} \lambda_j^{-2} < \infty$ by assumption, the continuity of $\tilde{\phi}$ on Ω_γ follows from (41). It is clear that $\phi(\omega) = \tilde{\phi}(\omega)$ for ν-a.e. $\omega \in \Omega$ because the Fourier expansion $\phi = \sum_{j=0}^{\infty} \langle \phi, e_j \rangle \, e_j$ converges in L^2-sense. We have thus proved (H1).

According to our convention, we do use the symbol ϕ for $\tilde{\phi}$ hereafter. The inequality (40) means that the evaluation $\delta_\omega : \phi \mapsto \phi(\omega)$ is a continuous linear function on $\mathcal{S}_A(\Omega)$. (In fact, $| \delta_\omega |_{-\alpha(\gamma)-1} \le \delta M_\gamma$ for $\omega \in \Omega_\gamma$.) Hence, $\delta_\omega \in \mathcal{S}_A^*(\Omega)$ and (H2) holds.

Finally we consider (H3). For a bounded subset $B \subset \mathcal{S}_A(\Omega)$ put

$$| B |_p = \sup \{ | \phi |_p \, ; \phi \in B \}$$

for simplicity. This is always finite. Note that

(42) $$\sup \{ | \langle \phi, e_j \rangle | \, ; \phi \in B \} \le \sup \{ | \phi |_0 | e_j |_0 \, ; \phi \in B \} = | B |_0 < \infty.$$

In view of (41) and (42), for $\omega_1, \omega_2 \in \Omega_\gamma$ we have

$$\sup\{|\langle \delta_{\omega_1} - \delta_{\omega_2}, \phi\rangle| ; \phi \in B\}$$
$$\leq |B|_0 \sum_{j=0}^{n} |e_j(\omega_1) - e_j(\omega_2)| + 2M_\gamma |B|_{\alpha(\gamma)+1} \left(\sum_{j>n} \lambda_j^{-2}\right)^{1/2}.$$

Hence the map $\omega \mapsto \delta_\omega \in \mathcal{S}_A^*(\Omega)$ is continuous on Ω_γ and therefore on Ω. Q.E.D.

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