ON THE ALGEBRA GENERATED BY A HERMITIAN OPERATOR

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The purpose of this note is to solve a problem of Dr A. M. Sinclair. Denote by $A^w(I, T)$ the algebra with identity generated by a bounded linear operator T in the weak operator topology. We prove the following result.

Theorem. *There is a separable reflexive complex Banach space X, and a hermitian operator H on X, such that the second commutant of $A^w(I, H)$ is strictly larger than $A^w(I, H)$.*

Throughout, if Y is a complex Banach space, $L(Y)$ denotes the Banach algebra of bounded linear operators on Y. Σ denotes the σ-algebra of Borel subsets of the complex plane. First, we describe a construction of Dieudonné (3).

Let K be a compact interval of the form $[0, y]$, where $y>0$. Two non-negative measurable functions f_1, f_2 on K are said to be *equimeasurable* if and only if for every $k \geqq 0$

$$m\{x \in K: f_1(x) \geqq k\} = m\{x \in K: f_2(x) \geqq k\},$$

where $m(\cdot)$ denotes Lebesgue measure on the real line. For each non-negative measurable function f on K, which is finite a.e., the decreasing rearrangement of f is the function f^* defined by

$$f^*(0) = \operatorname*{ess\text{-}sup}_{x \in K} f(x)$$

$$f^*(x) = \sup\{k \geqq 0: m\{y: f(y) \geqq k\} \geqq x\} \quad (x \in K, x \neq 0).$$

The function f^* is continuous on the left. Also f and f^* are equimeasurable by construction.

Now let w be a positive function, which is decreasing and integrable over K. The set of equivalence classes of complex measurable functions f on K such that $w(|f|^*)^2$ is integrable over K forms a separable reflexive Banach space L^2_w under the norm

$$\|f\|_w = \left(\int_0^y w(|f|^*)^2 dx\right)^{\frac{1}{2}} \quad (f \in L^2_w).$$

This class of Banach spaces has been studied by Halperin (5) and Lorentz (6), (7). It is possible to choose functions w_1, w_2 both positive, decreasing and integrable over K such that

$$\int_0^y w_1^2 dx = \int_0^y w_2^2 dx = +\infty$$

$$\int_0^y w_1 w_2 dx < +\infty.$$

Suppose that w_1, w_2 have been so chosen. Then $X = L_{w_1}^2 \oplus L_{w_2}^2$ is a separable reflexive Banach space. Define

$$\psi(f)(f_1, f_2) = (ff_1, ff_2) \quad (f_1 \in L_{w_1}^2, f_2 \in L_{w_2}^2)$$

for each f in $B(K)$, the Banach algebra of complex bounded Borel measurable functions on K under the supremum norm. Then ψ is a bounded algebra homomorphism of $B(K)$ into a subalgebra of $L(X)$. Moreover, the second commutant of $\{\psi(f)\colon f \in B(K)\}$ is strictly larger than $\{\psi(f)\colon f \in B(K)\}$.

In order to construct from this a hermitian operator with the desired properties, we require a preliminary result.

Lemma. *Let S be a scalar-type spectral operator with real spectrum on the separable complex Banach space Y. Suppose that $E(\cdot)$ is the resolution of the identity of S. Then $A^w(I, S)$ is precisely the algebra $A^u(E)$ generated by $\{E(\tau)\colon \tau \in \Sigma\}$ in the uniform operator topology.*

Proof. By Theorem 3 of (**4**, p. 592), $\{E(\tau)\colon \tau \in \Sigma\} \subseteq A^w(I, S)$. Since $A^w(I, S)$ is an algebra, it must also contain the algebra $A^u(E)$. To complete the proof it suffices to show that the last algebra is closed in the weak operator topology. Now $\{E(\tau)\colon \tau \in \Sigma\}$ is a σ-complete Boolean algebra of projections on Y. Since Y is separable, it is also countably decomposable, and so complete by (**1**, p. 350). The result now follows from Theorem 4.5 of (**1**, p. 358).

Returning now to Dieudonné's example, define

$$F(\tau) = \psi(\chi_{\tau \cap K}), \quad (\tau \in \Sigma).$$

Then $\{F(\tau)\colon \tau \in \Sigma\}$ is a bounded Boolean algebra of projections on X. By Lemmas 2.2 and 2.3 of (**2**, p. 367), X can be endowed with an equivalent norm with respect to which $\{F(\tau)\colon \tau \in \Sigma\}$ is a family of hermitian operators. With respect to the new norm, X is separable and reflexive. Hence $\{F(\tau)\colon \tau \in \Sigma\}$ is a σ-complete countably decomposable Boolean algebra of projections, and so is complete. Also $A^u(F)$ is strictly smaller than its second commutant. Define

$$S = \int_K \lambda F(d\lambda).$$

Then S is a hermitian scalar-type spectral operator with real spectrum. By the lemma, S has the property stated in the theorem.

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